Research paper

A numerical method for solving the underlying price problem driven by a fractional Levy process

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Abstract:
We consider European style options with risk-neutral parameters and time-fractional Levy diffusion equation of the exponential option pricing model in this paper. In a real market, volatility is a measure of the quantity of inflation in asset prices and changes. This makes it essential to accurately measure portfolio volatility, asset valuation, risk management, and monetary policy. We consider volatility as a function of time. Estimating volatility in the time-fractional Levy diffusion equation is an inverse problem. We use a numerical technique based on Chebyshev wavelets to estimate volatility and the price of European call and put options. To determine unknown values, the minimization of a least-squares function is used. Because the obtained corresponding system of linear equations is ill-posed, we use the Levenberg-Marquardt regularization technique. Finally, the proposed numerical algorithm has been used in a numerical example. The results demonstrate the accuracy and effectiveness of the methodology used.

Keywords: European options; Time-fractional Levy diffusion equation; Volatility; Chebyshev wavelets; Levenberg-Marquardt regularization.
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Introduction

Recently, financial mathematics has received a great deal of attention due to its extensive association with economics and financial markets. One of the most significant issues in finance is the pricing of financial instruments such as options and stocks. Merton [1] proposed the Black-Scholes (BS) model for this problem based on Brownian motion. However, empirical evidence has shown that recourse to Brownian motion as a driving process is too restrictive. E. Eberlein et al [2] considered Levy processes as the driving force and indicated that Levy processes-based models are more precise than models based on Brownian motion. In fact, Levy processes-based models can describe the observed reality of financial markets more accurately. In [3], the authors provided a modification of European style options under a risk-neutral probability condition for the stock-price assets based on Levy processes. They considered the following dynamic of the stock price driven by the α-stable Levy process $L_t^{\alpha,-1}$ with skew parameter β (see [3]) as below:

$$d^\gamma S_t = S_t\left((r-q)dt^\gamma + \sigma dL_t^{\alpha,-1}\right) + \lambda S_t d\beta_t^\gamma, \quad S_0 \text{ is initial price,} \tag{1}$$

where $1 < \alpha < 2$ and $d^\gamma S_t$ is the Caputo fractional derivative with order $0 < \gamma < 1$. The parameters $\sigma \geq 0$ and β_t^γ are volatility and the number of shares of the stock at time t, respectively. The term $\lambda S_t d\beta_t^\gamma, (\lambda \geq 0)$ refers to the price impact of the investor's trading (for the more details see [4]). In addition, r and q are the risk-less rate and dividend yield parameters, respectively. Also, they considered trading strategies as

$$d\beta_t^\gamma = \eta_t dt^\gamma + \zeta_t dL_t^{\alpha,-1}, \tag{2}$$

where $(\eta_t)_{t\geq 0}$ and $(\zeta_t)_{t\geq 0}$ are processes determined endogenously and β_0 is the initial number of shares in the stock. From Eqs. (1) and (2) we have

$$d^\gamma S_t = S_t\left((r-q)dt^\gamma + \sigma dL_t^{\alpha,-1}\right) + \lambda S_t(\eta_t dt^\gamma + \zeta_t dL_t^{\alpha,-1}). \tag{3}$$

The authors in [3] considered the wealth process $(V_t)_{t\geq 0}$ corresponds to a self-financing strategy $(\theta_t, \beta_t^\gamma)_{t\geq 0}$

$$V_t = \theta_t S_t^0 + \beta_t S_t, \quad \text{where} \quad dS_t^0 = rS_t^0 dt,\ S_0^0 = 1,$$

and proved that by using the change of variable $x = x_t := ln(S_t)$, the Eq. (3) converts to the following fractional partial differential equation (FPDE)

$$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma} + A\frac{\partial u(x,t)}{\partial x} + B\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = ru(x,t), \quad x \in [x_{min}, x_{max}], \quad (4)$$

with initial conditions $u(x, 0) = h(x)$ and $u_t(x, 0) = f(x)$. Obviously, the solution of this equation is the wealth process $V_t = u(S_t, t)$. Furthermore, we have

$$A = r - q + \lambda\eta + \frac{\sigma^\alpha}{2}\sec(\frac{\alpha\pi}{2}),$$

and

$$B = \frac{\sigma^\alpha}{2}\sec(\frac{\alpha\pi}{2}).$$

In Eq. (4), $\frac{\partial^\gamma u(x,t)}{\partial t^\gamma}$ and $\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}$ are Caputo time and Riemann-Liouville space fractional derivatives, respectively, that are defined as follows

$$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma} = \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{\partial u(x,\tau)}{\partial \tau}(t-\tau)^{-\gamma}d\tau, \quad 0 < \gamma < 1,$$

