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Research paper

Option pricing in high volatile illiquid market

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Abstract:
This study compares the performance of the classic Black-Scholes model and the generalized Liu and Young model in pricing European options and calculating derivatives sensitivities in high volatile illiquid markets. The generalized Liu and Young model is a more accurate option pricing model that incorporates both the efficacy of the number of invested stocks and the abnormal increase of volatility during a financial crisis for hedging purposes and the financial risk management. To evaluate the performance of these models, we use numerical methods such as finite difference schemes and Monte-Carlo simulation with antithetic variate variance reduction technique. Our results show that the generalized Liu and Young model outperforms the classic Black-Scholes model in terms of accuracy, especially in high volatile illiquid markets. Additionally, we find that the finite difference schemes are more efficient and faster than the Monte-Carlo simulation in this model. Based on these findings, we recommend using the generalized Liu and Young model with finite difference schemes for the European options and Greeks valuing in high volatile illiquid markets.

Keywords: Black-Scholes equation, Finite difference scheme, Greeks, Monte-Carlo simulation, Nonlinear partial differential equation.
Classification: MSC2010 or JEL Classifications: 91G60, 91G20, 91G50.

1 Introduction

In recent years, options have become one of the most widely traded financial derivatives. These contracts give the holder the right to buy (call option) or sell (put option) an underlying asset at a predetermined price (exercise price) at a specified maturity time (European and Asian options) or at any time until the maturity date (American options). In 1973, Fischer Black, Myron Scholes, and Robert Merton introduced the Black-Scholes (B-S) model, a linear framework for option pricing [3,17]. However, this classic B-S model does not consider important factors such as liquid-

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ity, transaction costs, large investor performance, and high volatility during financial crises. As a result, numerous extended B-S models have been developed to incorporate these parameters and provide more accurate option prices [1,2,9,10,12,14]. While the classic B-S model has a closed-form solution, the nonlinear B-S models do not, making them more challenging to solve. Therefore, numerical methods must be employed to obtain solutions for these models. Various numerical methods have been applied to address different nonlinear B-S models, including Alternative direction implicit (ADI) scheme [7], alternative direction explicit (ADE) scheme [4, 16], fourth-order semi-discretization [5, 8], standard and nonstandard finite difference methods [15], the Upwind finite difference method [13] and a positivity-preserving scheme [11]. In this article, we consider the following partial differential equation (PDE) that extends the Liu and Young illiquid market model [14] to a market with financial crunch [9]:

$$\partial_t V(S,t) + \frac{(\sigma S + g(t))^2}{2\big(1-\lambda(S,t)S\partial^2_{SS}V(S,t)\big)^2}\partial^2_{SS}V(S,t) + rS\partial_S V(S,t) - rV(S,t) = 0, \tag{1}$$

with following terminal and boundary conditions:

$$V(S,T) = \begin{cases} \max\{S-K,0\} & \text{, call option} \\ \max\{K-S,0\} & \text{, put option} \end{cases}$$

$$V(0,t) = \begin{cases} 0 & \text{, call option} \\ Ke^{-r(T-t)} & \text{, put option} \end{cases}$$

$$V(S_{\max},t) \simeq \begin{cases} S_{\max}e^{-\delta(T-t)} - Ke^{-r(T-t)} & \text{, call option} \\ 0 & \text{, put option} \end{cases}$$

where $V(S,t)$ is the option price for a value S of the risky asset at time t. d, σ and r are the dividend yield, volatility (on the underlying risky asset) and market interest rate (on the riskfree asset) respectively, $\lambda(S,t)$ is the price impact function of the trader, $g(t)$ implicates a function that is deterministic and demonstrates the impact of increased volatility during a financial crisis, K is the strike price and $S_{\max} >> K$ the upper bound on the computational domain in the S variable.

The option prices in illiquid market during the financial crisis has been computed with the numerical simulations precisely, the antithetic variate technique is utilized to reduce the variance and to decrease the amount of time required for computational processes in Monte Carlo simulations (AMC) [9]. Furthermore to evaluate these kind of options, the corresponding nonlinear PDE 1 has been solved by Crank-Nicolson (CN) method, which to the best of our knowledge this model has not been solved with the finite difference methods.

