Research paper

Dynamic behavior in a three coupled Kaldor-Kalecki delayed model

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Abstract:
In this paper, a three coupled Kaldor-Kalecki model with multiple delays is investigated. By means of the generalized Chafee's criterion, some sufficient conditions to guarantee the existence of oscillatory solution for the model are obtained. Computer simulations are provided to demonstrate the proposed results.

Keywords: three coupled Kaldor-Kalecki model, delay, instability, limit cycle
AMS Subject Classification 34K11

Introduction

It is known that the original Kaldor-Kalecki model of business cycle was an example of a difference-differential model [1,2]. The authors proposed and studied business models using ordinary differential equations, nonlinear investment and saving functions. They showed that periodic solutions exist under the assumption of nonlinearity. Since then, similar models were also analyzed by several researchers and the existence of limit cycles was established due to the nonlinearity, see [3, 4, 5]. There are many researchers who have studied the bifurcating periodic solutions of Kaldor-Kalecki models [6-14]. For example, Wang and Wu have investigated the

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following Kaldor-Kalecki model:

$$\begin{cases} y'(t) = \alpha(I(y(t), k(t))) - S(y(t), k(t)), \\ k'(t) = I(y(t-\tau)) - qk(t). \end{cases} \tag{1}$$

where y is the gross product, k is the capital stock, $\alpha > 0$ is the adjustment coefficient in the goods market, $o < q < 1$ is the depreciation rate of capital stock, $I(y, k)$ and $S(y, k)$ are investment and saving functions, and $\tau > 0$ is a time lag representing delay for the investment due to the past investment decision. Stability analysis for the equilibrium point was carried out. The authors showed that Hopf bifurcation occurred and periodic solutions emerged as the delay crosses some critical values. By deriving the normal forms for the system, the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions were established [6]. Cao and Sun have considered a Kaldor-Kalecki model of business cycle with two discrete time delays as follows:

$$\begin{cases} y'(t) = \alpha(I(y(t), k(t))) - S(y(t), k(t)), \\ k'(t) = I(y(t-\tau_1), k(t-\tau_2)) - qk(t). \end{cases} \tag{2}$$

By analyzing the corresponding characteristic equations, the local stability of the positive equilibrium was discussed. Choosing the adjustment coefficient in the goods market α as the bifurcation parameter, the existence of Hopf bifurcation was then investigated in detail. Secondly, by combining the normal form method with the center manifold theorem, the direction of the bifurcation and the stability of the bifurcated periodic solutions were determined [7]. Yu and Peng [8] introduced a distributed delay and modified the Kaldor-Kalecki model in the following form:

$$\begin{cases} y'(t) = \alpha(I(y(t), k(t))) - S(y(t), k(t)), \\ k'(t) = I(y(t-\tau), \int_{-\infty}^{t} F(t-s)k(s)ds) - qk(t). \end{cases} \tag{3}$$

With the corresponding characteristic equation analyzed, the local stability of the positive equilibrium was investigated. The authors found that there exist Hopf bifurcations when the discrete time delay passes a sequence of critical values in model (3). By applying the method of multiple scales, the explicit formulae which determine the direction of Hopf bifurcation and the stability of bifurcating periodic solutions were derived. Due to the importance of anticipation for making decisions and organizational transformations, the Kaldor-Kalecki model of business cycle was studied in view of showing its anticipatory capabilities [15]. The dynamics behaviors of Kaldor-Kalecki business cycle model with diffusion effect and time delay under the Neumann boundary conditions were investigated. The time delay can give rise to the Hopf bifurcation when the time delay passed a critical value [16]. For the diffusive Kaldor-Kalecki model with a delay included in both gross product and capital stock functions, the stability and Hopf bifurcation and the reaction-diffusion domain were considered [17]. For an extended version of Kaldor's economic growth model, the role of the government and its simultaneous monetary, fiscal policies can affect the economic stability [18]. In recent years, when local economies are subject to various economic dependencies due to various factors, we see that macroeconomic models cannot be treated as isolated systems anymore. Therefore, Zduniak et al. have investigated the two coupled Kaldor-Kalecki model with delay [19]:

$$\begin{cases} y_1'(t) = \alpha_1(F_1(y_1(t)) - \delta_1 y_2(t) - \gamma_1 y_1(t)), \\ y_2'(t) = F_1(y_1(t-\tau)) - \delta_1 y_2(t-\tau) - \delta y_2(t), \\ y_3'(t) = \alpha_2(F_2(y_2(t)) - \delta_2 y_4(t) - \gamma_2 y_3(t)), \\ y_4'(t) = F_2(y_2(t-\tau)) - \delta_2 y_4(t-\tau) - \delta y_4(t) + s(y_1(t) - y_3(t)). \end{cases} \tag{4}$$

