Journal of Mathematics and Modeling in Finance (JMMF)
Vol. 4, No. 1, Winter & Spring 2024
Research paper

On the numerical performance of the weak multilevel Monte-Carlo method for the Heston Model

Azadeh Ghasemifard[1], **Ali Valinejad**[2]

[1] Department of Applied Mathematics, University of Mazandaran
a.ghasemi@umz.ac.ir

[2] Department of Computer Science, University of Mazandaran
valinejad@umz.ac.ir

Abstract:
In this article, we discuss the numerical implementation of the Multilevel Monte-Carlo (MLMC) scheme for option pricing within the Heston asset model. The Heston model is a stochastic volatility model that captures the dynamics of the underlying asset price and its volatility. The MLMC method is a variance reduction technique that exploits the difference between two consecutive levels of discretization to estimate the expected value of a quantity of interest. We begin by providing an overview of the MLMC method, followed by an introduction to the weak methods used to approximate the Heston model. Weak methods are numerical schemes that preserve the distributional properties of the solution, rather than its path-wise behavior. Subsequently, we present the results of some numerical experiments conducted to evaluate the performance of the approach. Two different cases are surveyed.

Keywords: Heston model, Multilevel Monte-Carlo method, Weak approximation.
Classification: MSC2010: 65C05, 91G30.

1 Introduction

The Heston model is a well-known stochastic volatility model that describes the dynamics of an asset price S_t and its volatility v_t given by the following stochastic differential equations (SDEs):

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{v_t} S_t dW_t^1, \quad S_0 > 0, \quad 0 < t < T, \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^2, \quad v_0 > 0, \quad 0 < t < T, \end{aligned} \tag{1}$$

where:

- $0 < r < 1$ is the risk-free interest rate,

[1] Corresponding author
Received: 09/03/2024 Accepted: 08/05/2024
https://doi.org/10.22054/JMMF.2024.78741.1127

- $\kappa > 0$ is the mean reversion speed,
- $\theta > 0$ is the long-term volatility,
- $\sigma > 0$ is the volatility of volatility,
- W_t^1 and W_t^2 are two correlated Brownian motions with correlation $\rho \in [-1, 1]$,

and dW_t^1 and dW_t^2 are the corresponding increments of the Brownian motions. We denote the payoff of the option by $f : [0, \infty) \to \mathbb{R}$. Throughout this paper, the initial values S_0, v_0 are assumed to be deterministic.

While the Heston model lacks a closed-form solution, it is established that a strong solution exists and is unique, as per the Yamada-Watanabe theorem. We can rewrite the Heston model into one with two independent Brownian motions W_t^2 and B_t, by introducing an auxiliary variable $Z_t = \rho dW_t^2 + \sqrt{1-\rho^2} dB_t$. We get

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{v_t} S_t dZ_t, \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^2. \end{aligned} \tag{2}$$

A common numerical practice is to utilize the log-Heston model rather than the Heston model. The transformation $X_t = ln(S_t)$ yields

$$\begin{aligned} dX_t &= (r - \frac{1}{2}v_t)dt + \sqrt{v_t} dZ_t, \quad X_0 \in \mathbb{R}, \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t, \quad v > 0, \end{aligned} \tag{3}$$

where for notational convenience we have removed the superscript 2. Accordingly, the payoff function also incorporates the exponential term as $g : \mathbb{R} \to \mathbb{R}, \quad g(x) = f(\exp(x))$.

The Heston model finds extensive application in option pricing. The general form of the option price can be expressed as follows:

$$\mathbb{E}(f(S)),$$

where the expectation is calculated under the risk-neutral measure. Usually, one may resort to approximating it using Monte-Carlo simulation [10], coupled with a numerical method, preferably a weak one, to estimate S. Traditional time-discrete schemes, such as those outlined in Kloeden and Platen [11], frequently yield erratic outcomes when employed in the Heston model. This is primarily due to the model's diffusion term not satisfying the Lipschitz condition. Although several discretization schemes and simulation methods for the stochastic differential equation (3) have been proposed and numerically evaluated, studies analyzing the weak convergence rate are scarce. In [2], the authors proposed a numerical scheme, which uses the drift-implicit Milstein scheme for the volatility and a Euler discretisation for the log-Heston price. They show that under the assumption $\nu = \frac{2\kappa\theta}{\sigma^2} > 2$ on Feller index, and for sufficiently smooth payoffs, the method is weakly convergent of order $\alpha \in (0, 1)$. On the other hand, in [13], the author used the idea that the CIR variance process v_t can be simulated exactly. Specifically, the distribution of v_t conditioned on $v_u, u < t$ is Chi-squared and leveraging the known transition density of the variance process, some authors have explored an alternative approach for simulating the Heston model. In this approach, they either approximate the variance process using easily simulatable random variables or simulate it nearly exactly. Using the formula due to [4], the logarithmic asset process (3), conditioned on the variance process is simulated as