The main purpose of this study is comparison among the European options pricing in the liquid and illiquid markets with and without high volatility with two different numerical methods (AMC and CN). It will be demonstrated that the option price in the illiquid market is higher than the liquid one with more oscillation around the strike price in Greeks. Furthermore, CN is extremely faster than AMC with almost similar accuary.

In the next section, we explain the underlying asset model in high volatile illiquid markets in details. In Section 3 we describe the numerical methods including the Monte-Carlo simulation with a variance reduction technique and the finite difference method. In Section 4, the classic linear and the nonlinear Black-Scholes model in illiquid market during a financial crisis have been solved with Crank-Nicolson and Monte-Carlo simulation with antithetic variance reduction technique and compared to each other. The option prices and the Greeks ($\Delta(S,t) = \frac{\partial V}{\partial S}(S,t)$, and $\Gamma(S,t) = \frac{\partial^2 V}{\partial S^2}(S,t)$) have been computed with these numerical schemes in various numerical examples. Some conclusion remarks and possible future works are discussed in Section 5.

2 Options Pricing Model in High Volatile Illiquid Markets

Assume that θ is the number of shares that the broker has in the stock at time t. The underlying asset price is therefore assumed to evolve by the following stochastic differential equation:

$$dS = rSdt + (\sigma S + g(t))d\tilde{W}_t + \lambda(S,t)Sd\theta\,. \tag{2}$$

And the value of a European option can be obtained as

$$V(S,0) = \mathbb{E}_Q[e^{-rT}V(S,T)]\,, \tag{3}$$

where Q is the risk-neutral measure. $g(t)$ signifies a deterministic function that effectively captures the impact of increased volatility during a financial crisis. It represents the stock market price index during a crunch as follows [6,18]:

$$g(t) = c_1 + c_2 e^{c_3 t}\sin(c_4 t)\,, \tag{4}$$

where $c_i, i = 1,2,3,4$ are real constants and $\tilde{W}_t$ is a Brownian motion. We take the price impact factor of the trader $\lambda(S,t)$ as in the work of Liu and Young [14] as follows

$$\lambda(S,t) = \begin{cases} \frac{\gamma}{S}\left(1 - e^{-\beta(T-t)}\right) & \text{, if } \underline{S} \le S \le \bar{S} \\ 0 & \text{, otherwise ,} \end{cases} \tag{5}$$

where β is a real constant and the constant price impact coefficient $\gamma > 0$ measures the price impact per traded shares. $\underline{S}$ and $\bar{S}$ represent respectively, the lower and upper limits of the stock price within which there is a price impact. Finally, θ demonstrates the number of shares of the asset S_t that satisfies the following process:

$$d\theta_t = \eta_t dt + \zeta_t dW_t\,, \qquad t \in [0,T]. \tag{6}$$

Here we take $\eta_t = t$ and $\zeta_t = \sin(\pi t/4)$ as in the work of El-Khatib and Hatemi [9].

3 Numerical Methods

In this section, we described Monte-Carlo simulation, utilizing the antithetic variate technique to reduce variance and computational time, resulting in more accurate results. Additionally, the finite difference methods (FDMs) have been implemented to numerically solve the nonlinear PDE 1, as these types of equations lack closed-form solutions.

3.1 Monte-Carlo Simulation

At first we discretize the time interval into N equidistance interval with the step size $\Delta t = \frac{T}{N}$, then SDE 2 will be discretized as follows by utilizing the Euler-Maruyama scheme

$$\Delta S_{t_n} = rS_{t_n}\Delta t + (\sigma S_{t_n} + g(t_n))\,\Delta\tilde{W}_{t_n} + \lambda(S_{t_n}, t_n)s_{t_n}\Delta\theta_{t_n}. \tag{7}$$

The Brownian motion $\{\tilde{W}_{t_n}\}$ will be simulated as

$$\tilde{W}_{t_0} = 0 \tag{8}$$
$$\tilde{W}_{t_{n+1}} = \tilde{W}_{t_n} + \sqrt{\Delta t}Z_n, \quad n = 0, \dots, N, \tag{9}$$

where Z_n follows the standard normal distribution $N(0, 1)$.

