Research paper

Catastrophe Swap Valuation Based on Stochastic Damage and its Numerical Solution

Abdolsadeh Neisy[1], Nasrollah Mahmoudpour[2], Moslem Peymany[3], Meisam Amiri[4]

[1] Department of Mathematics, Faculty of Statistics,Mathematics & Computer, Allameh Tabataba'i University, Tehran, Iran
a_neisy@atu.ac.ir

[2] Department of Finance and Banking, Faculty of Management & Accounting, Allameh Tabataba'i University, Tehran, Iran
n_mahmoudpour@atu.ac.ir

[3] Department of Finance and Banking, Faculty of Management & Accounting, Allameh Tabataba'i University, Tehran, Iran
m.peymany@atu.ac.ir

[4] Department of Finance and Banking, Faculty of Management & Accounting, Allameh Tabataba'i University, Tehran, Iran
amiry@atu.ac.ir

Abstract:
Pricing catastrophe swap as an instrument for insurance companies risk management, has received trivial attention in the previous studies, but in most of them, damage severities caused by the disaster has been considered to be fixed. In this study, through considering jumps for modeling the occurrence of disasters as in Unger [32] and completing it through considering damages caused by natural disasters as stochastic, an integro-differential model was extracted to value catastrophe swap contracts. In determining the swap price changes, the Ito command was followed and to achieve the catastrophic swap model, the generalization of the Black and Scholes modeling method was used [3]. With regard to the initial and boundary conditions, extracted model does not have an analytical solution; thus, its answer was approximated using the finite difference numerical method and the effect of considering the damage as stochastic on swap value was analyzed. In addition, the model and the extracted numerical solution were separately implemented on the data about the

[2]Corresponding author
Received: 21/02/2022 Accepted: 28/06/2022
http://dx.doi.org/10.22054/jmmf.2022.66733.1050

earthquake damage in the United States and Iran. The results showed that prices will experience a regular upward trend until damage growth, damage severities, and occurrence probability of a catastrophe are not so high that the buyer of the swap is forced to pay compensation to the swaps seller. Of course, the prices will fall sharply as soon as they reach and cross the threshold.

Keywords: Catastrophe Swap, Stochastic Damage, Numerical Solution, Earthquake Damage.
JEL Classifications: G22.

Introduction

In recent years, the number and severity of natural disasters has increased so significantly that the number of such disasters from 1980 to 2014 doubled in comparison to the same period last year. As a result, it is predicted that the economic damage caused by natural disasters from 2005 to 2030 will be about twice more than that of the previous period, namely more than 300 billion dollars [30]. Also, the average loss of the insured in the first six months of the 30-year period from 1988 to 2017 compared to the average loss of the insured in the first six months of the ten-year period from 2007 to 2017 has increased from 17,500 million dollars to 30,600 million dollars (balanced with inflation), which indicates this figure has increased for 74 % [27].
In addition to its direct and indirect impact on economic wealth, this issue has posed many challenges for insurance companies to provide financial resources to compensate for these damages and manage their risk. In particular, some of these natural disasters are so devastating and catastrophic that cannot be coped with using traditional risk management mechanisms such as insurance and reinsurance companies. One way to manage this issue is to use the capacity of financial markets through a process called insurance securitization. Through this strategy, insurance companies reduce their need to maintain their capital and increase their ability to enter a new business by converting their insurance revenues into securities and selling these securities in financial markets and subsequently transferring the associated risks to investors. As a result, since the early 1990s, insurance companies, and subsequently reinsurance companies, have been using new financial instruments to cover the risk of major natural disasters in capital markets. This led to the creation of a