where F_i are investment functions, and it is taken from the published literature as $F_i(t) = \frac{e^{y_i(t)}}{1+e^{y_i(t)}}$. Note that s is a coupling coefficient, α_1 and α_2 are the adjustment coefficients (correction factors), $\delta \in (0,1)$ is the depreciation rate of capital stock, $\gamma_1, \gamma_2, \delta_1$ and δ_2 are constants, and τ denotes the time delay. The authors considered two types of investment functions that lead to different behavior of the system. The model with unidirectional coupling to investigate the influence of a global economy on a local economy was also considered. In the present paper, we extend model (4) to the three generalized coupled Kaldor-Kalecki model with delays:

$$\begin{cases} y_1'(t) = \alpha_1(F_1(y_1(t)) - \delta_1 y_2(t) - \gamma_1 y_1(t)), \\ y_2'(t) = F_1(y_1(t-\tau_1)) - \delta_1 y_2(t-\tau_2) - r_1 y_2(t) + s_1(y_1(t) - y_3(t)), \\ y_3'(t) = \alpha_2(F_3(y_3(t)) - \delta_2 y_4(t) - \gamma_2 y_3(t)), \\ y_4'(t) = F_3(y_3(t-\tau_3)) - \delta_2 y_4(t-\tau_4) - r_2 y_4(t) + s_2(y_3(t) - y_5(t)), \\ y_5'(t) = \alpha_3(F_5(y_5(t)) - \delta_3 y_6(t) - \gamma_3 y_5(t)), \\ y_6'(t) = F_5(y_5(t-\tau_5)) - \delta_3 y_6(t-\tau_6) - r_3 y_6(t) + s_3(y_5(t) - y_1(t)). \end{cases} \tag{5}$$

where $F_1(y_1(t)) = \frac{e^{y_1(t)}}{1+e^{y_1(t)}}, F_3(y_3(t)) = \frac{e^{y_3(t)}}{1+e^{y_3(t)}}$, and $F_5(y_5(t)) = \frac{e^{y_5(t)}}{1+e^{y_5(t)}}$. And, $\alpha_i, \delta_i, r_i, s_i$ are positive constants. Our goal is to consider the dynamic behavior of model (5). By means of the mathematical analysis method, the existence of periodic oscillatory solutions has been derived. We point out that the bifurcating method is hard to deal with in model (5) since there are six delays.

Preliminaries

If $y_1^*, y_2^*, \cdots, y_6^*$ is a positive equilibrium point of system (5), and make the change of $u_i(t) = y_i(t) - y_i^*$, then by linearizing system (5) around $(0, 0, \cdots, 0)$ we have

$$\begin{cases} u_1'(t) &=& \alpha_1 F_1'(y_1^*)u_1(t) - \alpha_1\delta_1 u_2(t) - \alpha_1\gamma_1 u_1(t), \\ u_2'(t) &=& F_1'(y_1^*)u_1(t-\tau_1) - \delta_1 u_2(t-\tau_2) + s_1 u_1(t) - r_1 u_2(t) - s_1 u_3(t), \\ u_3'(t) &=& \alpha_2 F_3'(y_3^*)u_3(t) - \alpha_2\delta_2 u_4(t) - \alpha_2\gamma_2 u_3(t), \\ u_4'(t) &=& F_3'(y_3^*)u_3(t-\tau_3) - \delta_2 u_4(t-\tau_4) + s_2 u_3(t) - r_2 u_4(t) - s_2 u_5(t), \\ u_5'(t) &=& \alpha_3 F_5'(y_5^*)u_5(t) - \alpha_3\delta_3 u_6(t) - \alpha_3\gamma_3 u_5(t), \\ u_6'(t) &=& F_5'(y_5^*)u_5(t-\tau_5) - \delta_3 u_6(t-\tau_6) + s_3 u_5(t) - r_3 u_6(t) - s_3 u_1(t). \end{cases} \tag{6}$$

The matrix form of system (6) is the following:

$$u'(t) = Au(t) + Bu(t-\tau). \tag{7}$$

where $u(t) = [u_1(t), u_2(t), \cdots, u_6(t)]^T$, $u(t-\tau) = [u_1(t-\tau_1), u_2(t-\tau_2), \cdots, u_6(t-\tau_6)]^T$. Both $A = (a_{ij})_{6\times 6}$ and $B = (b_{ij})_{6\times 6}$ are 6×6

matrices as follows:

$$A=(a_{ij})_{6\times6}=\begin{pmatrix} a_1 & -b_1 & 0 & 0 & 0 & 0\\ s_1 & -r_1 & -s_1 & 0 & 0 & 0\\ 0 & 0 & a_3 & -b_2 & 0 & 0\\ 0 & 0 & s_2 & -r_2 & -s_2 & 0\\ 0 & 0 & 0 & 0 & a_5 & -b_3\\ -s_3 & 0 & 0 & 0 & s_3 & -r_3 \end{pmatrix},$$

where $a_1 = \alpha_1 F_1'(y_1^*) - \alpha_1\gamma_1, b_1 = \alpha_1\delta_1, a_3 = \alpha_2 F_3'(y_3^*) - \alpha_2\gamma_2, b_2 = \alpha_2\delta_2, a_5 = \alpha_3 F_5'(y_5^*) - \alpha_3\gamma_3, b_3 = \alpha_3\delta_3$.