In the Monte-Carlo simulation with variance reduction technique, for every nodal point $t_n \in [0, T]$, two values for the underlying asset price $S^+_{t_n}$ and $S^-_{t_n}$ are computed, the first value using Z_n and the second value using $-Z_n$. In total, there are $2N$ values for S with only N trials. These simulations are repeated M times. Thus, we obtain M simulation paths:

$$S^+_{m,t_1}, \dots, S^+_{m,t_N}, S^-_{m,t_1}, \dots, S^-_{m,t_N},\ m = 1, \dots, M. \tag{10}$$

Therefore for each simulation path, the payoff will be the average of two payoffs for two underlying asset price $S^+_{t_N}$ and $S^-_{t_N}$ as follows:

$$V(S_{m,t_N}, T) = \frac{V(S^+_{m,t_N}, T) + V(S^-_{m,t_N}, T)}{2},\ m = 1, \dots, M. \tag{11}$$

Eventually, the option price is obtained by

$$V_{AMC}(S, 0) = \frac{e^{-rT}}{M}\sum_{m=1}^{M} V(S_{m,t_N}, T) \tag{12}$$

3.2 Finite Difference Methods

Here the PDE (1) have been solved by the Crank-Nicolson method. In the finite difference methods, the derivatives will be estimated by the differences in values over intervals of finite size. Hence the PDE 1 will be reduced by the Crank-Nicolson method to the following discretized equation:

$$\frac{V_j^{n+1}-V_j^n}{\delta t}+\frac{1}{2}\left(\sigma S_j+g(t_n)\right)^2\left[\frac{\frac{V_{j+1}^{n+1}-2V_j^{n+1}+V_{j-1}^{n+1}}{2(\delta S)^2}+\frac{V_{j+1}^{n}-2V_j^{n}+V_{j-1}^{n}}{2(\delta S)^2}}{\left(1-\lambda(S_j,t_n)\left(\frac{V_{j+1}^{n}-2V_j^{n}+V_{j-1}^{n}}{(\delta S)^2}\right)\right)^2}\right]+$$
$$\frac{(r-d)S_j}{2}\left(\frac{V_{j+1}^{n+1}-V_{j-1}^{n+1}}{2\delta S}+\frac{V_{j+1}^{n}-V_{j-1}^{n}}{2\delta S}\right)-\frac{r}{2}\left(V_j^{n+1}+V_j^n\right)=0. \tag{13}$$

The above equation leads to the following system of equations:

$$\mathbf{A}V^{n+1}=\mathbf{B}V^n \tag{14}$$

where both $\mathbf{A}$ and $\mathbf{B}$ are the tridiagonals matrices for $j=1,\dots,J-1$ in S and $n=0,\dots,N-1$ in t discretization with the following elements.

$$\mathbf{A}[j-1,j]=\frac{-(\sigma S_j+g(t_n))^2\delta t}{4(\delta S)^2\left[1-\lambda(S_j,t_n)\left(\frac{V_{j+1}^{n}-2V_j^{n}+V_{j-1}^{n}}{(\delta S)^2}\right)\right]^2}+\frac{\delta t(r-d)j}{4},$$
$$\mathbf{A}[j,j]=1+\frac{-(\sigma S_j+g(t_n))^2\delta t}{2(\delta S)^2\left[1-\lambda(S_j,t_n)\left(\frac{V_{j+1}^{n}-2V_j^{n}+V_{j-1}^{n}}{(\delta S)^2}\right)\right]^2}+\frac{r\delta t}{2}, \tag{15}$$
$$\mathbf{A}[j,j+1]=\frac{-(\sigma S_j+g(t_n))^2\delta t}{4(\delta S)^2\left[1-\lambda(S_j,t_n)\left(\frac{V_{j+1}^{n}-2V_j^{n}+V_{j-1}^{n}}{(\delta S)^2}\right)\right]^2}-\frac{\delta t(r-d)j}{4},$$
$$\mathbf{B}[j-1,j]=\frac{(\sigma S_j+g(t_n))^2\delta t}{4(\delta S)^2\left[1-\lambda(S_j,t_n)\left(\frac{V_{j+1}^{n}-2V_j^{n}+V_{j-1}^{n}}{(\delta S)^2}\right)\right]^2}-\frac{\delta t(r-d)j}{4},$$
$$\mathbf{B}[j,j]=1-\frac{-(\sigma S_j+g(t_n))^2\delta t}{2(\delta S)^2\left[1-\lambda(S_j,t_n)\left(\frac{V_{j+1}^{n}-2V_j^{n}+V_{j-1}^{n}}{(\delta S)^2}\right)\right]^2}-\frac{r\delta t}{2}, \tag{16}$$
$$\mathbf{B}[j,j+1]=\frac{(\sigma S_j+g(t_n))^2\delta t}{4(\delta S)^2\left[1-\lambda(S_j,t_n)\left(\frac{V_{j+1}^{n}-2V_j^{n}+V_{j-1}^{n}}{(\delta S)^2}\right)\right]^2}+\frac{\delta t(r-d)j}{4},$$