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\int_{-\infty}^x (x-\tau)^{1-\alpha}u(\tau,t)d\tau, \quad 1 < \alpha < 2.$$

Aljethi and Kilicman [5] solved Eq. (4) by an implicit scheme and obtained numerical values of the European-style option price and compared their results with those obtained by the Euler-Maruyama scheme for Eq. (1). However, experimental studies conducted by researchers have shown that the constant volatility in the option pricing models is not consistent with market data. Therefore, it makes more sense to consider volatility as a function of asset price and time, that is $\sigma = \sigma(S, t)$, and determine it. The inverse problem of determining volatility has been investigated by some researchers. In [6], the authors used the adjoint method to find implied volatility. They considered the case of $\sigma(S, t) = \sigma(S)$ on European call options. Egger and Engl [7] applied Tikhonov regularization to determine a stable and convergent solution to the inverse problem of option pricing. In [8], a linearized problem was considered and the uniqueness theorem for the state-dependent case $\sigma = \sigma(S)$ was proved. However, the purely time-dependent volatilities were investigated by Hein and Hofmann in [9] and Jin et al in [10].

In this paper, we discuss the case of $\sigma = \sigma(t)$ in Eq. (1). So, we have an inverse problem. Also, it is obvious that in the fractional partial differential equation (4), A and B are the functions of $\sigma(t)$. To find the volatility and u in Eq. (4), we use the additional condition (overlapping measured data)

$$u(x^*, t) = E(t), \qquad x_{min} < x^* < x_{max}. \tag{5}$$

Let us denote the solution of Eq. (4) under given conditions by $u(x, t; \sigma)$. Therefore, in order to determine the volatility, we find σ so that the following equality is satisfied

$$u(x^*, t; \sigma) = E(t), \qquad x_{min} < x^* < x_{max}. \tag{6}$$

In general, we solve the optimization problem in the below form

$$J(\sigma) = \sum_{i=1}^{I} (u(x^*, t_i; \sigma) - E(t_i))^2. \tag{7}$$

In this paper, we use Chebyshev wavelets to approximate the unknown volatility $\sigma(t)$. Then, the collocation points are used to obtain the price of European options.

The organization of the manuscript is as follows: In Section 2, we state the Chebyshev wavelets by using the concept of the Chebyshev polynomials. Section 3 is devoted to our computational procedure. In Section 4, we implement the mentioned numerical method and determine the volatility and the price of European call and put options.

The Chebyshev wavelets and their properties

Consider the first kind of Chebyshev polynomials of degree m defined as

$$T_m(x) = \cos(m\, arccos(x)), \quad x \in [-1, 1].$$

The Chebyshev polynomials are orthogonal with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ on the interval $[-1, 1]$ and satisfy the following recursive formula

$$T_0(x) = 1 \ , \ T_1(x) = x \ , \ T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x) \ , \ m = 1, 2, \cdots .$$

Chebyshev wavelets $\psi^C_{n,m}(x) = \psi^C(k, 2n-1, m, x)$ have four arguments; $k \in \mathbb{N},\ n = 1, 2, ..., 2^{k-1}$, $m = 0, 1, ..., M-1$ and x. These wavelets are defined on interval $[0, 1]$ based on Chebyshev polynomials as [11]

$$\psi^C_{n,m}(x) = \begin{cases} 2^{k/2}\widehat{T}_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}, \\ \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$

where

$$\widehat{T}_m(x) = \begin{cases} \frac{1}{\sqrt{\pi}} & m = 0, \\ \\ \sqrt{\frac{2}{\pi}}T_m(x) & m > 0. \end{cases}$$

Here, $T_m(x), m = 0, 1, \cdots, M-1$ are the Chebyshev polynomials of degree m. The Chebyshev wavelets are orthogonal with respect to the weight function $w_n(x) = w(2^k x - 2n + 1)$ instead of $\hat{w}(x) = w(2x - 1)$. Hence, any function $f(x) \in L_2[0, 1]$ can be expressed based on wavelets as

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} f_{n,m}\psi^C_{n,m}(x), \tag{9}$$

where

$$f_{n,m} = (f(x), \psi^C_{n,m}(x)) = \int_0^1 f(x)\psi^C_{n,m}(x)w_n(x)dx.$$

In practical, a finite series of (9) is used

$$f(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} f_{n,m}\psi^C_{n,m}(x).$$