new group of financial instruments whose cash flows depended on the occurrence or non-occurrence of large-scale natural disasters. For example, one of these securities is catastrophic bonds, in which payments are made to investors, unless a catastrophic event occurs and causes the loss of a part or all of the capital [7]. Other securities used in this regard are catastrophe derivatives, the variety of which is also expanding in official stock exchanges and over-the-counter markets such as Chicago Board of Trade (CBOT), Insurance Futures Exchange (IFEX), European Exchange (EUREX), and New York Mercantile Exchange (NYMEX) [15].
The use of catastrophe swap contracts as a derivative security, is also one of the developing strategies in insurance securities, in which the approach used in other swap contracts is modeled. In general, a swap contract is a type of derivative instrument, in which one party to the contract exchanges the revenues of its financial instruments with the revenues of the other party's financial instruments. Active participants in the swap contract market include financial institutions or companies. Thus, investors looking to cover the risk of price fluctuations choose a fixed cash flow, and other market participants, who take advantage of these opportunities by accepting risks because of their careers, choose a floating cash flow [22]. For example, Deutsche Bank Event Loss Swaps is one of the catastrophe swaps offered by Deutsche Bank in 2006 to protect customers from losses caused by natural disasters including floods and earthquakes in the United States. Accordingly, catastrophe swaps are a special type of swap contracts that allow the insurer (reinsurer) to take more risk by transferring part of the insurance risk to the other party. Thus, the investor's capital is exposed to the risk of natural disasters for certain revenue [7]. These securities, which are part of the contracts used in over-the-counter markets, will allow insurance companies (normal and reinsurance) to distribute the risk, finance and compensate for large damages.
Catastrophe swap is fundamentally different from other risk transfer instruments in contract design, areas of application and market development. In a typical contract, a support buyer (fixed payer- swap seller) agrees to pay the periodic premium to the seller of the support (floating payer- swap buyer) in return the predetermined compensation. All this is subject both to the occurrence of the trigger in the covered area and to the amount of damages exceeding the threshold specified in the swap contract. Of course, assuming no arbitrage, the catastrophe swap should act similarly to catastrophe bonds. However, unlike natural disaster bonds,

buying catastrophe swap does not require initial financial resources [7]. On the other hand, unlike natural disaster bonds and other insurance derivatives, the catastrophe swap market is very new, and its trading is based on the quoting method because of being in over-the-counter markets [15]. In addition, industry experts have stated that the size of the market is growing rapidly [12].
Despite the development of the use of catastrophe swaps, especially in recent years, researchers have paid more attention to other catastrophe bonds. For instance, many studies have examined catastrophe bonds (e.g., the experimental study of pricing of catastrophe bonds by Lane [24], Lane and Mahul [25] and Young, [37] computes the indifference price of cat bonds based on exponential utility investor preferences, investigating that by Egami and Young [17], and applying the basics of catastrophe bonds in analysis of reinsurance contracts [26]. In other studies, instead of swap bonds, other catastrophe derivative bonds have been investigated (e.g., using Markov model for pricing catastrophe derivatives [1], for pricing catastrophe futures using a jump diffusion process [13,14], for modeling catastrophe options [9,19], for extracting an analytical solution for pricing catastrophe options using Fourier transform [4], for providing a pricing model for catastrophe call options based on the compound Poisson process [29], for pricing Asian-style cat options [10,11], and some other studies such Härdle and Cabrera [20] and Chang [10]).
Few people have examined catastrophe swap, especially the way it is valued. For example, Borden and Sarkar [6] and Canter et al. [8] have referred to catastrophe swap in insurance-linked bonds. Cummins [12] and Cummins and Weiss [15] briefly described the general mechanism for contracting catastrophe swaps. In addition, Braun [7] is one of the few scholars to discuss the value of catastrophe swap. Using various parametric distributions to normalize the historical data of storm and earthquake damage in the United States, he concluded that in catastrophe swap pricing model, the heavy-tailed Burr distribution is appropriate to estimate damage severities and OrnsteinUhlenbeck process is appropriate to dynamize the intensity of Poisson's distribution. What is as important as other derivatives in catastrophe swap securities modeling is the assumed process of how the underlying asset behaves (i.e., the amount of damage caused by natural disasters in the catastrophe swap). Given the nature of natural disasters, which are low in frequency but high in severity, researchers use random processes with mutations stochastic processes with

jump sentences to model the value of catastrophe swaps, which indicates that catastrophes are occurring [32]. However, in all the conducted research in this domain, damage severities have been considered fixed in case of occurrence of natural disasters. Of course, the amount of damage caused by natural disasters is not a fixed amount and can, in itself, follow a separate random process, which is considered in this study. In this way, in order to model the catastrophe swap value, it has been assumed that the catastrophe followed the Poisson process, but damage severities in the event of a catastrophe are not constant and will be a random process.
Accordingly, in the second section of the article, by estimating the severity of catastrophic damages in the event of an accident and assuming that the price of the swap is a function of the damage, fluctuations, and time, the value of the catastrophe swap is extracted. In the third section, to solve the extracted model, a partial integro-differential equation is solved using the finite difference numerical and Euler methods. In the fourth section, the price sensitivity of the catastrophe swap to the newly added variable, namely the damages caused by the catastrophe, is analyzed. In the fifth section, the results obtained on the real data of earthquake damages in the United States and Iran are implemented and the relevant results are presented. Finally, in the sixth section, the results will be summarized and analyzed.