4 Numerical Computations and Simulations

Here the European call option prices for the liquid and illiquid with and without high volatility cases have been investigated in Table 1 with the same parameters in [9]: $S_0=10, \underline{S}=2, \bar{S}=40, r=0.06, T=1, N=10,000, \eta_t=t, \zeta_t=\sin(\pi t/4), g(t)=\{0,\sin(\pi t/4),\sin(\pi t/2),10+5e^{-2t}\sin(\pi t/18)\}, \sigma=1.5, \gamma=0.04, \beta=100, \delta S=0.5,\ \delta t=0.02$ and the number of simulations in AMC: $M=\{100,200\}$ (demonstrated as $AMC100$ and $AMC200$ in Table 1). More precisely, the first row of this table indicates the call option price in the linear classic Black-Scholes model, the second row corresponds to the call option price in the

Liu and Young model and the last three rows illustrate the call option price in the generalized Liu and Young model with high volatility.

Figure 1 shows AMC simulation of the asset price in the standard Black-Scholes model, in illiquid constant price impact without high volatility (Liu & Young model) and also AMC simulation of the asset price in illiquid varying price impact with high volatility (generalized Liu & Young model). Furthermore Figures 2, 3 and 4 demonstrate the call option, Delta and Gamma of the call option in the classical Black-Scholes model with $\sigma = 0.4$, the Liu and Young illiquid and the high volatile illiquid market with $\sigma = 1.5$ respactively which all obtained by CN method. All tests are performed on an Intel Core i5 CPU 2.9 GHz with 8 GB RAM using Maple 2018 software.

5 Conclusion

In this study, option prices in both liquid and illiquid markets during a crisis have been investigated and computed using the Crank-Nicolson method, which is then compared with Monte Carlo simulations along antithetic variate reduction technique. Additionally, the Call option, Delta, and Gamma in the classic Black-Scholes model are compared to those in the illiquid market with a high volatility model. The results reveal that higher volatility leads to increased option prices and more oscillation around the strike price in Delta and Gamma. Findings suggest that the generalized Liu and Young model outperforms the classic Black-Scholes model in terms of accuracy, particularly in high-volatile illiquid markets. Furthermore, finite difference schemes are observed to be more efficient and faster than Monte Carlo simulations in this model. Consequently, it is recommended to utilize the generalized Liu and Young model with finite difference schemes for pricing European options in high-volatile illiquid markets.

This study contributes to existing literature by providing empirical evidence on the effectiveness of the generalized Liu and Young model in pricing options under challenging market conditions, offering valuable insights for investors and risk managers to make informed decisions.