Remark 0.7. To apply the Chebyshev wavelets on the $[x_{min}, x_{max}]$, we use the change of variable $x = (x_{max} - x_{min})\eta + x_{min}, 0 \leq \eta \leq 1$. Therefore, the shifted Chebyshev wavelets are defined on $[x_{min}, x_{max}]$ as follows

$$\psi_{n,m}(x) = \psi^C_{n,m}(\frac{x - x_{min}}{x_{max} - x_{min}}).$$

In this paper, we expand the function $u(x,t)$ by the Chebyshev wavelets as follows

$$u(x,t) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}(t)\psi_{n,m}(x),$$

where

$$C_{n,m}(t) = (u(x,t), \psi_{n,m}(x))$$

The computational procedure

At first, notice that by [5], the price of the call and put European option corresponding with Eq. (3) is as follows

$$u(S_t,t) = \mathbb{E}^Q[h(S_T)|\mathcal{F}_t], \tag{10}$$

where $u(x,t)$ is the solution of time-fractional partial differential equation

$$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma} = A\frac{\partial u(x,t)}{\partial x} + B\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + ru(x,t), \quad x \in [x_{min}, x_{max}], \tag{11}$$

with the initial conditions $u(x,0) = h(x)$ and $u_t(x,0) = f(x)$. Here, $A = A(\sigma(t))$ and $B = B(\sigma(t))$ are the functions of the volatility σ. Also, the boundary conditions $u(x_{min},t) = 0$ and $u(x_{max},t) = e^{x_{max}} - Ke^{-r(T-t)}$ is used for a call option and for a put option, we have $u(x_{max},t) = 0$ and $u(x_{min},t) = Ke^{-r(T-t)} - e^{x_{min}}$, where K is the strike price, $x_{min} = -\log(4K)$ and $x_{max} = \log(4K)$. Note that if σ is known, then the (11) with given conditions is a called direct problem (DP).

Remark 0.8. In this paper, we assume that $A(\sigma(t)) = a_0 + a_1\sigma(t) + a_2\sigma^2(t)$ and $B(\sigma(t)) = b_0 + b_1\sigma(t) + b_2\sigma^2(t)$ where a_0, a_1, a_2 and b_0, b_1, b_2 are constants that will be determined by the least-squares method.

Now, we use the Chebyshev wavelets to find the volatility σ and u in (11).
Consider an approximation of the volatility σ as follows

$$\bar{\sigma}(t) = \sum_{n_1=1}^{2^{k_1-1}} \sum_{m_1=0}^{M_1-1} d_{n_1,m_1}\psi_{n_1,m_1}(t). \tag{12}$$

Also, Let the approximation solution of the DP (11) is as

$$\bar{u}(x,t) = \sum_{n_2=1}^{2^{k_2-1}} \sum_{m_2=0}^{M_2-1} c_{n_2,m_2}(t)\psi_{n_2,m_2}(x). \tag{13}$$

Now, considering the following discretization scheme

$$\Delta t = \frac{T}{L},\ \forall L \in \mathbb{N} \quad \text{and} \quad t_i = i\Delta t,\ i = 0, 1, \cdots, L.$$

$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma}$ is discretized as the following [12]

$$\frac{\partial^\gamma u(x,t_i)}{\partial t^\gamma} = \frac{\Delta t^{-\gamma}}{\Gamma(2-\gamma)} \sum_{k=0}^{i-1} b_k[u(x,t_{i-k}) - u(x,t_{i-k-1})] \tag{14}$$

where $b_k = (k+1)^\gamma - k^{1-\gamma},\ b_0 = 1$.
Substituting the relations (13) and (14) in Eq. (11), we have