$$B=(b_{ij})_{6\times6}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ a_2 & -\delta_1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & a_4 & -\delta_2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & a_6 & -\delta_3 \end{pmatrix}.$$

where $a_2 = F_1'(y_1^*), a_4 = F_3'(y_3^*), a_6 = F_5'(y_5^*)$.

Definition 1 The trivial solution of system (6) is unstable, if there exists at least one component of the trivial solution which is unstable.

Lemma 1 Assume that matrix $C = (A + B)$ is a nonsingular matrix, then system (5) has a unique positive equilibrium point.

Proof An equilibrium point $u^* = [u_1^*, u_2^*, \cdots, u_6^*]^T$ of system (7) is a constant solution of the following algebraic equation

$$Au^* + Bu^* = Cu^* = \mathbf{0}. \tag{8}$$

Since C is a nonsingular matrix, based on the basic algebraic knowledge, system (8) has a unique trivial solution, implying that system (5) has a unique equilibrium point $y_1^*, y_2^*, \cdots, y_6^*$. Indeed, if $\bar{y}_1^*, \bar{y}_2^*, \cdots, \bar{y}_6^*$ is another set of equilibrium point of system (5), then we have

$$\begin{cases} \alpha_1(F_1(y_1^*) - F_1(\bar{y}_1^*)) - \alpha_1\delta_1(y_2^* - \bar{y}_2^*) - \alpha_1\gamma_1(y_1^* - \bar{y}_1^*) = 0,\\ F_1(y_1^*) - F_1(\bar{y}_1^*) - \delta_1(y_2^* - \bar{y}_2^*) - r_1(y_2^* - \bar{y}_2^*) + s_1(y_1^* - \bar{y}_1^*) - s_1(y_3^* - \bar{y}_3^*) = 0,\\ \alpha_2(F_3(y_3^*) - F_3(\bar{y}_3^*)) - \alpha_2\delta_2(y_4^* - \bar{y}_4^*) - \alpha_2\gamma_2(y_3^* - \bar{y}_3^*) = 0,\\ F_3(y_3^*) - F_3(\bar{y}_3^*) - \delta_2(y_4^* - \bar{y}_4^*) - r_2(y_4^* - \bar{y}_4^*) + s_2(y_3^* - \bar{y}_3^*) - s_2(y_5^* - \bar{y}_5^*) = 0,\\ \alpha_3(F_5(y_5^*) - F_5(\bar{y}_5^*)) - \alpha_3\delta_3(y_6^* - \bar{y}_6^*) - \alpha_3\gamma_3(y_5^* - \bar{y}_5^*) = 0,\\ F_5(y_5^*) - F_5(\bar{y}_5^*) - \delta_3(y_6^* - \bar{y}_6^*) - r_3(y_6^* - \bar{y}_6^*) + s_3(y_5^* - \bar{y}_6^*) - s_3(y_1^* - \bar{y}_1^*) = 0. \end{cases} \tag{9}$$

Noting that $F_i(y_i(t)) = \frac{e^{y_i(t)}}{1+e^{y_i(t)}}$, then $F_i'(y_i(t)) = \frac{e^{y_i(t)}}{(1+e^{y_i(t)})^2}(i = 1, 3, 5)$. Therefore, $0 < F_i(y_i(t)) < 1$ are monotone increasing functions. By the mean value theorem, $F_i(y_i^*) - F_i(\bar{y}_i^*) = F_i'(\eta_i)(y_i^* - \bar{y}_i^*)$, where $\eta_i \in (y_i^*, \bar{y}_i^*)(i = 1, 3, 5)$. Thus, from system (9) we have