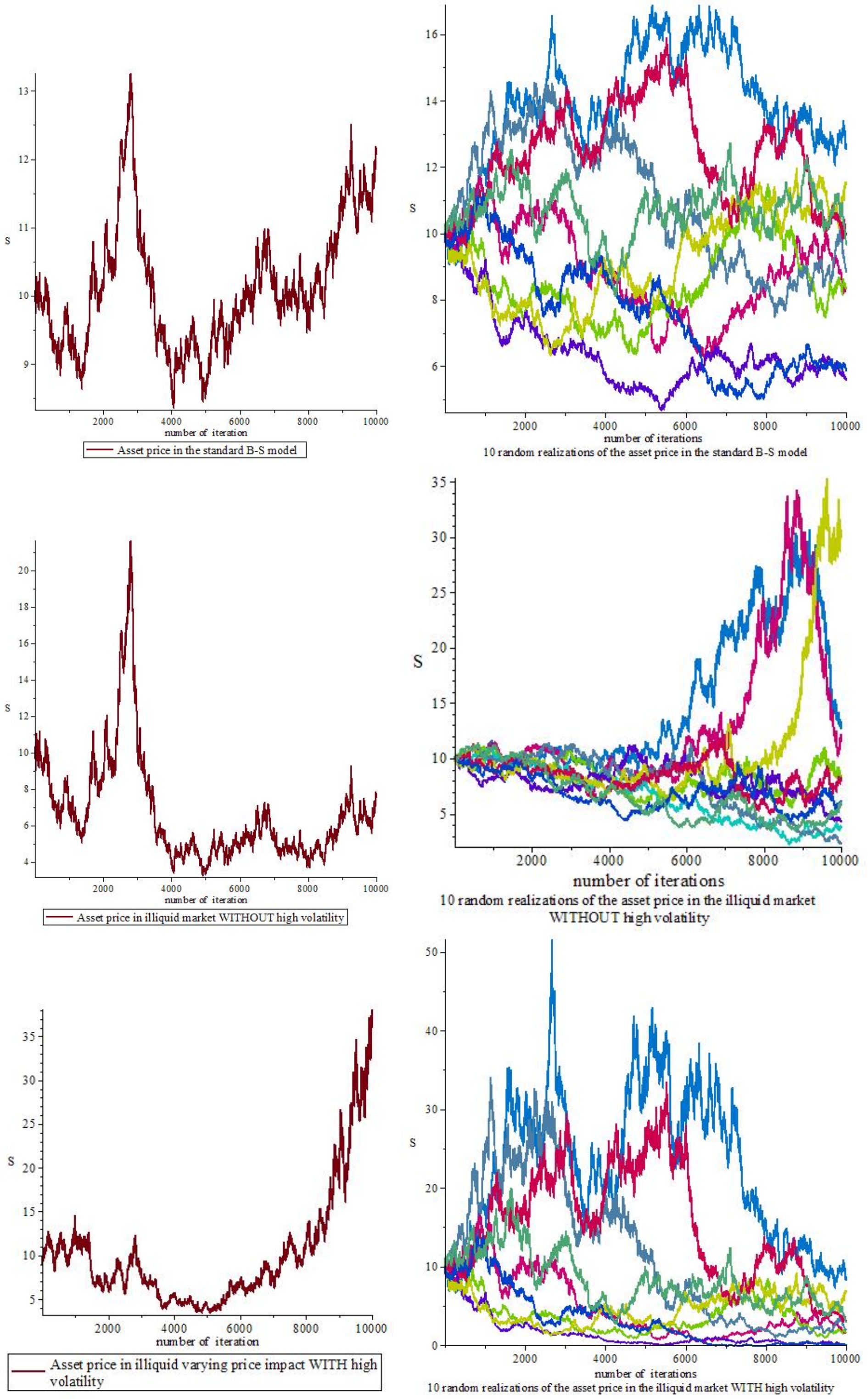

Figure 1: The last run of the simulation and 10 random realizations of the asset price in the classic B-S model (top row), in illiquid market without high volatility (middle row), in illiquid market with high volatility (last row) with these parameters: $S_0 = 10, \underline{S} = 2, S \leq \bar{S} = 40, N = 10000, M = 100, r = 0.06, \sigma = 0.4$(1st row) $\sigma = 1.5$(2nd and 3rd row)