$$\begin{aligned}
&\frac{\Delta t^{-\gamma}}{\Gamma(2-\gamma)} \sum_{k=0}^{i-1} \sum_{n_2=1}^{2^{k_2-1}} \sum_{m_2=0}^{M_2-1} b_k \left(c_{n_2,m_2}(t_{i-k}) - c_{n_2,m_2}(t_{i-k-1})\right) \psi_{n_2,m_2}(x) \\
&\quad = A(\sum_{n_1=1}^{2^{k_1-1}} \sum_{m_1=0}^{M_1-1} d_{n_1,m_1}\psi_{n_1,m_1}(t)) \sum_{n_2=1}^{2^{k_2-1}} \sum_{m_2=0}^{M_2-1} c_{n_2,m_2}(t_i)\psi'_{n_2,m_2}(x) \\
&\qquad + B(\sum_{n_1=1}^{2^{k_1-1}} \sum_{m_1=0}^{M_1-1} d_{n_1,m_1}\psi_{n_1,m_1}(t)) \sum_{n_2=1}^{2^{k_2-1}} \sum_{m_2=0}^{M_2-1} c_{n_2,m_2}(t_i)\frac{\partial^\alpha \psi_{n_2,m_2}(x)}{\partial x^\alpha} \\
&\qquad + r\sum_{m_2=0}^{M_2-1} c_{n_2,m_2}(t_i)\psi_{n_2,m_2}(x)
\end{aligned} \tag{15}$$

Moreover, the boundary conditions for the call and put options are obtained as follows

$$\begin{aligned}
&\sum_{n_2=1}^{2^{k_2-1}} \sum_{m_2=0}^{M_2-1} c_{n_2,m_2}(t_i)\psi_{n_2,m_2}(x_{min}) = 0, \\
&\sum_{n_2=1}^{2^{k_2-1}} \sum_{m_2=0}^{M_2-1} c_{n_2,m_2}(t_i)\psi_{n_2,m_2}(x_{max}) = e^{x_{max}} - Ke^{-r(T-t_i)},
\end{aligned} \tag{16}$$

$$\sum_{n_2=1}^{2^{k_2-1}}\sum_{m_2=0}^{M_2-1} c_{n_2,m_2}(t_i)\psi_{n_2,m_2}(x_{min}) = Ke^{-r(T-t_i)} - e^{x_{min}},$$
$$\sum_{n_2=1}^{2^{k_2-1}}\sum_{m_2=0}^{M_2-1} c_{n_2,m_2}(t_i)\psi_{n_2,m_2}(x_{max}) = 0, \tag{17}$$

respectively. In Eq. (15), there are $2^{k_2-1}M_2$ unknown c_{n_2,m_2} where $n_2 = 1, 2, \cdots, 2^{k_2-1}, m_2 = 0, 1, \cdots, M_2 - 1$; $2^{k_1-1}M_1$ unknown values of d_{n_1,m_1} where $n_1 = 1, 2, \cdots, 2^{k_1-1}, m_1 = 0, 1, \cdots, M_1 - 1$, $2^{k_1-1}M_1$ and six unknown constants a_0, a_1, a_2, b_0, b_1 and b_2. Considering Eqs. (16), (17) and taking the collocation points

$$x_j = \frac{2j-1}{2^{k_2}M_2}, j = 1, 2, \cdots, 2^{k_2-1}M_2 - 2, \tag{18}$$

we can obtain the unknowns c_{n_2,m_2} based on the unknown constants $d_{n_1,m_1}, a_0, a_1, a_2, b_0, b_1$ and b_2 for call and put option respectively. So, the approximate solution of DP (11) is obtained based on $d_{n_1,m_1}, a_0, a_1, a_2, b_0, b_1$ and b_2. Now, to find these unknown constants , we minimize the following functional

$$S(d) = \sum_{i=1}^{I}(\bar{u}(x^*, t_i; \sigma) - E(t_i))^2. \tag{19}$$

Given that the obtained system of equations is ill-conditioned, so we use the Levenberg-Marquardt regularization [13].
It has been shown that the matrix form of the functional (19) is

$$S(d) = [E - \bar{U}(d)]^T[E - \bar{U}(d)], \tag{20}$$

where

$$[E - \bar{U}(d)]^T \equiv [E_1 - \bar{U}_1, E_2 - \bar{U}_2, \cdots, E_I - \bar{U}_I],$$

in which $E_i = E(t_i)$, $\bar{U}_i = \bar{u}(x^*, t_i; \sigma)$, $i = 1, 2, \cdots, I$ and

$$d = [d_{1,0}, d_{1,1}, \cdots, d_{1,M_1-1}, \cdots, d_{2^{k_1-1},1}, d_{2^{k_1-1},2}, \cdots, d_{2^{k_1-1},M_1-1}, a_0, a_1, a_2, b_0, b_1, b_2].$$