Evaluation of the catastrophe swap by stochastic damages

Pricing catastrophe swaps, like many other derivative bonds, is based on assuming a specific model for analyzing the underlying asset behavior. Of course, In catastrophe swaps, damages from natural disasters will replace this underlying asset. Various studies have been done to model this damage. One study in this field is that of Xu Yue [36]. He examined the distribution of damages caused by catastrophes in Norway. In this regard, Zolfaghari and Campbell [38] used historical data to provide an analysis of the earthquake damage model. Vickery et al. [33] also provided a model for damages caused by tornados. In 2010, Unger also considered the damage model as a geometric Brownian motion (GBM) and jump diffusion and used it to price catastrophe bonds. However, what is important in the assumed process for damage behavior is the fact that the damage model should measure small changes and large jumps. For example, in the model introduced by Unger [32], the following random process is considered to explain the behavior of catastrophe damage:

$$ds = \alpha s dt + \sigma s dw + \eta C dN \tag{1}$$

where α is the growth rate of damage, σ is small fluctuations (indicating small damage), ηC is the jump caused by major accidents and crises and the result of severe damage, and N is a Poisson process. The characteristic of Ungers [32] assumed model is that the severity of large damage added to the model by a Poisson process is considered fixed [28]. However, in the real world, damage severities can be variable and, as a result, a stochastic process. Accordingly, in this study, a damage model is designed that considers large damage severities (λ) in the assumed process for damage behavior as a stochastic process. In this regard, based on Ungers [32], a model is introduced as follows for damage behavior (s), that large damage severities (λ) also follows a stochastic process:

$$\begin{aligned} ds &= \alpha s dt + \sigma s dw_t^1 + \mathbb{C} e^{\lambda} \eta dN \\ d\lambda &= \mu \lambda dt + \gamma \lambda dw_t^2 \end{aligned} \tag{2}$$

where μ and γ are the drift rate and the fluctuation of damage severities, namely λ, and w_t^1 and w_t^2 are Winer's processes correlated with ρ, and the other variables are the same as those used in Ungers model. As it can be seen, the new model introduced in this paper is a two-factor model for damages, which simultaneously includes occurrence probabilities and damage severities. This is because instead of considering a fixed value of C as in Ungers model, the $\mathbb{C}e^{\lambda}$ is used as the amount of damage severities, and λ changes randomly in the geometric Brownian motion (GBM). This view in the damage model can be distinguished from Ungers damage model. By assuming the process introduced in equation 2, it can be stated that the value of the catastrophe swap shown in $c(t, s, \lambda)$ will be a function of time, damage severities, and the fluctuations in large damage severities. Therefore, its changes, based on Ito formula, will be as follows:

$$\begin{aligned} dc = &\left(c_t + \alpha s c_s + \mu \lambda c_\lambda + \frac{1}{2}\sigma^2 s^2 c_{ss} + \frac{1}{2}\gamma^2 \lambda^2 c_{\lambda\lambda} + \rho \sigma s \gamma \lambda c_{s\lambda} \right) dt \\ &+ (\gamma \lambda c_\lambda + \sigma s c_s) dw_t + [c(t, s + \mathbb{C}e^{\lambda}\eta, \lambda) - c(t, s, \lambda) dN \end{aligned} \tag{3}$$

To make it even simple, after placing

$$\begin{gathered}\varphi = c_t + \sigma s c_2 + \mu\lambda c_\lambda + \frac{1}{2}\sigma^2 s^2 c_{ss} + \frac{1}{2}\gamma^2 c_{\lambda\lambda} + \rho\sigma s\gamma\lambda c_{s\lambda} \\ \Delta = \gamma\lambda c_\lambda + \sigma s c_s\end{gathered} \tag{4}$$

We can state

$$dc = \varphi dt + \Delta dw_t + [c(t, s + \mathbb{C}e^{\lambda}\eta, \lambda) - c(t, s, \lambda)]dN \tag{5}$$

Now, similar to Black and Scholes [3] and the their delta risk hedging method, assume that is a portfolio containing two catastrophe swaps (for example, a catastrophe swap for an earthquake called c_1 and a catastrophe swap for a flood called c_2). Therefore, the value of the portfolio will be as follows:

$$\pi = x_1 c_1 + x_2 c_2 \tag{6}$$

where x_1 and x_2 will be the weight of each swap in the portfolio. It should be noted that in the process of making the above portfolio, the choice of the type of security is optional. To reach the model for valuing the catastrophe swap, we calculate changes in the value of the mentioned portfolio:

$$\begin{gathered}d\pi = x_1 dc_1 + x_2 dc_2 \\ dc_i = \varphi_i dt + \Delta_i dW_t + [c_i(t, s + \mathbb{C}e\lambda\eta, \lambda) - c_i(t, s, \lambda)]dN, i = 1, 2\end{gathered} \tag{7}$$