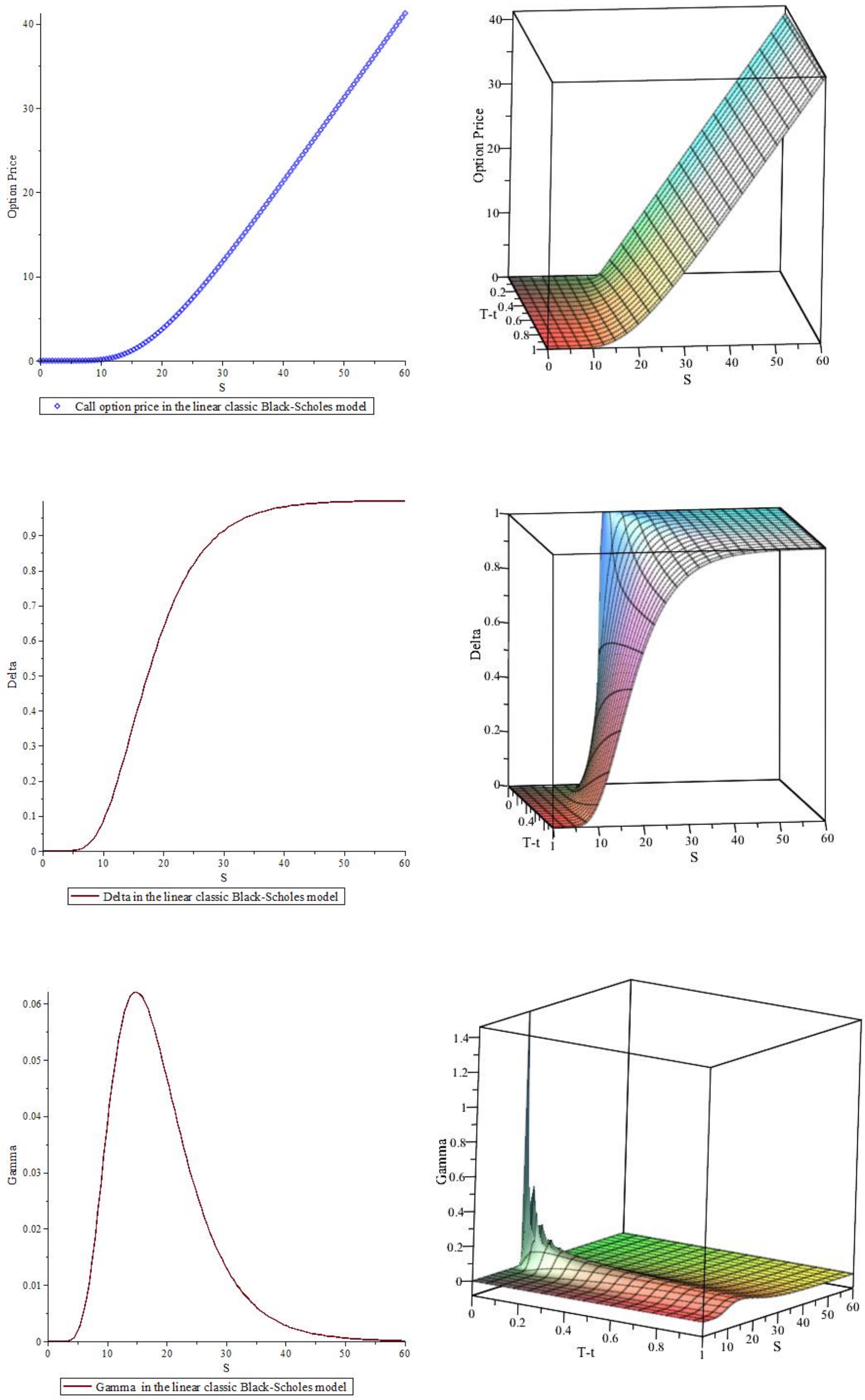

Figure 2: Call option (first row), Delta of the call option (midle row) and Gamma of the call option (last row) in the classical Black-Scholes model with parameters $\sigma = 0.4, K = 20$, $Smax = 60$, $\delta S = 0.5, T = 1, \delta t = 0.02$