Now, we minimize the least squares norm by equating the derivatives of $S(d)$ to zero, that is

$$\nabla S(d) = 2\left[-\frac{\partial \bar{U}^T(d)}{\partial d}\right][E - \bar{U}(d)] = 0,$$

in which

$$\frac{\partial \bar{U}^T(d)}{\partial d} = \begin{bmatrix} \frac{\partial}{\partial d_{1,0}} \\ \frac{\partial}{\partial d_{1,1}} \\ \vdots \\ \frac{\partial}{\partial d_{2^{k_1-1},M_1-1}} \\ \frac{\partial}{\partial a_0} \\ \vdots \\ \frac{\partial}{\partial b_2} \end{bmatrix} [\bar{U}_1 \quad \bar{U}_2 \quad \cdots \quad \bar{U}_I].$$

Therefore, we can write the sensitivity matrix as [13]

$$J(d) = \left[\frac{\partial \bar{U}^T(d)}{\partial d}\right]^T = \begin{bmatrix} \frac{\partial \bar{U}_1}{\partial d_{1,0}} & \frac{\partial \bar{U}_1}{\partial d_{1,1}} & \cdots & \frac{\partial \bar{U}_1}{\partial d_{2^{k_1-1},M_1-1}} & \frac{\partial \bar{U}_1}{\partial a_0} & \cdots & \frac{\partial \bar{U}_1}{\partial b_2} \\ \frac{\partial \bar{U}_2}{\partial d_{1,0}} & \frac{\partial \bar{U}_2}{\partial d_{1,1}} & \cdots & \frac{\partial \bar{U}_2}{\partial d_{2^{k_1-1},M_1-1}} & \frac{\partial \bar{U}_2}{\partial a_0} & \cdots & \frac{\partial \bar{U}_2}{\partial b_2} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial \bar{U}_I}{\partial d_{1,0}} & \frac{\partial \bar{U}_I}{\partial d_{1,1}} & \cdots & \frac{\partial \bar{U}_I}{\partial d_{2^{k_1-1},M_1-1}} & \frac{\partial \bar{U}_I}{\partial a_0} & \cdots & \frac{\partial \bar{U}_I}{\partial b_2}. \end{bmatrix} \tag{21}$$

When the sensitivity coefficients of the elements of the matrix $J(d)$, are small, the determinant of the matrix $(J(d))^T J(d)$ is approximately zero and the inverse problem is ill-conditioned. Also, if any column of $J(d)$ is a linear combination of the other columns, the determinant of the matrix $(J(d))^T J(d)$ is null [13].

Now, we use the Levenberg-Marquardt regularization with the following computational algorithm [13].

Consider an initial guess for the vector of unknown coefficients d and denote it with $d^{(0)}$.

1) Allocate a value for the regularization parameter and denote it with μ_0 and set $k = 0$.

2) Compute $\bar{U}(d^{(0)})$ and $S(d^{(0)})$.

3) Using the current values of $(d^{(0)})$, compute the sensitivity matrix J^k and $\Omega^k = diag[(J^k)^T J^k]$.

4) Solve the following system of algebraic equations

$$\left[(J^k)^T J^k + \mu^k \Omega^k\right] \Delta(d^{(0)})^k = \left(J^k\right)^T \left[E - \bar{U}\left((d^{(0)})^k\right)\right].$$

5) Compute $(d^{(0)})^{k+1} = \Delta(d^{(0)})^k + (d^{(0)})^k$.

6) If $S\left((d^{(0)})^{k+1}\right) \geq S\left((d^{(0)})^k\right)$, set $10\mu^k$ instead of μ^k and go to step 4.

7) If $S\left((d^{(0)})^{k+1}\right) < S\left((d^{(0)})^k\right)$, accept $(d^{(0)})^{k+1}$ and set $0.1\mu^k$ instead of μ^k.

8) If $\|(d^{(0)})^{k+1} - (d^{(0)})^k\| < tol$, so an acceptable approximation is obtained where the value of tol is given. Otherwise, set $k+1$ instead of k and go to step 3.

Numerical simulations

In this section, we provide some numerical simulations to illustrate the efficiency of the method and obtain the prices of the call and put European options. To find volatility σ and the price of European options by the provided numerical method, we consider the overposed measured data $u(0.1, t_i) = 0$ where $t_i = i\Delta t$ and assume $k_1 = k_2 = 2$, $M_1 = M_2 = 5$. The used parameters' value are reported in Table 1. To investigate the