Assuming a risk-neutral portfolio and considering the return on pi stock portfolio equal to the risk-free interest rate r, we consider the expected return as follows [5]:

$$d\pi = r\pi dt \tag{8}$$

Now, using the pattern of Neisy and Salmani [31] and by placing equation (7) in equation (8), the random part of the model will be removed and the following equation will be obtained:

$$\begin{aligned}&\frac{\varphi_1 - (r + \lambda)c_1 + \lambda E[c_1(t, s + \mathbb{C}e^{\lambda}\eta, \lambda)]}{\Delta_1} \\ = &\frac{\varphi_2 - (r + \lambda)c_2 + \lambda E[c_2(t, s + \mathbb{C}e^{\lambda}\eta, \lambda)]}{\Delta_2}\end{aligned} \tag{9}$$

This equation is called market risk price, which is denoted by q. Continuing the modeling process, as in Unger [32], the following partial integro-differential equation will be obtained:

$$\begin{aligned} c_t + (\alpha s - q\sigma s)c_s + (\mu\lambda - q\gamma\lambda)c_\lambda + \frac{1}{2}\sigma^2 s^2 c_{ss} \\ + \frac{1}{2}\gamma^2\lambda^2 c_{\lambda\lambda} + \rho\sigma s\gamma\lambda c_{\gamma\lambda} + \mathcal{J} - (r+\lambda)c = 0 \end{aligned} \tag{10}$$

where

$$\mathcal{J} = \lambda \int_0^\infty c(t, s + \mathbb{C}e^\lambda, \lambda) f(\mathbb{C}e^\lambda \eta) d\eta \tag{11}$$

exists, and in $f(x) = \frac{1}{\nu\sqrt{2\pi}} e^{-0.5\left(\frac{Lnx-\chi}{\nu}\right)^2}$, χ is mean, and ν is variance. Furthermore, to solve the above model, the required final and boundary conditions will be as follows:

$$c(T, s, \lambda) = \begin{cases} B_{fix} & s < Z \\ 0 & s \geq Z \end{cases} \tag{12}$$

where Z is the threshold and

$$\begin{aligned} \lim_{s\to\infty} c(t, s, \lambda) = 0 \\ \lim_{\lambda\to\infty} c(t, s, \lambda) = \left(B_{fix} * e^{-rt}\right) I_{s<Z} \end{aligned} \tag{13}$$

where $I_{s<Z}$ indicates the characteristic function. Now, by placing $\lambda = 0$ and once again $s = 0$ in the original model, the boundary conditions will be completed as follows:

$$c_t + (\mu\lambda - q\gamma\lambda)c_\lambda + \frac{1}{2}\gamma^2\lambda^2 c_{\lambda\lambda} + \mathcal{J} - (r+\lambda)c = 0 \; ifs \to 0 \tag{14}$$

and

$$c_t + (\alpha s - q\sigma s)c_s + \frac{1}{2}\sigma^2 s^2 c_{ss} - (r)c = 0 \; if\lambda \to 0 \tag{15}$$

Numerical Solution of the Model

The partial integro-differential equation extracted in the previous section (Equation 10) is an initial and boundary value problem, which is called the new catastrophe swap model based on stochastic models in this study.

Because the initial and boundary value problem has an answer [23], the new swap model will also have a unique answer. However, this answer is not a closed and analytical answer, so it is necessary to get an estimate of the answer by numerical methods. This numerical method is presented in this section. To do so, a finite difference semi-discretization method will be used. Therefore, in order to increase the accuracy of convergence, first s and are discretization and we will reach an ordinary differential equation system based on time. Then the equation will be solved by Euler's method. To achieve this, the range of s , λ changes is shown as $[0, s_{max}]$ and $[0, s_{max}]$, respectively and $0 < s < s_{max}$ and $0 < \lambda < \lambda_{max}$ become discrete as follows:

$$\begin{aligned} 0 < s_0 < s_1 < s_2 < s_max \\ 0 < \lambda_0 < \lambda_1 < \lambda_2 < \lambda_max \end{aligned} \tag{16}$$

Then by changing $\tau = T - t$ and placing $C(t, s, \lambda) = U(\tau, s, \lambda)$ to achieve one of the convergence conditions and considering the following equations:

$$\begin{aligned} \frac{\partial C}{\partial t} = -\frac{\partial U}{\partial \tau}, \frac{\partial C}{\partial s} = \frac{\partial C}{\partial s}, \frac{\partial C}{\partial \lambda} = \frac{\partial U}{\partial \lambda}, \frac{\partial^2 C}{\partial s^2} = \frac{\partial^2 U}{\partial s^2}, \frac{\partial^2 C}{\partial \lambda^2} \\ = \frac{\partial^2 U}{\partial \lambda^2}, \frac{\partial^2 C}{\partial s \partial \lambda} = \frac{\partial^2 U}{\partial s \partial \lambda} \end{aligned} \tag{17}$$