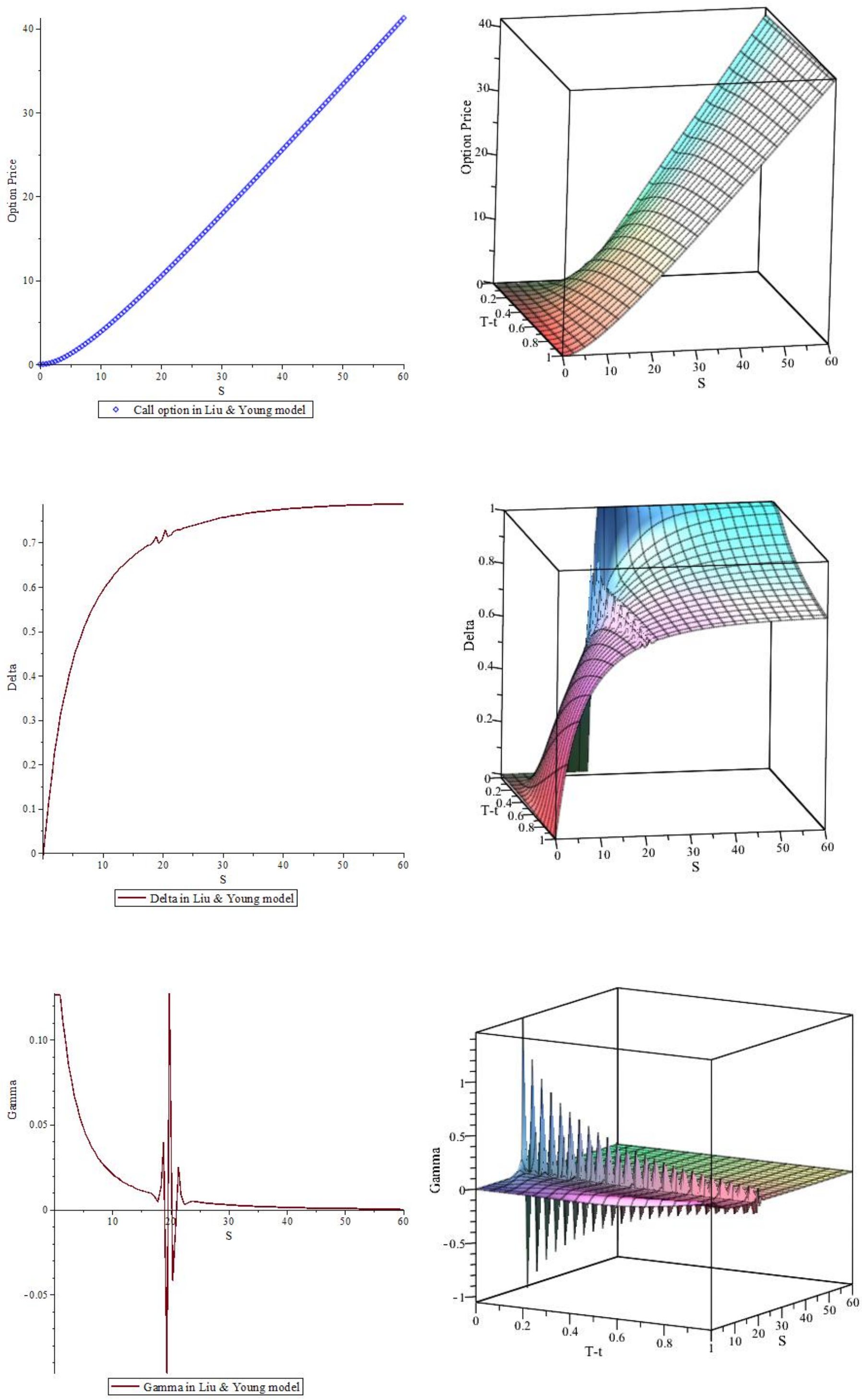

Figure 3: Call option (first row), Delta of the call option (midle row) and Gamma of the call option (last row) in the Liu and Young model with parameters $\lambda(S,t)$ =(5), $\sigma = 1.5, K = 20$, $Smax = 60$, $\delta S = 0.5, T = 1, \delta t = 0.02$