$$\begin{cases} \alpha_1 F_1'(\eta_1)(y_1^* - \bar{y}_1^*) - \alpha_1\delta_1(y_2^* - \bar{y}_2^*) - \alpha_1\gamma_1(y_1^* - \bar{y}_1^*) = 0, \\ F_1'(\eta_1)(y_1^* - \bar{y}_1^*) - \delta_1(y_2^* - \bar{y}_2^*) - r_1(y_2^* - \bar{y}_2^*) + s_1(y_1^* - \bar{y}_1^*) - s_1(y_3^* - \bar{y}_3^*) = 0, \\ \alpha_2 F_3'(\eta_3)(y_3^* - \bar{y}_3^*) - \alpha_2\delta_2(y_4^* - \bar{y}_4^*) - \alpha_2\gamma_2(y_3^* - \bar{y}_3^*) = 0, \\ F_3'(\eta_3)(y_3^* - \bar{y}_3^*) - \delta_2(y_4^* - \bar{y}_4^*) - r_2(y_4^* - \bar{y}_4^*) + s_2(y_3^* - \bar{y}_3^*) - s_2(y_5^* - \bar{y}_5^*) = 0, \\ \alpha_3 F_5'(\eta_5)(y_5^* - \bar{y}_5^*) - \alpha_3\delta_3(y_6^* - \bar{y}_6^*) - \alpha_3\gamma_3(y_5^* - \bar{y}_5^*) = 0, \\ F_5'(\eta_5)(y_5^* - \bar{y}_5^*) - \delta_3(y_6^* - \bar{y}_6^*) - r_3(y_6^* - \bar{y}_6^*) + s_3(y_5^* - \bar{y}_6^*) - s_3(y_1^* - \bar{y}_1^*) = 0. \end{cases} \tag{10}$$

Noting that if y_i^* is close sufficiently to $\bar{y}_i^*$, then $F_i'(\eta_i) = F_i(y_i^*), i = 1, 3, 5$. Therefore, the coefficient matrix of system (10) about variables $(y_i^* - \bar{y}_i^*)$ is exactly $C = A + B$. Since C is a nonsingular matrix, implies that $y_i^* - \bar{y}_i^* = 0$. Namely $y_i^* = \bar{y}_i^*$. So system (5) has a unique positive equilibrium point. Our simulation also indicates the uniqueness of the positive equilibrium point.

Lemma 2 Assume that $\alpha_i > 0, r_i > 0, \delta_i > 0, \gamma_i > 0$ $(i = 1, 2, 3)$, then all solutions of system (5) are bounded.

Proof To prove the boundedness of the solutions in system (5), we construct a Lyapunov function $V(t) = \sum_{i=1}^{6} \frac{1}{2} y_i^2$. Noting that $F_i(y_i(t)) \le 1$, calculating the derivative of $V(t)$ through system (5) we get

$$\begin{aligned} V'(t)|_{(5)} &= \sum_{i=1}^{6} y_i y_i' = y_1[\alpha_1(F_1(y_1(t)) - \delta_1 y_2(t) - \gamma_1 y_1(t))] \\ &+ y_2[F_1(y_1(t-\tau_1)) - \delta_1 y_2(t-\tau_2) - r_1 y_2(t) + s_1(y_1(t) - y_3(t))] \\ &+ \cdots + y_6[F_5(y_5(t-\tau_5)) - \delta_3 y_6(t-\tau_6) - r_3 y_6(t) + s_3(y_5(t) - y_1(t))] \\ &\le -\alpha_1\gamma_1 y_1^2 - r_1 y_2^2 - \alpha_2\gamma_2 y_3^2 - r_2 y_4^2 - \alpha_3\gamma_3 y_5^2 - r_3 y_6^2 - \alpha_1\delta_1 y_1 y_2 - \cdots \\ &\quad + \alpha_1 y_1 + \alpha_2 y_3 + \alpha_3 y_5 \end{aligned} \tag{11}$$

Obviously, there exists a positive number L such that $V'(t)|_{(5)} < 0$ when $y_i > L$. This means that the all solutions of system (5) are bounded.

Lemma 3 For each eigenvalue λ of matrix $A \in R^{n\times n}$, define $\mu(A) = \lim_{\theta\to 0^+} \frac{\|I+\theta A\|-1}{\theta}$, then the inequality holds:

$$Re\lambda_i(A) \le \mu(A), i \in \{1, 2, \cdots, n.\} \tag{12}$$

Proof See [20].

Existence of oscillatory solutions

Theorem 1 Assume that the conditions of Lemma 1 and Lemma 2 hold. Let $\alpha_1, \alpha_2, \cdots, \alpha_6$ represent the eigenvalues of matrix A, and $\beta_1, \beta_2, \cdots, \beta_6$ the eigenvalues of matrix B. If there exists one eigenvalue, say α_1 which is a positive real number, or α_1 is a complex number which has a positive real part, then the trivial solution of system (6) is unstable, implying that the unique equilibrium point $y_1^*, y_2^*, \cdots, y_6^*$ of system (5) is unstable, and system (5) generates a limit cycle, namely, a periodic solution.

