

Research paper

A Numerical solution for the new model of time-fractional bond pricing: Using a multi-quadric approximation method

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Abstract:

The bond market is an important part of the financial markets. The coupon bonds are issued by companies or banks for increasing capital, and the interest is paid by banks or companies, periodically. In terms of maturities, bonds are divided into three categories as follows: short term, medium term, and long term. In this paper, we model the fractional bond pricing under fractional stochastic differential equation. We implement the multiquadric approximation for solving the fractional bond pricing equation. The equation is discretized in the time direction based on modified Riemann-Liouville derivative and finite difference methods and is approximated by using the multiquadric approximation method in the space direction which achieves the semi-discrete solution. We investigate the unconditional stability and convergence of the proposed method. The method presented in the article has been implemented on two examples with different values, which confirm the results of the effectiveness of the method and show that appropriate results can be obtained with the MQ method. It should be noted that all calculations were done with the help of matlab software. Numerical results demonstrate the efficiency and ability of the presented method.

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Received: 06/06/2022 Accepted: 08/09/2022

<http://dx.doi.org/10.22054/jmmf.2022.68274.1056>

Keywords: Fractional derivative, Fractional interest rate, Time-fractional bond pricing, Multiquadric approximation method

Mathematics Subject Classification: 65M70, 91G80

Introduction

Fractional Differential Calculus (FDC) was born in the 17th century and its initial discussions were related to the works of Leibniz, Euler, Lagrange, Laplace, Abel, Liouville, Riemann, and others. In recent decades, the fractional differential equations have been considered in different fields such as fluid flow, engineering, electromagnetics, economics, and finance [1]. They have been also used in many different issues, such as telegraph, diffusion, and the diffusion–wave fractional equation [2–4, 4, 5]. In finance, Wyss (2000) provided the time–fractional Black–Scholes equation for pricing European call option, and Jumarie (2010) presented time–fractional and space–fractional Black–Scholes equations [6, 7]. Many researchers have solved the fractional Black–Scholes equation and examined its stability and convergence [8–10, 12, 13]. In this paper, we intend to present a model for the fractional bond pricing.

The bond market is an important part of the financial markets. The coupon bonds are issued by companies or banks for increasing capital, and the interest is paid by banks or companies, periodically. When, the bond has no interest, it will be a zero coupon bond. In terms of maturities, bonds are divided into three categories as follows: short term, medium term, and long term [24]. In the classical model of bond pricing, the short interest rate is a function of standard Brownian motion. This motion lacks long–range dependence and loses its previous condition. The existence of a long–range dependence in asset returns has important applications in examining the market, bond pricing, and selecting asset portfolio [14]. Mandelbrot was the first one that offers the idea of the long–range dependence in asset returns [15].

The fractional Brownian motion has stationary increments and long–range dependence which is compatible with market reality. Existence of this process in the fractional pricing model will make the price more realistic [12]. According to the mentioned contents in this work, we use the fractional interest rate instead of the standard interest rate. Therefore, the fractional bond pricing equation becomes a generalized classical bond

pricing equation by replacing the fractional interest rate in the standard interest rate.

The fractional bond pricing model can be solved by different methods. In this paper, we use the multi-quadric (MQ) approximation method because this method of solution works well, in particular when the data points are scattered. This method was introduced by Hardy (1971). He did not relate the base function to the coordinates (space) of the points and set it as a function of distance (radial) by using the concept of norm. The proposed function was $\varphi(p) = \sqrt{(p^2 + C^2)}$. It is the best method for the numerical solution of ordinary differential equations and partial differential equations [21, 23]. We will describe some advantages of this method in section 3.1.

The paper is organized as follows: In section 2, we model the fractional bond pricing. In section 3, first, the multiquadric approximation method is introduced. Then, we describe the time discretization process and a boundary value problem (BVP) is obtained which is solved by the multiquadric approximation method. In section 4, we investigate the unconditional stability and convergence. In section 5, the numerical result is described. The last section includes conclusion and recommendation for future study.

The model

The classical short interest rate model described by the following stochastic differential equation [13, 24]:

$$dr = \mu(r, t)dt + \rho(r, t)dW_t,$$

where dW_t is the standard Brownian motion and $\mu(r, t)$ and $\rho(r, t)$ are drift and variance parameters, respectively. In the fractional condition, such as the fractional underlying asset in [7, 25], suppose that $r = r_t = r(t)$ follows the fractional stochastic differential equation:

$$dr = \mu(r, t)dt + \rho(r, t)w(t).(dt)^{\frac{\alpha}{2}}, \quad (1)$$

where $w(t)$ is a normalized Gaussian white noise with unit variance and zero mean. Unfortunately, the exponents 1 and $\frac{\alpha}{2}$ of dt are not consistent with fractional Taylor's series. So, we use two formulas to solve this

problem as follows:

$$d^\alpha r = \Gamma(1 + \alpha)dr, \quad 0 < \alpha \leq 1, \quad (2)$$

$$\frac{d^\alpha r}{dr^\alpha} = \frac{1}{\Gamma(2 - \alpha)}r^{(1-\alpha)}, \quad 0 < \alpha \leq 1. \quad (3)$$

For the proof of (1) and (2), see [25]. By substituting (2) in (1), we have

$$dr = \mu(r, t)(\Gamma(1 + \alpha))^{(-1)}d^\alpha t + \rho(r, t)\omega(t)(dt)^{\frac{\alpha}{2}}; \quad (4)$$

now, by substituting (3) in (4), we obtain

$$dr = \frac{1}{\Gamma(1 + \alpha)\Gamma(2 - \alpha)}\mu(r, t)t^{(1-\alpha)}(dt)^\alpha + \rho(r, t)\omega(t)(dt)^{\frac{\alpha}{2}}, \quad 0 < \alpha \leq 1. \quad (5)$$

According to Lemma 3.2 in [7], equation (5) is consistent with fractional Taylor's series. By combining (2) and (3), we obtain

$$dr = \frac{r^{(1-\alpha)}}{\Gamma(1 + \alpha)\Gamma(2 - \alpha)}(dr)^\alpha, \quad 0 < \alpha \leq 1. \quad (6)$$

Now, we assume that the bond pricing $V(r, t)$ is twice differentiable with respect to r and has a fractional derivative of order α with respect to t .