The following PIDE will be reached:

$$U_\tau = g_1(s)U_s + g_2(\lambda)U_\lambda + g_3(s)U_{ss} + g_4(\lambda)U_{\lambda\lambda} + g_5(s, \lambda)U_{s\lambda} + \mathcal{J} + g_6(\lambda)U \tag{18}$$

where $g_i, i \in \{1, \ldots, 6\}$ are coefficients and

$$\mathcal{J} = \lambda \int_0^x U(\tau, s + \mathbb{C}e^\lambda \eta, \lambda) f(\mathbb{C}e^\lambda \eta) d\eta \tag{19}$$

Also, the initial and boundary conditions will be as follows:

$$\begin{aligned} U_(0, s, \lambda) = \begin{cases} B_{fix} & s < Z \\ 0 & s \geq Z \end{cases} \\ \lim_{s \to \infty} U(\tau, s, \lambda) = 0 \\ \lim_{\lambda \to \infty} U(\tau, s, \lambda) = \left(B_{fix} * e^{-r(T-\tau)}\right) I_{s<Z} \end{aligned} \tag{20}$$

The partial integro-differential equation (18) is now written as follows:

$$U_\tau = \Im_d U + \Im_j U \tag{21}$$

where

$$\Im_d U = g_1(s)U_s + g_2(\lambda)U_\lambda + g_3(s)U_{ss} + g_4(\lambda)U_{\lambda\lambda} + g_5(s,\lambda)U_s\lambda + g_6(\lambda)U \tag{22}$$

and

$$\Im_j U = \mathcal{J} \tag{23}$$

Now, by assuming $u_{i,j} = U(\tau, s_i, \lambda_j)$ and using the finite difference method, we will have:

$$\begin{gathered} U_s \approx \frac{u_{i+1,j} - u_{i-1,j}}{2ds}, U_\lambda \approx \frac{u_{i,j+1} - u_{i,j-1}}{2d\lambda}, \\ U_{ss} \approx \frac{u_{i+1,j} - u_{i,j} + u_{i-1,j}}{(ds)^2} U_{\lambda\lambda} \approx \frac{u_{i,j+1} - u_{i,j} + u_{i,j+1}}{(d\lambda)^2}, \\ U_{s\lambda} \approx \frac{u_{i+1,j+1} - u_{i-1,j+1} + u_{i+1,j-1} + u_{i-1,j-1}}{4d\lambda ds} \end{gathered} \tag{24}$$

Now assume:

$$u = [u_{1,1} \dots u_{1,M-1} \dots u_{N-1,1} \dots u_{N-1,M-1}]^{Tr.} \tag{25}$$

Using the above assumptions, matrices A and B are considered in such a way that can justify operator (24).

$$\Im_d U \to Au + B \tag{26}$$

Matrix A is known as the sparse matrix, which, according to discrete (26) and vector B, includes boundary conditions and is easily obtained using Hirsas [21] method. Also, considering $x = s + \mathbb{C}e^{\lambda}\eta \ and \ f_{i,j} = f(s_i - s_j)$ for the integral part, we will have the following:

$$\begin{gathered} \frac{1}{\mathbb{C}e^{\lambda_j}} \int_0^\infty U(\tau, x, \lambda_j) f(x - s_i) dx \\ \approx \frac{1}{\mathbb{C}e^{\lambda_j}} \frac{ds}{2} \left[f_{i,0}u_{0,j} + f_{i,N}u_{N,j} + 2\sum_{p=1}^{N-1} f_{i,p}u_{p,j} \right] \end{gathered} \tag{27}$$

A numerical method is proposed to solve the new pricing model, for which the semi-discretization finite difference method will be used. Therefore, to

increase the convergence accuracy, we first discretize S and λ and arrive at a system of ordinary differential equations in terms of time, then the equation will be solved by the Euler method. To achieve this, the change intervals S and λ are $[0, s_{max}]$ and $[0, \lambda_{max}]$, respectively, in which s_{max} and λ_{max} are derived from market data and insurance companies. The intervals $0 < s < s_{max}$ and $0 < \lambda < \lambda_{max}$ are discretized as follows:
$0 < s_0 < s_1 < s_2 < s_{max}$
$0 < \lambda_0 < \lambda_1 < \lambda_2 < \lambda_{max}$