Proof It is known that the trivial solution of the linearized system (6) is unstable, then the positive equilibrium point of original system (5) is unstable. Therefore, for proving the instability of the unique positive equilibrium point of system (5) we only need to prove the instability of the trivial solution of system (6). Considering an auxiliary equation of the system (6) as follows:

$$u'(t) = Au(t) + Bu(t - \tau_*) \tag{13}$$

where $\tau_* \leq \min\{\tau_1, \tau_2, \cdots, \tau_6\}), u(t-\tau_*) = [u_1(t-\tau_*), u_2(t-\tau_*), \cdots, u_6(t-\tau_*)]^T$. Based on the property of delayed differential equation, if the trivial solution of (13) is unstable then the trivial solution of system (6) is unstable [21]. Thus in the following we discuss the instability of the trivial solution of system (13). Since the eigenvalues of matrix A are $\alpha_1, \alpha_2, \cdots, \alpha_6$, and the eigenvalues of matrix B are $\beta_1, \beta_2, \cdots, \beta_6$, system (13) has the following characteristic equation:

$$\prod_{i=1}^{6}(\lambda - \alpha_i - \beta_i e^{-\lambda\tau_*}) = 0 \tag{14}$$

Noting that there exist three row entries of matrix B which are zeros, there is a characteristic value, say $\beta_1 = 0$. Hence, we have

$$\lambda - \alpha_1 - \beta_1 e^{-\lambda\tau_*} = \lambda - \alpha_1 = 0 \tag{15}$$

This means that there exists an eigenvalue which is a positive number or is a complex number that has a positive real part, implying that the trivial solution of system (13) is unstable. This suggests that the trivial solution of system (6) is unstable, implying that the unique positive equilibrium point of system (5) is unstable. The instability of the unique positive equilibrium point with the boundedness of the solution will force system (5) to generate a limit cycle, namely, a periodic solution [22, 23].

Theorem 2 Assume that the conditions of Lemma 1 and Lemma 2 hold. If the following condition holds

$$0 < \mu(A) + \parallel B \parallel . \tag{16}$$

where $\parallel B \parallel = \max_j \sum_{i=1}^{6} |b_{ij}|$. Then the trivial solution of system (6) is unstable, implying that the unique positive equilibrium point of system (5) is unstable, and system (5) generates a limit cycle, namely, a periodic solution.

Proof We must prove that the trivial solution of auxiliary system (13) is unstable. The characteristic equation associated with system (13) is the following:

$$\det(\lambda I_6 - A - Be^{-\lambda\tau_*}) = 0 \tag{17}$$

where I_6 is the 6×6 identity matrix. Set

$$f(\lambda) = \det(\lambda I_6 - A - Be^{-\lambda\tau_*}) \tag{18}$$

If the trivial solution of auxiliary system (13) is unstable, based on Theorem 1 there exists a root of $f(\lambda)$ satisfying $Re(\lambda) > 0$. From lemma 3, we get

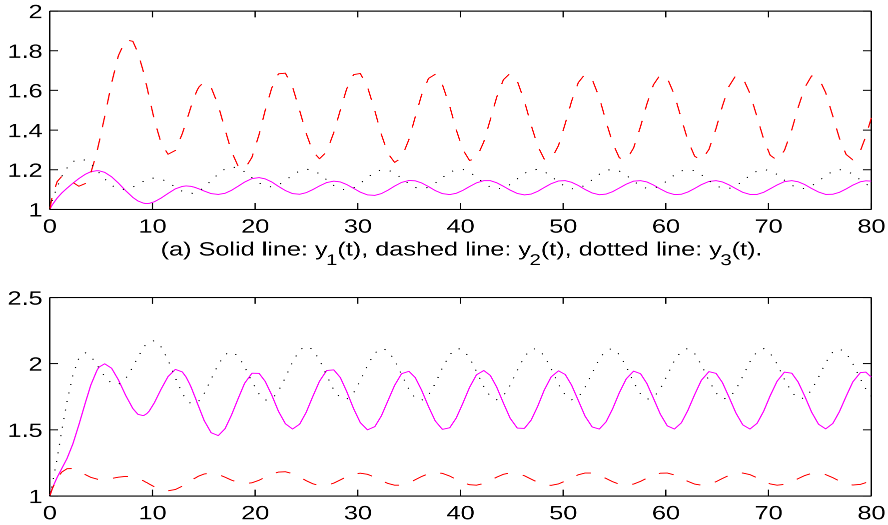
$$\begin{aligned} & Re(\lambda) \leq \mu(A + Be^{-\lambda\tau_*}) \\ = & \lim_{\theta \to 0^+} \frac{\|I + \theta(A + Be^{-\lambda\tau_*})\| - 1}{\theta} \\ \leq & \mu(A) + \|B\| \max_{1 \leq k \leq 6} |e^{(-\lambda_k \tau_*)}| \\ \leq & \mu(A) + \|B\| \end{aligned} \tag{19}$$

Thus, condition (16) holds. The trivial solution of auxiliary system (13) is unstable, implying that the trivial solution of system (6) is unstable, this means that the unique positive equilibrium point of system (5) is unstable. Based on the extended Chafee's criterion, system (5) generates a limit cycle, namely, a periodic solution [23]. The proof is completed.