When the interest rate follows equation (1), the bond has the price of $V(r, t)$. According to the assumptions, fractional Taylor's formula, and Itô's lemma, it can be written as follows [7]:

$$dV = \frac{1}{\Gamma(1 + \alpha)}V_t^{(\alpha)}(dt)^\alpha + V_r dr + \frac{1}{2}V_{rr}(dr)^2; \quad (7)$$

by replacing the equation (7) by (1), we have

$$dV = \frac{1}{\Gamma(1 + \alpha)}V_t^{(\alpha)}(dt)^\alpha + V_r \mu dt + V_r \rho w(t).(dt)^{\frac{\alpha}{2}} + \frac{1}{2}V_{rr}\rho^2(dt)^\alpha. \quad (8)$$

It cannot be used to hedge with the bond because the interest rate is not a traded security. So, we use different maturities (T_1 and T_2) for hedging bonds. We buy a bond of value V_1 with a maturity T_1 and sell the another bond of value V_2 with a maturity T_2 , where $V_1 = V(r, t; \Pi_1)$ and $V_2 = V(r, t; \Pi_2)$ [24].

Therefore, the portfolio (Π) value is calculated as follows [13]:

$$\Pi = V_1 - \Delta V_2, \quad (9)$$

where Δ is a covering risk.

The change of portfolio value in time dt is as follows:

$$d\Pi = dV_1 - \Delta dV_2. \quad (10)$$

Then we substitute (8) into (10),

$$\begin{aligned} d\Pi &= \left(\frac{1}{\Gamma(1+\alpha)} V_{1t}^{(\alpha)} (dt)^\alpha + V_{1r} \mu dt + V_{1r} \rho w(t) (dt)^{\left(\frac{\alpha}{2}\right)} + \frac{1}{2} V_{1rr} \rho^2 (dt)^\alpha \right) \\ &\quad - \Delta \left(\frac{1}{\Gamma(1+\alpha)} V_{2t}^{(\alpha)} (dt)^\alpha + V_{2r} \mu dt + V_{2r} \rho w(t) (dt)^{\left(\frac{\alpha}{2}\right)} + \frac{1}{2} V_{2rr} \rho^2 (dt)^\alpha \right). \end{aligned} \quad (11)$$

By selecting $\Delta = \frac{V_{1r}}{V_{2r}}$ in (11), the stochastic component is eliminated:

$$\begin{aligned} d\Pi &= \frac{1}{\Gamma(1+\alpha)} V_{1t}^{(\alpha)} (dt)^\alpha + \frac{1}{2} V_{1rr} \rho^2 (dt)^\alpha \\ &\quad - \frac{V_{1r}}{V_{2r}} \frac{1}{\Gamma(1+\alpha)} V_{2t}^{(\alpha)} (dt)^\alpha - \frac{1}{2} \frac{V_{1r}}{V_{2r}} V_{2rr} \rho^2 (dt)^\alpha. \end{aligned} \quad (12)$$

Since the portfolio is instantaneously riskless, we should obtain the riskless short interest rate, which is as follows:

$$d\Pi = r\Pi dt = r \left(V_1 - \frac{V_{1r}}{V_{2r}} V_2 \right) dt. \quad (13)$$

Replacing (6) into (41), implies

$$d\Pi = r \left(V_1 - \frac{V_{1r}}{V_{2r}} V_2 \right) \frac{t^{1-\alpha}}{\Gamma(1+\alpha)\Gamma(2-\alpha)} (dt)^\alpha, \quad (14)$$

now, by replacing (12) by (14), we obtain

$$\begin{aligned} &\left(\frac{1}{\Gamma(1+\alpha)} V_{1t}^{(\alpha)} (dt)^\alpha + \frac{1}{2} V_{1rr} \rho^2 (dt)^\alpha - r \frac{t^{1-\alpha}}{\Gamma(1+\alpha)\Gamma(2-\alpha)} V_1 (dt)^\alpha \right) \\ &+ \left(-\frac{V_{1r}}{V_{2r}} \frac{1}{\Gamma(1+\alpha)} V_{2t}^{(\alpha)} (dt)^\alpha - \frac{1}{2} \frac{V_{1r}}{V_{2r}} V_{2rr} \rho^2 (dt)^\alpha + r \frac{t^{1-\alpha}}{\Gamma(1+\alpha)\Gamma(2-\alpha)} \frac{V_{1r}}{V_{2r}} V_2 (dt)^\alpha \right) = 0. \end{aligned} \quad (15)$$

Multiplying (15) by $\frac{\Gamma(1+\alpha)}{(dt)^\alpha}$ yields

$$\begin{aligned} &\left(V_{1t}^{(\alpha)} + \frac{\Gamma(1+\alpha)}{2} V_{1rr} \rho^2 - r \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} V_1 \right) \\ &= \frac{V_{1r}}{V_{2r}} \left(V_{2t}^{(\alpha)} + \frac{\Gamma(1+\alpha)}{2} V_{2rr} \rho^2 - r \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} V_2 \right), \end{aligned} \quad (16)$$

with gathering together all V_1 terms on the left-hand side and all V_2 terms on the right-hand side, we find

$$\frac{\left(V_{1t}^{(\alpha)} + \frac{\Gamma(1+\alpha)}{2}V_{1rr}\rho^2 - r\frac{t^{(1-\alpha)}}{\Gamma(2-\alpha)}V_1\right)}{V_{1r}} = \frac{\left(V_{2t}^{(\alpha)} + \frac{\Gamma(1+\alpha)}{2}V_{2rr}\rho^2 - r\frac{t^{(1-\alpha)}}{\Gamma(2-\alpha)}V_2\right)}{V_{2r}}. \quad (17)$$

Now both sides of (17) are independent of the maturity time, by assuming $a(r, t)$ as follows:

$$\frac{\left(V_t^{(\alpha)} + \frac{\Gamma(1+\alpha)}{2}V_{rr}\rho^2 - r\frac{t^{(1-\alpha)}}{\Gamma(2-\alpha)}V\right)}{\left(\frac{\partial V}{\partial r}\right)} = a(r, t), \quad (18)$$

according to [24]

$$a(r, t) = \rho\lambda - \mu, \quad (19)$$

where λ is the market price of risk. So, we have the fractional zero-coupon bond pricing equation

$$V_t^{(\alpha)} + \frac{\Gamma(1+\alpha)}{2}\rho^2V_{rr} - r\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}V + (\mu - \lambda\rho)V_r = 0, \quad (20)$$

where $0 < t < T$.