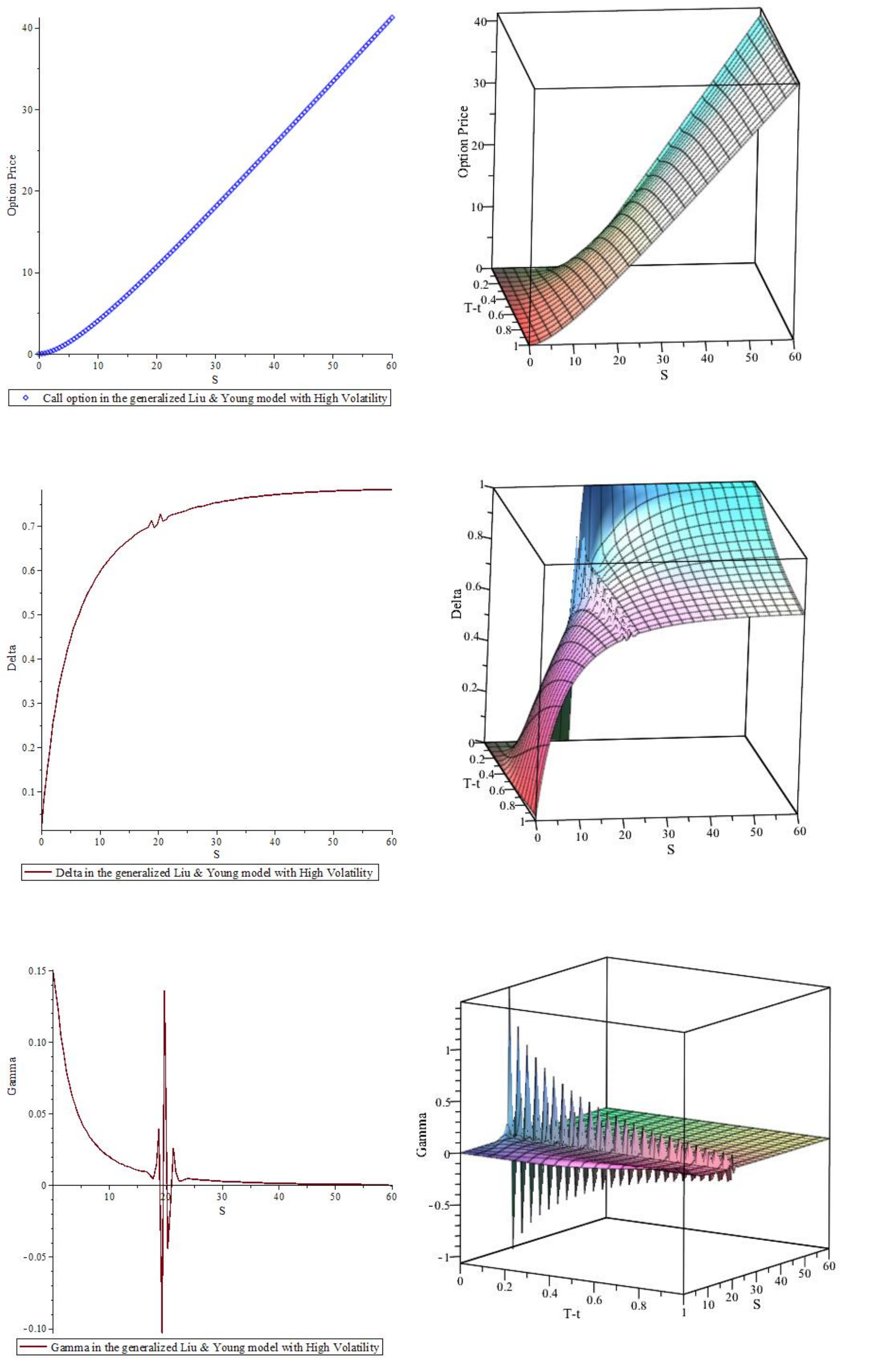

Figure 4: Call option (first row), Delta of the call option (midle row) and Gamma of the call option (last row) in the high volatile illiquid market with parameters $\lambda(S,t) =$(5) ,$g(t) = \sin(\pi t/2)$, $\sigma = 1.5, K = 20$, $Smax = 60$, $\delta S = 0.5, T = 1, \delta t = 0.02$

Table 1: European call option in an illiquid market with and without high volatility with AMC and CN

$\lambda(S,t)$	g(t)	AMC100	AMC200	CN	AMC100 time	AMC200 time	CN Time
0	0	3.54544	3.32807	3.80609	147.164	224.749	5.294
(5)	0	3.54493	3.32838	3.81358	208.95	279.673	4.011
(5)	$\sin(\pi t/4)$	3.64951	3.40646	3.91276	247.329	356.674	3.159
(5)	$\sin(\pi t/2)$	3.72952	3.47135	3.98774	270.633	408.272	2.901
(5)	$10+5\sin(\pi t/18)e^{-2t}$	6.68363	6.14072	6.04179	346.665	402.576	4.602

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