Table 1: Parameters' value

S_0	*Strike K*	r	γ	α	T	L
100	250	0.02	0.75	1.76	100	4

accuracy of numerical solutions, we assume that the true volatility $\sigma(t)$ is defined as

$$\sigma_{ex}(t) = 0.1\cos(4\pi t) - 0.1t + 0.2,$$

and compare the obtained volatility by numerical method for FPDE $\sigma_{es}(t)$ with the true volatility $\sigma_{ex}(t)$. Figures 1(a), 1(b) and 1(c) show the comparison between the true volatility $\sigma_{ex}(t)$ and the estimated volatility by the numerical method. Also, we compare the obtained values for $A(\sigma(t))$ and $B(\sigma(t))$ based on $\sigma_{ex}(t)$ and $\sigma_{es}(t)$. In Figures 2(a) and 2(b), we represent these estimated values. Also, in Table 2, we present the price of call and put European option for the true volatility σ_{ex} and the obtained volatility by numerical method for FPDE.

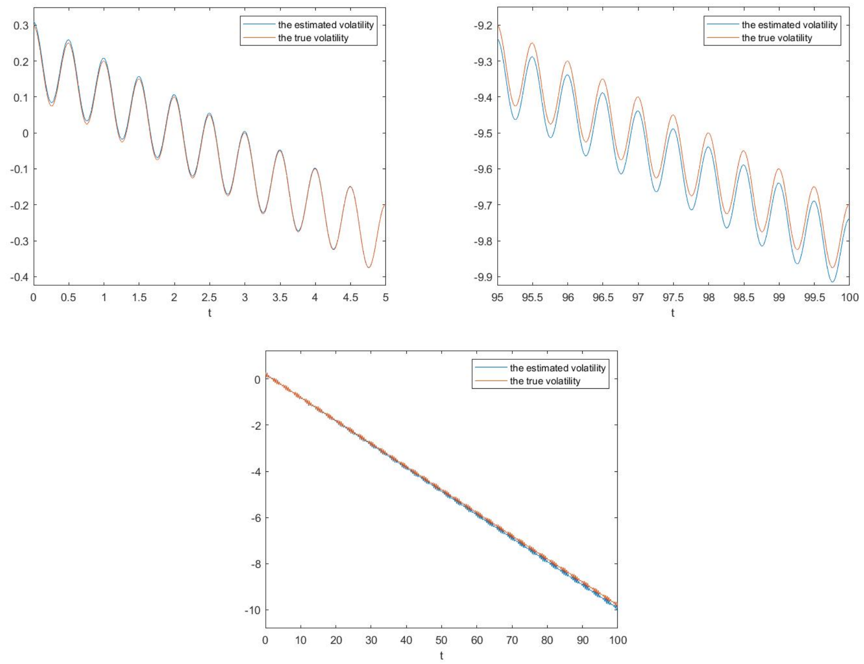

Figure 1: The comparison between the true volatility $\sigma_{ex}(t)$ and the estimated volatility by the numerical method

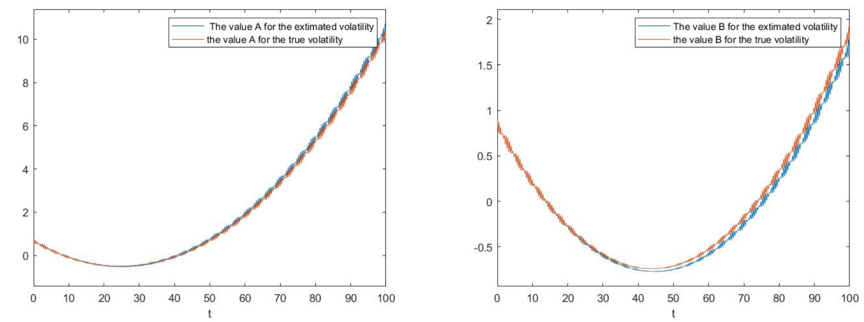

Figure 2: The comparison between the obtained values for A and B for $\sigma_{ex}(t)$ and $\sigma_{es}(t)$

Conclusions

In this study, the inverse fractional diffusion equation in terms of the Levy process is solved by Chebyshev wavelets. We also obtained the un-

Table 2: The prices of call and put European option with true volatility and the obtained volatility by numerical method for inverse FPDE

	Strike K	$\sigma_{ex}(t)$	$\sigma_{es}(t)$
Call price	250	18.9431	18.9321
Put price	250	1.1364	1.0225

known volatility for solving the price of fractional financial derivatives of the European options price. Moreover, we made a comparison between European call and put option prices based on the provided numerical solutions of the inverse fractional partial differential equation and solutions corresponding to the true volatility. The numerical results illustrated that the method is effective and accurate to approximate the price of call and put European options.

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