In equation (20), we can use different interest rate models for μ and ρ , which lead to different dynamics of instant interest rate. In this article we use the Vasicek mean-reversion interest rate which is corresponding to $\mu(r, t) = \nu(\gamma - r)$ and $\rho(r, t) = \rho$ in (1) [13]. Thus the fractional bond pricing model can be obtained as follows:

$$V_t^{(\alpha)} + \frac{\Gamma(1+\alpha)}{2}\rho^2V_{rr} - r\frac{t^{(1-\alpha)}}{\Gamma(2-\alpha)}V + (\nu(\gamma - r) - \lambda\rho)V_r = 0, \quad (21)$$

where $0 < t < T$. We need initial and boundary conditions to solve (21). According to [26], we have

$$\begin{cases} V(r, T) = 1, & 0 < r < R, \quad t = T, \\ \frac{\partial V(0, t)}{\partial t} + \mu(0)\frac{\partial V(0, t)}{\partial r} = 0, & r = 0, \quad 0 < t < T, \\ \frac{\partial V(R, t)}{\partial t} + \mu(R)\frac{\partial V(R, t)}{\partial r} = RV, & r = R, \quad 0 < t < T. \end{cases} \quad (22)$$

Equation (21) with conditions of (22) can be solved by different methods. In the next section, we solve this equation by the multiquadric-RBF approximation method.

Numerical method

In this section, we first define the multi-quadric approximation method. Then, equation (21) with conditions of (22) is solved.

Multi-quadric approximation method

The MQ approximation method is a suitable method for numerical solution of the initial and boundary value problem. Some advantages of this method are as follows:

- This method is very flexible because it does not depend on the positions of the points and it can be used as an interpolation method for scattered points.
- It depends on the shape parameter, and we can obtain a convergence of high accuracy for the pricing problem by setting this parameter and obtaining its different values.

We assume that each function can be expanded as a finite series of upper hyperboloids, which is written as follows [23]:

$$V(p) = \sum_{j=1}^N b_j \varphi(p - p_j), \quad (23)$$

where

$$\varphi(p - p_j) = (\|p - p_j\|^2 + C^2)^{\frac{1}{2}}, \quad j = 1, \dots, N, \quad (24)$$

where $\|p - p_j\|^2$ is the squared Euclidean distances in \mathbb{R} and C is a non-zero parameter that is determined by the user. The value of this parameter affects the shape; hence it is known as the shape parameter. The function φ is continuously differentiable. The unknown coefficients $\{b_j\}_{j=1}^N$ will be determined by solving the following linear equation:

$$V(p_i) = \sum_{j=1}^N b_j \varphi(p_i - p_j), \quad i = 1, \dots, N. \quad (25)$$

The shape parameter plays an important role in the multi-quadric approximation, and it affects the accuracy and stability of the approximation. There are two kinds of shape parameters: Constant shape parameter and variable shape parameter.

Many people use the constant shape parameter due to its simple analysis in approximation [18, 19]. But multiple results of large set of applications show the advantages of using variable shape parameter; for example, using of variable shape parameter creates more distinct entries of discretization matrix, which reduces the rounding error in calculations [20–22].

In this paper, we use the exponential shape parameter, which is given by Kansa to get more accurate solutions in numerical results [21]:

$$\varphi(p - p_j) = (\|p - p_j\|^2 + C_j^2)^{\frac{1}{2}}, \quad j = 1, \dots, N,$$

$$C_j^2 = C_{\min}^2 \left(\frac{C_{\max}^2}{C_{\min}^2} \right)^{\frac{j-1}{N-1}}, \quad j = 1, 2, \dots, N,$$

where C_{\max} and C_{\min} are the selected input parameters; so that the following ratio is in the given range:

$$\frac{C_{\max}}{C_{\min}} \simeq 10 \text{ to } 10^6.$$

Discretization in time

The fractional derivative in (21) is a modified right Riemann–Liouville derivative which is defined as follows:

$$\frac{\partial^\alpha V(r, t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \frac{V(r, \xi) - V(r, T)}{(\xi - t)^\alpha} d\xi, & 0 < \alpha < 1, \\ \frac{\partial V(r, t)}{\partial t}, & \alpha = 1. \end{cases} \quad (26)$$

When $\alpha = 1$, the model (21) becomes the classical bond pricing model.

Let $t = T - \tau$; for $0 < \alpha < 1$, we have

$$\begin{aligned} \frac{\partial^\alpha V(r, t)}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \frac{V(r, \xi) - V(r, T)}{(\xi - t)^\alpha} d\xi \\ &= -\frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau} \int_{T-\tau}^T \frac{V(r, \xi) - V(r, T)}{(\xi - (T-\tau))^\alpha} d\xi \\ &= -\frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau} \int_0^\tau \frac{V(r, T-\eta) - V(r, T)}{(\tau-\eta)^\alpha} d\eta. \end{aligned} \quad (27)$$