Simulation result

This simulation is performed based on system (5). Firstly we select the parameters $\alpha_1 = 0.98, \alpha_2 = 1.25, \alpha_3 = 1.26, \delta_1 = 0.15, \delta_2 = 0.16, \delta_3 = 0.17$;

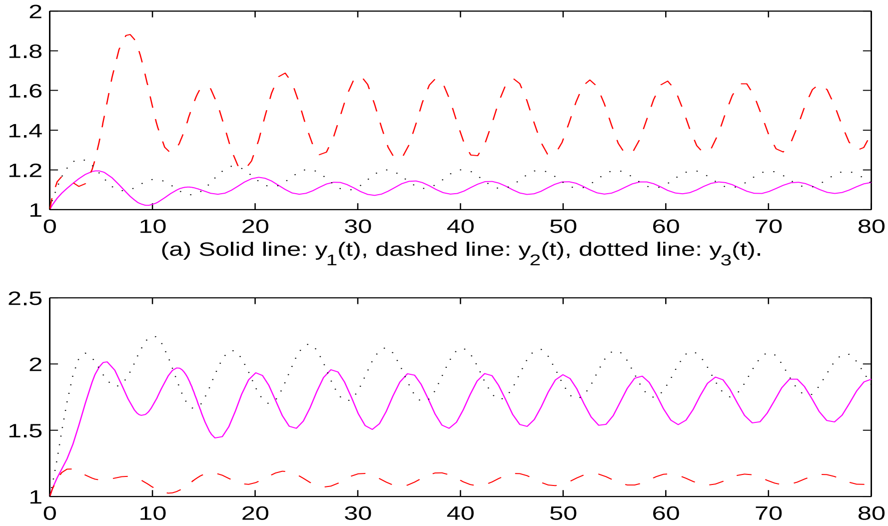
Fig. 1 Periodic oscillatory solutions, delays: 4.75, 4.65, 4.85, 4.55, 4.62, 4.68.



(a) Solid line: $y_1(t)$, dashed line: $y_2(t)$, dotted line: $y_3(t)$.

(b) Solid line: $y_4(t)$, dashed line: $y_5(t)$, dotted line: $y_6(t)$.

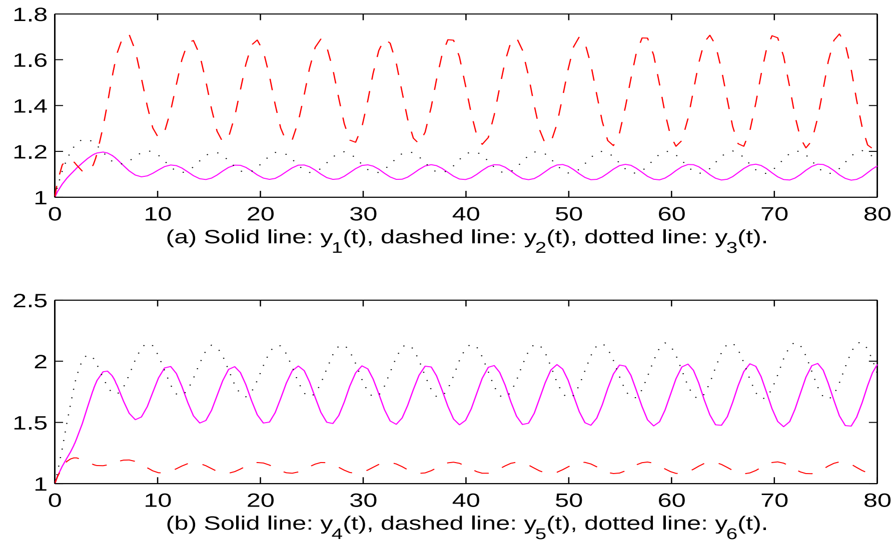
Fig. 2. Periodic oscillatory solutions, delays: 4.35, 5.25, 4.46, 5.18, 4.12, 4.38.



(a) Solid line: $y_1(t)$, dashed line: $y_2(t)$, dotted line: $y_3(t)$.

(b) Solid line: $y_4(t)$, dashed line: $y_5(t)$, dotted line: $y_6(t)$.