The Integral operator of matrices F and G are as follows:

$$\Im_j U \rightarrow Fu + G \tag{28}$$

According to Equations (24) and (25), operator I will be considered as follows:

$$\Im =: \Im_d + \Im_j \tag{29}$$

Finally, to solve PIDE (18), it is enough to use the following repetitive method:

$$u^{k+1} = \Delta u^k + \vartheta_k \tag{30}$$

where

$$\begin{aligned} u^{k+1} &= [u_{11}^k \dots u_{N-1M-1}^k]' \\ \Delta &= A + \lambda F \\ \vartheta_k &= B_k + \lambda G_k \end{aligned} \tag{31}$$

To test general futures of extracted solution, following figure shows the behavior of the swap price by changing the values of λ and s the remaining time until the maturity of the swap agreement.

According to the diagram on the left, the swap price will decrease as the amount of damage increases, and this decrease is very large at the threshold value. In this figure, the fluctuation of the damage is assumed to be constant and the price of the catastrophic swap sheet is examined in relation to the amount of damage. In the diagram on the right, disaster swap prices are considered in terms of time. Also, for fixed amounts for λ and damages, the price of the catastrophe swap is converged with fixed payments, and finally at maturity date, the price of the swap is equal to the price of fixed payments, which is the same settlement in the maturity date. The reason for this lies in that the closer we get to maturity time, or in other words, the shorter our distance from the maturity time, the

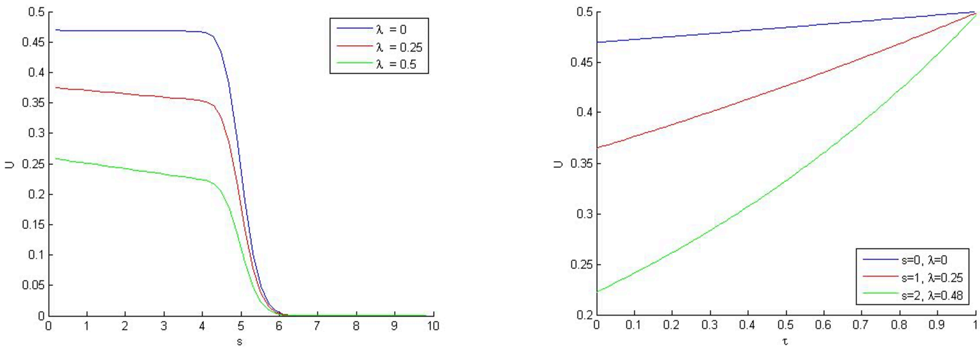

Figure 1: The effect of the extracted model variables on swap value

less likely the damage will occur. As a result, the price of swaps will be equal to fixed payments, which is fully consistent with the general characteristics of catastrophe swaps.

Sensitivity analysis of the extracted model

In this section, the sensitivity analysis of λ will be performed for the catastrophe swap model. For this purpose, the effect of the slight change of $d\lambda$ in the λ on the price changes of the catastrophe swap is analyzed to show what effect λ will have on the extracted price if it is not accurate. To begin, by changing the symbolism, it is assumed that u_k^{λ} is the vector price of the catastrophe swap at the time of k is equal to λ, and $u_k^{\lambda+d\lambda}$ is the vector price of the catastrophe swap at the time of k is equal to $\lambda+d\lambda$.. Therefore, the repetitive process (30) for each one is performed as follows:

$$u_{k+1}^{\lambda} = \psi u_k^{\lambda} + \theta_k \tag{32}$$

$$u_{k+1}^{\lambda+d\lambda} = \psi' u_k^{\lambda+d\lambda} + \theta'_k \tag{33}$$

As a result, the difference between the two prices will be equal to:

$$D_{k+1} = \overline{\psi} D_k + \overline{\theta}_k \tag{34}$$

where $D_k = u_k^{\lambda+d\lambda} - u_k^{\lambda}\lambda$, $\overline{\psi} = \psi' - \psi$, and $\overline{\theta}_k = \theta'_k - \theta_k$ and its equivalent matrix form is as follows:

$$\overline{\psi} = d\lambda F - d\lambda d\tau I_{(M-1)(M-1)} \tag{35}$$

$$\overline{\theta}_k = d\lambda G_k \tag{36}$$

Because the initial conditions are independent and the same for both cases ($i.e., u_0^{\lambda+d\lambda} = u_0^{\lambda}$), the value of D_1 will be as follows:

$$D_1 = \overline{\psi}_0 + \overline{\theta}_0 = \overline{\theta}_0 \tag{37}$$

By placing D_1 in the repetitive process (34), D_2 will also be obtained as follows:

$$D_2 = \overline{\psi}\ \overline{\theta}_0 + \overline{\theta}_1 \tag{38}$$

As this trend continues, the price difference at K will be as follows:

$$D_K = \sum_{l=i}^{K} \overline{\psi}^{K-i} \overline{\theta}_{i-1} \tag{39}$$

By changing the value of K, the difference in other times can be seen. In fact, the sum indicates the difference between two different values for λ. Therefore, if we do not pay attention to λ, the real price of the swap will not be achieved.