Then (21) can be rewritten as:

$$\begin{aligned} {}_0D_\tau^\alpha U(r, \tau) &= \frac{\Gamma(1+\alpha)}{2} \rho^2 \frac{\partial^2 U(r, \tau)}{\partial r^2} - r \frac{(T-\tau)^{(1-\alpha)}}{\Gamma(2-\alpha)} U(r, \tau) \\ &\quad + (\nu(\gamma - r) - \lambda\rho) \frac{\partial U(r, \tau)}{\partial r}, \\ U(r, 0) &= 1, \quad 0 < r < R, \\ \frac{\partial U(0, \tau)}{\partial \tau} - \mu(0) \frac{\partial U(0, \tau)}{\partial r} &= 0, \quad r = 0, \quad 0 < \tau < T, \\ \frac{\partial U(R, \tau)}{\partial \tau} - \mu(R) \frac{\partial U(R, \tau)}{\partial r} &= -RU(R, \tau), \quad r = R, \quad 0 < \tau < T, \end{aligned} \quad (28)$$

where the fractional derivative is

$${}_0D_\tau^\alpha U(r, \tau) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau} \int_0^\tau \frac{U(r, \eta) - U(r, 0)}{(\tau - \eta)^\alpha} d\eta, \quad 0 < \alpha < 1. \quad (29)$$

For $0 < \alpha \leq 1$ and $U(r, \tau) \in C^1$, the modified Riemann–Liouville derivative is as follows:

$$\begin{aligned} {}_0D_\tau^\alpha U(r, \tau) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau} \int_0^\tau \frac{U(r, \eta) - U(r, 0)}{(\tau - \eta)^\alpha} d\eta \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau} \int_0^\tau \frac{U(r, \eta)}{(\tau - \eta)^\alpha} d\eta - \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau} \int_0^\tau \frac{U(r, 0)}{(\tau - \eta)^\alpha} d\eta \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau} \int_0^\tau \frac{U(r, \eta)}{(\tau - \eta)^\alpha} d\eta - \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)} U(r, 0) \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^\tau \frac{\partial U(r, \eta)}{\partial \eta} (\tau - \eta)^{-\alpha} d\eta \\ &= {}_0^C D_\tau^\alpha U(r, \tau), \end{aligned} \quad (30)$$

where ${}_0^C D_\tau^\alpha U(r, \tau)$ is a Caputo derivative [1]. In order that discretize the problem for $0 < \alpha < 1$ in time direction, we substitute τ^{n+1} into (30), then we have:

$${}_0D_\tau^\alpha U(r, \tau^{n+1}) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\tau^{n+1}} \frac{\partial U(r, \eta)}{\partial \eta} (\tau^{n+1} - \eta)^{-\alpha} d\eta, \quad 0 < \alpha \leq 1, \quad (31)$$

where $\tau^{n+1} = \tau^n + \delta\tau$, $\tau^0 = 0$, $n = 0, 1, \dots, N$, $\delta\tau N = T$. Approximation of the first order derivative in (31) by the following finite difference formula

$$\frac{\partial U(r, \tau)}{\partial \tau} \simeq \frac{U(r, \tau^{n+1}) - U(r, \tau^n)}{\delta\tau}, \quad \tau \in (\tau^n, \tau^{n+1}). \quad (32)$$

By replacing equation (32) into (31), we obtain

$$\begin{aligned} \frac{\partial^\alpha U(r, \tau^{n+1})}{\partial \tau^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \int_0^{\tau^{n+1}} \frac{\partial U(r, \eta)}{\partial \eta} (\tau^{n+1} - \eta)^{-\alpha} d\eta \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^n \int_{\tau^k}^{\tau^{k+1}} \frac{\partial U(r, \eta)}{\partial \eta} (\tau^{n+1} - \eta)^{-\alpha} d\eta \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^n \frac{U^{k+1} - U^k}{\delta\tau} \int_{\tau^k}^{\tau^{k+1}} (\tau^{n+1} - \eta)^{-\alpha} d\eta, \end{aligned} \quad (33)$$

where $U^k = U(r, \tau^k)$, $k = 0, 1, \dots, N$.

Then, the integral in the equation (33) is solved and we have

$$\frac{\partial^\alpha U^{n+1}}{\partial \tau^\alpha} = a_0 \left(U^{n+1} - U^n + \sum_{k=1}^n b_k (U^{n-k+1} - U^{n-k}) \right), \quad (34)$$

where $a_0 = \frac{\delta\tau^{-\alpha}}{\Gamma(2-\alpha)}$, $b_k = (k+1)^{1-\alpha} - (k)^{1-\alpha}$ and $n = 0, 1, \dots, N$.

By replacing equation (34) into equation (14), we obtain

$$\begin{aligned} & a_0 (U^{n+1} - U^n + \sum_{k=1}^n b_k (U^{n-k+1} - U^{n-k})) \\ &= \frac{\Gamma(1+\alpha)}{2} \rho^2 \frac{\partial^2 U^{n+1}}{\partial r^2} - r \frac{(T-(n+1)\delta\tau)^{(1-\alpha)}}{\Gamma(2-\alpha)} U^{n+1} \\ & \quad + (\nu(\gamma - r) - \lambda\rho) \frac{\partial U^{n+1}}{\partial r}, \\ & U^{n+1}(r, 0) = 1, \quad 0 < r < R, \\ & U^{n+1}(0, \tau) - \mu(0)\delta\tau U_r^{n+1}(0, \tau) = U^n(0, \tau), \quad r = 0, \quad 0 < \tau < T, \\ & U^{n+1}(R, \tau) - \mu(R)\delta\tau U_r^{n+1}(R, \tau) + R\delta\tau U^{n+1}(R, \tau) = U^n(R, \tau), \\ & r = R, \quad 0 < \tau < T. \end{aligned} \tag{35}$$

Now, we solve (35) by the MQ approximation method. Therefore, it can be written as follows:

For $n = 0$,

$$\begin{aligned} & a_0 U^1 - \frac{\Gamma(1+\alpha)}{2} \rho^2 \frac{\partial^2 U^1}{\partial r^2} + r \frac{(T-\delta\tau)^{(1-\alpha)}}{\Gamma(2-\alpha)} U^1 - (\nu(\gamma - r) - \lambda\rho) \frac{\partial U^1}{\partial r} = a_0 U^0, \\ & U^1(r, 0) = 1, \quad 0 < r < R, \\ & U^1(0, \tau) - \mu(0)\delta\tau U^1(0, \tau) = U^0(0, \tau), \quad r = 0, \quad 0 < \tau < T, \\ & U^1(R, \tau) - \mu(R)\delta\tau U^1(R, \tau) + R\delta\tau U^1(R, \tau) = U^0(R, \tau), \\ & r = R, \quad 0 < \tau < T. \end{aligned} \tag{36}$$

For $n \geq 1$,

$$\begin{aligned} & a_0 U^{n+1} - \frac{\Gamma(1+\alpha)}{2} \rho^2 \frac{\partial^2 U^{n+1}}{\partial r^2} + r \frac{(T-(n+1)\delta\tau)^{(1-\alpha)}}{\Gamma(2-\alpha)} U^{n+1} - (\nu(\gamma - r) - \lambda\rho) \frac{\partial U^{n+1}}{\partial r} \\ &= a_0 U^n - a_0 \sum_{k=1}^n b_k (U^{n-k+1} - U^{n-k}), \\ & U^{n+1}(r, 0) = 1, \quad 0 < r < R, \\ & U^{n+1}(0, \tau) - \mu(0)\delta\tau U_r^{n+1}(0, \tau) = U^n(0, \tau), \quad r = 0, \quad 0 < \tau < T, \\ & U^{n+1}(R, \tau) - \mu(R)\delta\tau U_r^{n+1}(R, \tau) + R\delta\tau U^{n+1}(R, \tau) = U^n(R, \tau), \\ & r = R, \quad 0 < \tau < T. \end{aligned} \tag{37}$$

Now we approximate $U^n(r)$ by the equation (23) as follows:

$$U^n(r) = \sum_{j=1}^N b_j^n \varphi(r - r_j),$$

where b_1, b_2, \dots, b_N are unknowns. Then we consider N collocation points to obtain the values of b_j , $j = 1, \dots, N$ in the interpolant $U^n(r)$

as:

$$U_i^{n+1} = U^{n+1}(r_j) \simeq \sum_{j=1}^N b_j^{n+1} \varphi(r_i - r_j).$$

By reconstruction of equation (25) in the matrix form, we obtain

$$[U]^{n+1} = A[b]^{n+1}, \quad (38)$$

where $[V]^{n+1} = [V_1^{n+1}, V_2^{n+1}, \dots, V_N^{n+1}]^T$, $[b]^{n+1} = [b_1^{n+1}, b_2^{n+1}, \dots, b_N^{n+1}]^T$, and A is an $N \times N$ matrix

$$A = \begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1j} & \cdots & \varphi_{1N} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \varphi_{i1} & \cdots & \varphi_{ij} & \cdots & \varphi_{iN} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varphi_{N1} & \cdots & \varphi_{Nj} & \cdots & \varphi_{NN} \end{bmatrix}.$$

Now, by considering equations (36) and (37) in a matrix form, we obtain

$$[D]^1 = B[b]^1,$$

where

$$B = \begin{bmatrix} G(\varphi_{11}) & \cdots & G(\varphi_{1j}) & \cdots & G(\varphi_{1N}) \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ G(\varphi_{i1}) & \cdots & G(\varphi_{ij}) & \cdots & G(\varphi_{iN}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G(\varphi_{N1}) & \cdots & G(\varphi_{Nj}) & \cdots & G(\varphi_{NN}) \end{bmatrix},$$

$$G(*) = \begin{cases} (1 - \mu(0)\delta\tau \frac{d}{dr})(*), & i = 1, \\ (a_0 - \frac{\Gamma(1+\alpha)}{2}\rho^2 \frac{d^2}{dr^2} + r \frac{(T-\delta\tau)^{1-\alpha}}{\Gamma(2-\alpha)}) & 1 < i < N, \\ -(\nu(\gamma - r) - \lambda\rho) \frac{d}{dr})(*), & \\ (1 - \mu(R)\delta\tau \frac{d}{dr} + R\delta\tau)(*), & i = N, \end{cases}$$

and $[D]^1 = [D_1^1, D_2^1, \dots, D_N^1]^T$,

$$D_i^1 = \begin{cases} U_1^0, & i = 1, \\ a_0 U_i^0, & 1 < i < N, \\ U_N^0, & i = N. \end{cases}$$

Similarly, equation (37) can be written as:

$$[D]^{n+1} = B[b]^{n+1},$$

where

$$G(*) = \begin{cases} (1 - \mu(0)\delta\tau \frac{d}{dr})(*), & i = 1, \\ (a_0 - \frac{\Gamma(1+\alpha)}{2}\rho^2 \frac{d^2}{dr^2} + r \frac{(T-(n+1)\delta\tau)^{1-\alpha}}{\Gamma(2-\alpha)} \\ - (\nu(\gamma - r) - \lambda\rho) \frac{d}{dr})(*), & 1 < i < N, \\ (1 - \mu(R)\delta\tau \frac{d}{dr} + R\delta\tau)(*), & i = N, \end{cases}$$

and $[D]^{n+1} = [D_1^{n+1}, D_2^{n+1}, \dots, D_N^{n+1}]^T$,

$$D_i^{n+1} = \begin{cases} U_1^n, & i = 1, \\ a_0 U_i^n - a_0 \sum_{k=1}^n b_k (U_i^{n-k+1} - U_i^{n-k}), & 1 < i < N, \\ U_N^n, & i = N. \end{cases}$$

The stability and convergence analysis

According to section (), we discuss the unconditional stability and convergence of equation (14). We consider the following equation:

$$\frac{\partial^\alpha U}{\partial \tau^\alpha} = a_0 \left(\sum_{k=0}^n b_k (U^{n-k+1} - U^{n-k}) \right) + \eta_1^n, \quad (39)$$

according to [3], $|\eta_1^n| \leq c(\delta\tau)^2$ and

$$\begin{cases} b_k > 0, & k = 0, 1, \dots, n, \\ 1 = b_0 > b_1 > \dots > b_n, & b_n \rightarrow 0, \quad n \rightarrow \infty, \\ \sum_{k=0}^n (b_k - b_{k+1}) + b_{n+1} = (1 - b_1) + \sum_{k=1}^{n-1} (b_k - b_{k+1}) + b_n = 1. \end{cases} \quad (40)$$