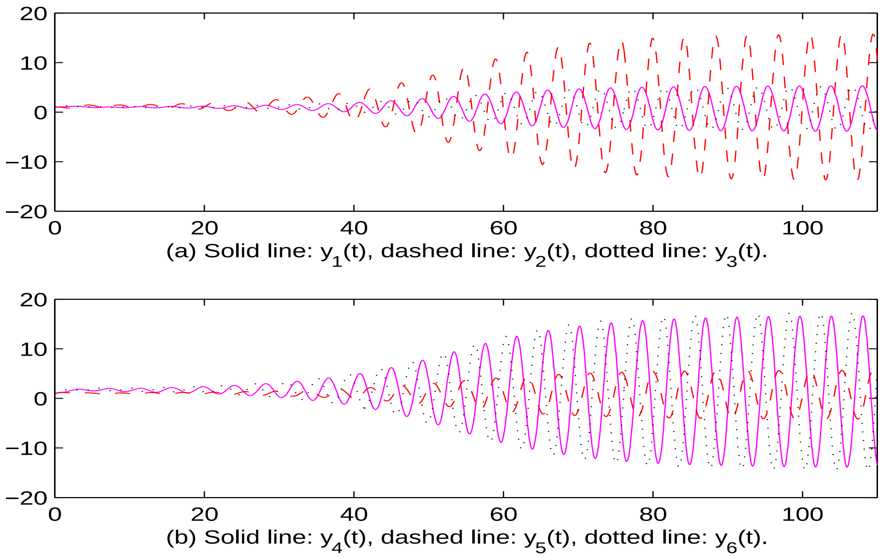
Fig. 3. Periodic oscillatory solutions, delays: 1.95, 1.85, 1.88, 1.86, 1.92, 1.98; $alpha_1$=2.45, $alpha_2$=2.65, $alpha_3$=2.55.



(a) Solid line: $y_1(t)$, dashed line: $y_2(t)$, dotted line: $y_3(t)$.

(b) Solid line: $y_4(t)$, dashed line: $y_5(t)$, dotted line: $y_6(t)$.

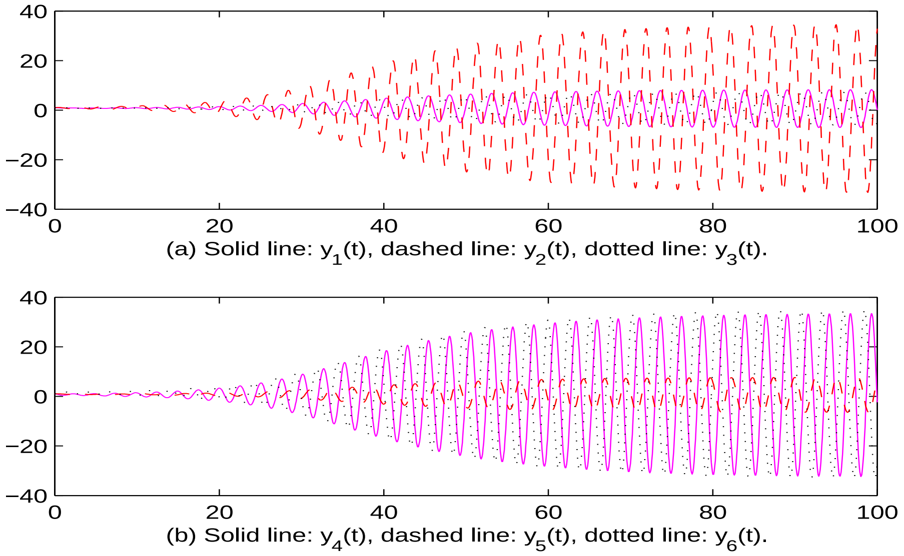
Fig. 4 Periodic oscillatory solutions, $delta_1$=0.22, $delta_2$=0.18, $delta_3$=0.20.



(a) Solid line: $y_1(t)$, dashed line: $y_2(t)$, dotted line: $y_3(t)$.

(b) Solid line: $y_4(t)$, dashed line: $y_5(t)$, dotted line: $y_6(t)$.

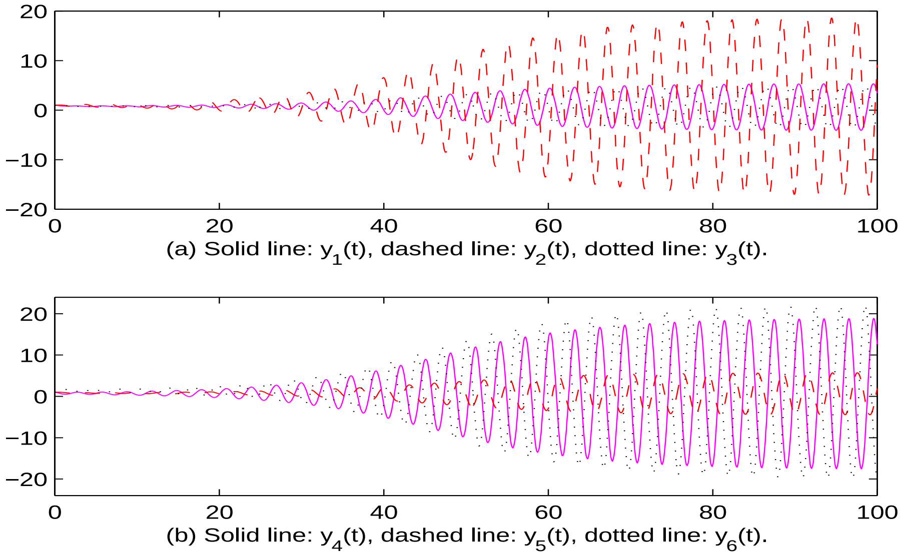
Fig. 5 Periodic oscillatory solutions, delays: 1.75, 1.78, 1.68, 1.76, 1.72, 1.65.