Numerical Results

In this section, using the stated method, the extracted model was implemented on Vranes and Pielkes [34] real earthquake damage dataset from 1900 to 2005 as well as on Irans earthquake damage data from 1910 to 2019, which were extracted EM-DAT database and the model parameters will be estimated. Vranes and Pielke [34] estimated the economic damage (normalized based on inflation, wealth, and population) to be 435 billion dollars. During this period, 13 earthquakes with a damage

It should be noted that because information on economic losses of 112 earthquakes recorded in Iran did not exist for all periods, this amount, for periods that faced this problem, was estimated using regression models with explanatory power of about 47 % and based on the information available for each earthquake (magnitude of the earthquake in Richter, the number of people killed, and the number of people affected by the earthquake) of more than 1 billion dollars and 5 earthquakes with a damage of more than 10 billion dollars were recorded. This huge difference between the amount of damage caused by severe earthquakes is a good justification for preferring a random value for the severity of the damage over a fixed value for it. This is also true for the damage caused by the earthquakes occurring in Iran. Another revealing point about the earthquake damage information in both data sets is the presence of unstable fluctuations in both time series. This also explains how the modeling is implemented in this study.

Based on the mentioned data and to obtain the parameters of the equation related to λ, the discretization method provided by Azizi and Neisy [2] and Bjorks [5] estimation method were used. By repeating the above-mentioned method, the values of other parameters such as μ and γ were also determined. The estimated results are presented in the following table:

Table 1: Results of estimating model parameters based on data from the US and Iran

Estimated parameter	the US	Iran
α	0.38350	0.0028
σ	0.76690	0.00570
μ	0.00009	0.00010
γ	0.00190	0.00170
q	0.00190	0.00170
χ	0.50006	0.49120
ν	2.45000	0.23300

Notes: This table shows estimation results of extracted model parameters for described data sets. In estimating these parameters, is considered to be 2.5, λ *is limited to zero and a half and the threshold value is 0.5 (these values are determined by the terms of the swap contract). Also, in the implementation of the discretization process, the values of* N*,* M*, and* dt*are considered to be 50, 30, and 0.01, respectively.*

This has been verified by the Arch test in both data sets

In short, in this method, the data are categorized based on the minimum amount of time, and an average and a fluctuation are estimated for each time period. Then, using the least squares method and $n$numbers of α and σ values, a time-dependent function will be created for each

Earthquake damage data is divided into catastrophic and non-catastrophic. Catastrophe damages are data that amount to 4.8776 (approximately 0.5) or more, and other data less than that will be considered non-catastrophic. This number is the threshold of the contract, and if the amount of damage caused by the earthquake is more than this amount, the cash flow from the swap seller to the swap buyer will change. With the economic growth of the society, the damages caused by the insured property and assets will increase. Therefore, there is a close and direct relationship between the amount of damage and growth. The growth rate of the damage will be indicated by α. Events may occur that deviate from the specified value α, which is considered to be σ in the damage model. α and σ are obtained from non-catastrophic damage data. μ is growth rate fluctuations of damage and γ is standard deviations of damage fluctuations, both of which result from catastrophic damage data. χ and ν respectively are average and variance damage data, When the data is not yet divided into catastrophic and non-catastrophic.
Based on the above estimated parameters, in the following two figures, the price of the catastrophe swap at the outset of the contract is drawn for different values of λ:

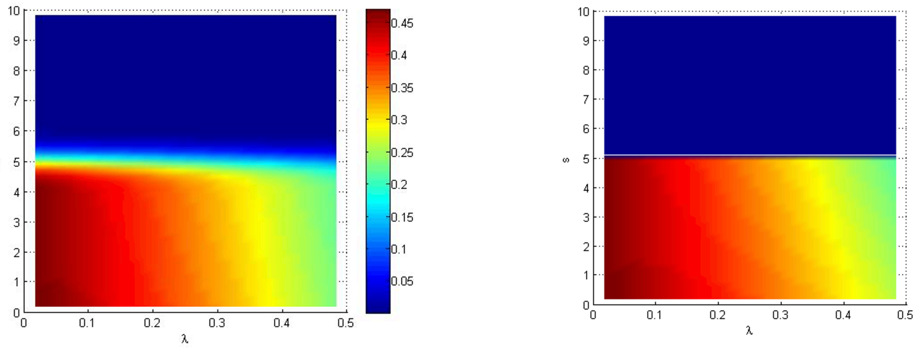

Figure 2: Catastrophe Swap Price at time zero based on the estimated Parameters

In the above diagrams, the horizontal axis indicates the severity of the damage and the probability of occurrence and the vertical axis indicates the damage. Due to the scattering of data and for ease of analysis, the damage data is assumed to be in the range of zero to ten, and the fluctuations of damage are segmented in the range of 0 to 0.5. each of these diagrams point represents a number, which creates from the intersection of λ and S. As can be seen, when the damage is about 0.5, due to the starting threshold and the reversal of the cash flow of the swap, the price tends to be zero. A change in the direction of cash flow means that in the event of a catastrophe, the support buyer or swap seller pays a fixed amount of cash to the support seller or swap buyer. However, after the occurrence of the catastrophe and overtaking the damage threshold, the direction of cash flow will change and the investor or support buyer will be required to compensate for the damage and pay the rates at a floating rate. when using the Eulers repetitive method in numerical methods, there are unique S and λ at each point in any time and in each step. If at different times and with different S and λs, the swap security is purchased and the damage has not exceeded the threshold, Whatever we closer to maturity or settlement (time one), Prices will converge to one point, this is the catastrophe swap security yield at maturity, which is approximately equal to the amount of fixed rate payments by the support buyer.

Results and Conclusion

The potential of financial markets has led various institutions, including insurance and reinsurance companies to increasingly use it and manage their risks, especially in the event of severe and catastrophic damages by financial solutions. One of the solutions used in this field is catastrophe swap contracts. Despite the capabilities of the catastrophe swap and its expanding use in financial markets, little research has dealt with the valuation of these securities. In the few conducted studies in this domain, the assumed random process to model the behavior of catastrophe damages contained only one statement for occurrence probabilities, but the amount of this damage was considered fixed. In this study, this shortcoming was addressed by considering the amount of stochastic catastrophic damages. To do so, the occurrence probabilities and the resulting damage are considered as a stochastic process of two-factor diffusion jump, and by changing the fixed amount of catastrophe damage in previous studies (i.e., parameter c) to ce^{λ} and considering a random behavior for λ, the damage severities changed from a fixed variable to a random one. In other words, changes in the amount of the catastrophe damage were added to previous models, such as Ungers [32] model, as a separate random process.

In order to determine the swap price, a partial integro-differential equation was extracted to value catastrophe swaps using Black and Scholes [3] model and Ito formula. Thereafter, the answer of the extracted model was estimated by numerical methods, and the equation turned into an ordinary differential equation using semi-discretization. Then, the finite difference numerical method and Eulers method were used to solve the extracted catastrophe swap pricing model. In addition, the model and extracted numerical solution were implemented on the data related to earthquake damages in the United States and Iran.
In general, the results of modeling and implementation of the model indicate that in this model, as expected, the price trend for damage below the threshold is a regular trend that is commensurate with the damage occurrence probability and severity. In other words, prices will continue to rise regularly until the damage growth, severity, and occurrence probability have reached the point where the swap buyer is forced to pay compensation to the swap seller. However, prices will fall sharply as soon as the specified threshold in the swap agreement is reached and exceeded.
In conclusion, it is recommended that in future research, the extracted model be applied to other natural disasters and other countries. In addition, if there are reliable price data for catastrophe swap contracts in a market, the price of the catastrophe swap contracts could be compared to the results of the extracted model in this paper and previous model with fixed catastrophe damage. It is also possible to modify this model and expanded to include pricing for other items, such as catastrophe bonds, if the yield function and the initial boundary conditions change.

Notes: In this figure, left diagram shows effect of s on swap values (U) for different values of λ assuming that other factors are constant. In the right diagram, the effect of τ on U is presented for different combinations of s and λ. (Figure caption 1)

Notes: In these diagrams, the current catastrophe swap price is presented, based on extracted model and estimated parameters of data sets of US and Iran. The left diagram shows results for US and right diagram shows results of Iran. In both diagrams, the horizontal axis shows the value of λ and the vertical axis is the amount of damage, and the estimated value of the catastrophe swap is determined by the color of the diagram, based on the sidebar. In fact, considering the value of λ and the damage, the coordinates of a given point in the diagram will be located, *and the color of that given point will be compared with that of the sidebar and finally the swaps value will be obtained. (Figure caption 2)*

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