Now, we can reconstruct (14) by (34),

$$\begin{aligned} & a_0 U^{n+1} - \frac{\Gamma(1+\alpha)}{2}\rho^2 \frac{\partial^2 U^{n+1}}{\partial r^2} + r \frac{(T-(n+1)\delta\tau)^{1-\alpha}}{\Gamma(2-\alpha)} U^{n+1} - (\nu(\gamma - r) - \lambda\rho) \frac{\partial U^{n+1}}{\partial r} \\ & = a_0 U^n - a_0 \sum_{k=1}^n b_k (U^{n-k+1} - U^{n-k}), \end{aligned} \quad (41)$$

the right-hand side of equation (41) can be rewritten as follows:

$$\begin{aligned} a_0 U^n - a_0 \sum_{k=1}^n b_k (U^{n-k+1} - U^{n-k}) \\ &= a_0 (b_0 U^n - \sum_{k=1}^n b_k (U^{n-k+1} - U^{n-k})) \\ &= a_0 (b_0 U^n - \sum_{k=0}^{n-1} b_{k+1} U^{n-k} + \sum_{k=1}^n b_k U^{n-k}). \end{aligned} \quad (42)$$

We have

$$(u, v) = \int_{\Omega} (uv) dx,$$

and $\|v\| = (v, v)^{\frac{1}{2}}$ in \mathcal{L}^2 .

By the inner product of equation (41) in v , we obtain

$$\begin{aligned} a_0 (U^{n+1}, v) - \frac{\Gamma(1+\alpha)}{2} \rho^2 \left(\frac{\partial U^{n+1}}{\partial r}, \frac{\partial v}{\partial r} \right) \\ + r \frac{(T-(n+1)\delta\tau)^{1-\alpha}}{\Gamma(2-\alpha)} (U^{n+1}, v) - (\nu(\gamma - r) - \lambda\rho) (U^{n+1}, \frac{\partial v}{\partial r}) \\ &= a_0 (b_0 (U^n, v) - \sum_{k=0}^{n-1} b_{k+1} (U^{n-k}, v) \\ &\quad + \sum_{k=1}^n b_k (U^{n-k}, v)). \end{aligned} \quad (43)$$

Theorem 0.1. *The semidiscrete equation (41) is an unconditional stable for $\delta\tau > 0$ and*

$$\|U^{n+1}\| \leq \|U^0\|, \quad n = 0, 1, \dots, N-1.$$

Proof. Consider equation (43), and let $n = 0$; then

$$\begin{aligned} a_0 (U^1, v) - \frac{\Gamma(1+\alpha)}{2} \rho^2 \left(\frac{\partial U^1}{\partial r}, \frac{\partial v}{\partial r} \right) \\ + r \frac{(T-\delta\tau)^{1-\alpha}}{\Gamma(2-\alpha)} (U^1, v) - (\nu(\gamma - r) - \lambda\rho) (U^1, \frac{\partial v}{\partial r}) \\ &= a_0 b_0 (U^0, v), \end{aligned} \quad (44)$$

by substituting $v = U^1$ in (44) and by using Schwartz inequality, we obtain

$$\|U^1\|^2 \leq \|U^1\| \|U^0\| \Rightarrow \|U^1\| \leq \|U^0\|.$$

Now, suppose

$$\|U^j\| \leq \|U^0\|, \quad j = 1, 2, \dots, n; \quad (45)$$

we want to prove $\|U^{n+1}\| \leq \|U^0\|$. Replacing $v = U^{n+1}$ in (43), implies

$$\begin{aligned} a_0 (U^{n+1}, U^{n+1}) - \frac{\Gamma(1+\alpha)}{2} \rho^2 \left(\frac{\partial U^{n+1}}{\partial r}, \frac{\partial U^{n+1}}{\partial r} \right) \\ + r \frac{(T-(n+1)\delta\tau)^{1-\alpha}}{\Gamma(2-\alpha)} (U^{n+1}, U^{n+1}) - (\nu(\gamma - r) - \lambda\rho) (U^{n+1}, \frac{\partial U^{n+1}}{\partial r}) \\ &= a_0 (b_0 (U^n, U^{n+1}) - \sum_{k=0}^{n-1} b_{k+1} (U^{n-k}, U^{n+1}) \\ &\quad + \sum_{k=1}^n b_k (U^{n-k}, U^{n+1})). \end{aligned} \quad (46)$$

The right-hand side of equation (46) can be rewritten as follows:

$$\begin{aligned}
 & a_0(b_0(U^n, U^{n+1}) - \sum_{k=0}^{n-1} b_{k+1}(U^{n-k}, U^{n+1}) + \sum_{k=1}^n b_k(U^{n-k}, U^{n+1})) \\
 &= a_0((1 - b_1)(U^n, U^{n+1}) - \sum_{k=1}^{n-1} b_{k+1}(U^{n-k}, U^{n+1}) \\
 &\quad + \sum_{k=1}^n b_k(U^{n-k}, U^{n+1})) \\
 &= a_0((1 - b_1)(U^n, U^{n+1}) + \sum_{k=1}^{n-1} (b_k - b_{k+1})(U^{n-k}, U^{n+1}) \\
 &\quad + b_n(U^0, U^{n+1})).
 \end{aligned} \tag{47}$$

By using the Schwartz inequality, we have

$$\|U^{n+1}\|^2 \leq (1 - b_1)\|U^n\|\|U^{n+1}\| + \sum_{k=1}^{n-1} (b_k - b_{k+1})\|U^{n-k}\|\|U^{n+1}\| + b_n\|U^0\|\|U^{n+1}\|; \tag{48}$$

therefore, equation (48) leads to

$$\|U^{n+1}\| \leq (1 - b_1)\|U^n\| + \sum_{k=1}^{n-1} (b_k - b_{k+1})\|U^{n-k}\| + b_n\|U^0\|. \tag{49}$$