(a) Solid line: $y_1(t)$, dashed line: $y_2(t)$, dotted line: $y_3(t)$.

(b) Solid line: $y_4(t)$, dashed line: $y_5(t)$, dotted line: $y_6(t)$.

Fig. 6 Periodic oscillatory solutions, delays: 1.15, 1.25, 1.35, 1.35, 1.28, 1.25.



(a) Solid line: $y_1(t)$, dashed line: $y_2(t)$, dotted line: $y_3(t)$.

(b) Solid line: $y_4(t)$, dashed line: $y_5(t)$, dotted line: $y_6(t)$.

$\gamma_1 = 0.48, \gamma_2 = 0.42, \gamma_3 = 0.38, r_1 = 0.28, r_2 = 0.32, r_3 = 0.25, s_1 = 2.85, s_2 = 2.78, s_3 = 2.76$. The unique positive equilibrium point is $y_1^* = 1.1096, y_2^* = 1.1632, y_3^* = 1.4216, y_4^* = 1.7243, y_5^* = 1.1283, y_6^* = 1.9222$. The characteristic values of matrix C are $0.3145, -1.0345, -0.2850 \pm 0.7562i, -0.1450 \pm 0.6955i$. Therefore, C is a nonsingular matrix. The characteristic values of matrix A are $0.0778 \pm 0.8937i$, $-0.4321 \pm 0.8973i$, $-0.0556, -0.3058$. Since there is a positive real part of a complex characteristic value of matrix A, the conditions of Theorem 1 are satisfied. When we select time delays as $\tau_1 = 4.75, \tau_2 = 4.65, \tau_3 = 4.85, \tau_4 = 4.55, \tau_5 = 4.62, \tau_6 = 4.68$, and $\tau_1 = 4.35, \tau_2 = 5.25, \tau_3 = 4.46, \tau_4 = 5.18, \tau_5 = 4.12, \tau_6 = 4.38$, respectively. There are periodic oscillatory solutions of system (5) (see Fig. 1 and Fig. 2).

Then we only change $\alpha_1 = 2.45, \alpha_2 = 2.65, \alpha_3 = 2.55$, the other parameters are the same as in figure 1, we see that the oscillatory frequency is changed when time delays are selected as $\tau_1 = 1.95, \tau_2 = 1.85, \tau_3 = 1.88, \tau_4 = 1.86, \tau_5 = 1.92, \tau_6 = 1.98$ (see Fig. 3). Then we select $\delta_1 = 0.22, \delta_2 = 0.18, \delta_3 = 0.20$ and the other parameters are kept as figure 3. We see that the unique positive equilibrium point is very close to zero and the oscillatory behavior still maintains. However, the oscillatory frequency and amplitude both are changed (see Fig. 4). This means that the values of δ_i are strong effect the oscillatory behavior. Now we select another set of parameters as $\alpha_1 = 2.15, \alpha_2 = 2.18, \alpha_3 = 2.25, \delta_1 = 0.28, \delta_2 = 0.25, \delta_3 = 0.24$, $\gamma_1 = 0.56, \gamma_2 = 0.64, \gamma_3 = 0.45, r_1 = 0.38, r_2 = 0.50, r_3 = 0.48, s_1 = 7.65, s_2 = 7.85, s_3 = 6.75$. The unique positive equilibrium point is $y_1^* = 0.7158, y_2^* = 0.7012, y_3^* = 0.8665, y_4^* = 0.7049, y_5^* = 0.8149, y_6^* = 1.3876$. Then $\mu(A) + \| B \| = 0.8424 > 0$. Therefore, the conditions of Theorem 2 are satisfied. The delays are selected as $\tau_1 = 1.75, \tau_2 = 1.78, \tau_3 = 1.68, \tau_4 = 1.76, \tau_5 = 1.72, \tau_6 = 1.65$, and $\tau_1 = 1.15, \tau_2 = 1.25, \tau_3 = 1.35, \tau_4 = 1.35, \tau_5 = 1.28, \tau_6 = 1.25$, respectively. There exist periodic oscillatory solutions (see Fig. 5 and Fig. 6).

Conclusion

In this paper, we have discussed the dynamical behavior of a three coupled Kaldor-Kalecki model with time delays. The existence of a limit cycle which is easy to check, as compared to the general bifurcating method. Some simulations are provided to indicate the result of the criterion. Time delays only affect the oscillatory frequency when there exists a limit cycle of the system. The simulations also indicate that the present theorems are only sufficient conditions.

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