According to equation (45),

$$\|U^{n+1}\| \leq (1 - b_1)\|U^0\| + \sum_{k=1}^{n-1} (b_k - b_{k+1})\|U^0\| + b_n\|U^0\|, \tag{50}$$

by considering equation (40), we obtain

$$\|U^{n+1}\| \leq ((1 - b_1) + \sum_{k=1}^{n-1} (b_k - b_{k+1}) + b_n)\|U^0\| \leq \|U^0\|.$$

□

Theorem 0.2. Let $(U(r, t^n))_{n=0}^N$ be the exact solution of (14), and let $(U^n)_{n=0}^N$ be the discrete time solution of (14) with initial $U(r, 0) = 1$; then we have the following error estimates,

$$\|U(r, t^n) - U^n\| \leq \frac{c}{1 - \alpha} T^\alpha (\delta\tau)^{2-\alpha}, \quad 0 < \alpha < 1. \tag{51}$$

Proof. By replacing equation (39) into (14)

$$\begin{aligned}
& a_0((U(r, t^{n+1}), v)) - \frac{\Gamma(1+\alpha)}{2} \rho^2 \left(\frac{\partial U(r, t^{n+1})}{\partial r}, \frac{\partial v}{\partial r} \right) \\
& + r \frac{(T-(n+1)\delta\tau)^{1-\alpha}}{\Gamma(2-\alpha)} (U(r, t^{n+1}), v) - (\nu(\gamma - r) - \lambda\rho) (U(r, t^{n+1}), \frac{\partial v}{\partial r}) \\
& = a_0 \left(b_0(U(r, t^n), v) - \sum_{k=0}^{n-1} b_{k+1}(U(r, t^{n-k}), v) \right. \\
& \quad \left. + \sum_{k=1}^n b_k(U(r, t^{n-k}), v) \right) + (\eta_1^{n+1}, v).
\end{aligned} \tag{52}$$

By subtracting equation (52) from equation (43), we define $e^{n+1} = U(r, t^{n+1}) - U^{n+1}$; we have

$$\begin{aligned}
& a_0(e^{n+1}, v) - \frac{\Gamma(1+\alpha)}{2} \rho^2 \left(\frac{\partial e^{n+1}}{\partial r}, \frac{\partial v}{\partial r} \right) \\
& + r \frac{(T-(n+1)\delta\tau)^{1-\alpha}}{\Gamma(2-\alpha)} (e^{n+1}, v) - (\nu(\gamma - r) - \lambda\rho) (e^{n+1}, \frac{\partial v}{\partial r}) \\
& = a_0 \left((1 - b_1)(e^n, v) + \sum_{k=1}^{n-1} (b_k - b_{k+1})(e^{n-k}, v) \right. \\
& \quad \left. + b_n(e^0, v) \right) + (\eta_1^{n+1}, v).
\end{aligned} \tag{53}$$

By substituting $v = e^{n+1}$ into equation (53), we obtain

$$\begin{aligned}
& a_0(e^{n+1}, e^{n+1}) - \frac{\Gamma(1+\alpha)}{2} \rho^2 \left(\frac{\partial e^{n+1}}{\partial r}, \frac{\partial e^{n+1}}{\partial r} \right) \\
& + r \frac{(T-(n+1)\delta\tau)^{1-\alpha}}{\Gamma(2-\alpha)} (e^{n+1}, e^{n+1}) - (\nu(\gamma - r) - \lambda\rho) (e^{n+1}, \frac{\partial e^{n+1}}{\partial r}) \\
& = a_0 \left((1 - b_1)(e^n, e^{n+1}) + \sum_{k=1}^{n-1} (b_k - b_{k+1})(e^{n-k}, e^{n+1}) \right. \\
& \quad \left. + b_n(e^0, e^{n+1}) \right) + (\eta_1^{n+1}, e^{n+1}),
\end{aligned} \tag{54}$$

and for $n = 0$, we have

$$\begin{aligned}
& a_0(e^1, e^1) - \frac{\Gamma(1+\alpha)}{2} \rho^2 \left(\frac{\partial e^1}{\partial r}, \frac{\partial e^1}{\partial r} \right) \\
& + r \frac{(T-(n+1)\delta\tau)^{1-\alpha}}{\Gamma(2-\alpha)} (e^1, e^1) - (\nu(\gamma - r) - \lambda\rho) (e^1, \frac{\partial e^1}{\partial r}) \\
& = a_0 \left((1 - b_1)(e^0, e^1) \right) + (\eta_1^1, e^1).
\end{aligned} \tag{55}$$

The Schwartz inequality gives

$$\|e^1\|^2 \leq \|e^1\| \|e^0\| + \|\eta_1^1\| \|e^1\|; \tag{56}$$

according to (40) and $|\eta_1^n| \leq c(\delta\tau)^2$,

$$\|e^1\| \leq cb_0^{-1}(\delta\tau)^2.$$

Now, we suppose that inequality $e^n \leq cb_{n-1}^{-1}(\delta\tau)^2$ holds for n ; we want to prove $\|e^{n+1}\| \leq cb_n^{-1}(\delta\tau)^2$.

By using the Schwartz inequality in equation (54), we have

$$\|e^{n+1}\|^2 \leq (1-b_1)\|e^n\|\|e^{n+1}\| + \sum_{k=1}^{n-1} (b_k - b_{k+1})\|e^{n-k}\|\|e^{n+1}\| + \|\eta_1^{n+1}\|\|e^{n+1}\|, \quad (57)$$

$$\|e^{n+1}\| \leq c(\delta\tau)^2((1-b_1)b_{n-1}^{-1} + \sum_{k=1}^{n-1} (b_k - b_{k+1})b_{n-k-1}^{-1}) + c(\delta\tau)^2, \quad (58)$$

by applying the induction assumption and $\frac{b_k}{b_{k+1}} < 1$, we obtain

$$\begin{aligned} \|e^{n+1}\| &\leq c(\delta\tau)^2 b_n^{-1} \left((1-b_1) + \sum_{k=1}^{n-1} (b_k - b_{k+1}) \right) + c(\delta\tau)^2 \\ &= c(\delta\tau)^2 b_n^{-1} \left((1-b_1) + \sum_{k=1}^{n-1} (b_k - b_{k+1}) \right) + c b_n b_n^{-1} (\delta\tau)^2 \\ &= \left((1-b_1) + \sum_{k=1}^{n-1} (b_k - b_{k+1}) \right) + b_n \quad c b_n^{-1} (\delta\tau)^2 \\ &\leq c b_n^{-1} (\delta\tau)^2. \end{aligned} \quad (59)$$

Therefore, the estimation is proved. By using the definition of b_k , we have

$$n^{-\alpha} b_{n-1}^{-1} \leq \frac{1}{1-\alpha};$$

consequently, we have

$$\begin{aligned} \|U(r, t^n) - U^n\| &\leq c b_{n-1}^{-1} (\delta\tau)^2 \leq c b_{n-1}^{-1} n^{-\alpha} n^\alpha (\delta\tau)^2 \\ &\leq \frac{c}{1-\alpha} (n\delta\tau)^\alpha (\delta\tau)^2 (\delta\tau)^{-\alpha} \leq \frac{c}{1-\alpha} T^\alpha (\delta\tau)^{2-\alpha}, \end{aligned}$$

for all n , such that $n\delta\tau \leq T$. \square

Numerical result

In this section, we investigate the efficiency of the presented method. The method is implemented for solving some examples with different parameters.

Example 0.3. Consider equation (21) with the following parameters [27]:

$$\rho = 0.0126, \quad \nu = 0.025, \quad \gamma = 0.15339, \quad C_{\max} = 50, \quad C_{\min} = 1.9.$$

Figure 1 shows the numerical solution, which is obtained by the multi-quadratic approximation method for the value of 2, 3, and 5 years bonds based on the Vasicek model. It depicts the interest rate from 0 to 1 and the price of bond limits to 0.

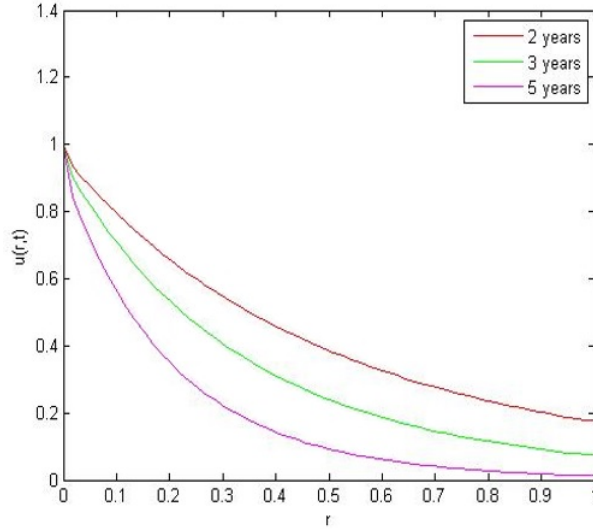


Figure 1: Numerical solution of fractional bond pricing of 5, 7, and 10 years and $\alpha=1$.

Example 0.4. In this example, we consider the actual data of the treasury Bills of the central bank of the Islamic republic of Iran in 2016 and 2017. By using actual data, the parameters of equation (21) are estimated as follows:

$$\rho = 0.0299, \quad \nu = 1.2859, \quad \gamma = 0.2009.$$

Now we plot the treasury Bills figure by considering the other variables and parameters and by using the numerical method for $\alpha = 1, 0.75, 0.5$ at $t = 1$ as follows:

$$r_{\max} = R = 0.25, \quad r_{\min} = 0.18, \quad T = 2,$$

$$C_{\max} = 50, \quad C_{\min} = 1.9.$$

According to the figure 2, we can find that the price of bonds using the numerical method for different α is limited to the price of the classical bonds. It indicates that the numerical method is efficient.

Conclusion and recommendation

The main focus of this paper was to model fractional bond price and to solve it based on the MQ method. In order to reach this goal, first we

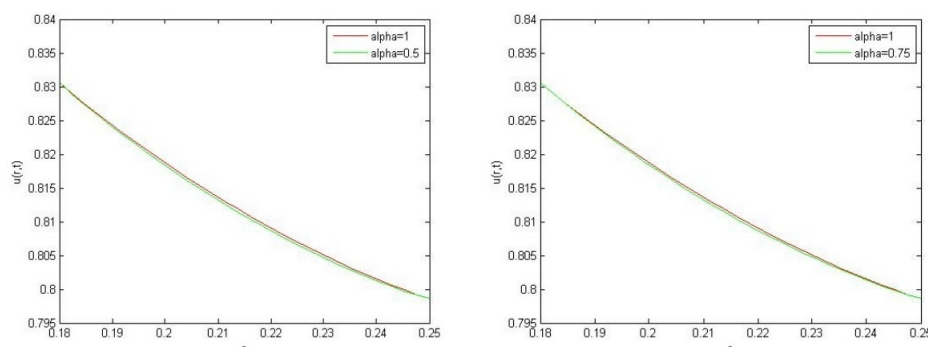


Figure 2: Numerical solution of the bond pricing at $t = 1$.

modelled the bond pricing under vasicek short interest rate model, then we solved this equation by MQ method in which the shape parameter was variable.

The numerical result emphasizes that when the interest rate raises, the price of the bond reduces. Also, by increasing the maturity time of the bonds, the possibility of fluctuating in prices will be higher, due to the interest rate.

Finally, for future research, we suggest to add the jump term to our recommended model to obtain a new model. Also, we can use other interest rate models in this equation and compare the behavior of the models. Other numerical methods can be used for this recommended model.

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