$$\begin{aligned} X_{t+h} =& X_t + \frac{\rho}{\sigma}\left(v_{t+h} - v_t - k\theta h\right) + \left(\frac{\rho k}{\sigma} - \frac{1}{2}\right)\int_t^{t+h} v_s \, \mathrm{d}s \\ &+ \sqrt{1-\rho^2}\sqrt{\int_t^{t+h} v_s \, \mathrm{d}s} N, \end{aligned} \tag{4}$$

where N is a standard normal random variable, independent of the variance process and h is the step size. Then, the time integral is estimated by the stochastic trapezoidal rule

$$\int_t^{t+h} v_s \, \mathrm{d}s \approx \frac{v_t + v_{t+h}}{2} h.$$

In [13], the weak convergence order 2 for polynomial payoffs is calculated for this method. The advantage is this method has no restriction on parameter regimes.

In a parallel development, Giles [8] introduced the multilevel Monte-Carlo (MLMC) method as a variance reduction technique to reduce the computational cost of the Monte-Carlo method. Hence, a significant challenge arises in integrating the MLMC method with an efficient numerical scheme tailored for the Heston model.

Recently, [14], combined the multilevel Monte-Carlo method with the semi exact numerical scheme for the Heston model (4), that either simulates the variance process exactly or nearly exactly. Additionally, the stochastic trapezoidal rule employed to approximate the time-integrated variance process within the stochastic differential equation governing the logarithmic asset process. Although, applying the MLMC for Heston type models is rare in the literature, for more examples consult [1], [8], [9] and [6].

While weak convergence holds traditional significance in financial mathematics due to its focus on expectations of functionals of the solutions, strong convergence assumes a pivotal role in Multilevel Monte-Carlo methods. Therefore, we need to also examine pathwise convergence, along with techniques that maintain the positivity of the solutions. Subsequently, [9] and [3] devised methods to bypass this stringent requirement for strong convergence and the idea of weak MLMC is born. They achieved this by introducing antithetic paths and a coupling mechanism between the levels, respectively.

In this article, we consider two different approaches to apply the weak MLMC method, by combining MLMC with the weak numerical scheme for the Heston model. In section 2, the weak MLMC scheme is reviewed and section 3 contains the numerical experiments.

2 Multilevel Monte-Carlo method

The Multilevel Monte-Carlo (MLMC) method, introduced by M. Giles [8], provides an efficient approach for managing the computational complexity arising from variance and bias across multiple levels. This scheme employs successive corrections to estimate $\mathbb{E}[f(X_T)]$, facilitating independent estimation of the mean value for each correction. As a result, significant reduction in computational complexity is achieved compared to the Monte-Carlo (MC) method.

Consider the numerical approximation Y_N^l, where $N = 2^l$, of X_T for each level $l = 0, \dots, L$, with a step size $_l = (T - t_0)/2^l$. Let P denote a functional of X_T, $P = f(X_T)$, and P_l denote the same functional of Y_N^l, $P_l = f(Y_N^l)$, for each $l = 0, \dots, L$. Expanding $[P_L]$ into a telescopic sum, we obtain

$$\mathbb{E}[P_L] = \mathbb{E}[P_0] + \sum_{l=1}^{L} \mathbb{E}[P_l - P_{l-1}].$$

Instead of directly computing the expectation, we approximate it using θ_l, with M_l paths. For instance, $\theta_0 = \frac{1}{M_0} \sum_{i_0=1}^{M_0} P_0(\omega_{0,i_0})$, and $\theta_l = \frac{1}{M_l} \sum_{i_l=1}^{M_l} (P_l - P_{l-1})(\omega_{l,i_l})$, where ω_{l,i_l} corresponds to the i_l-th path of Y_N^l. The optimal M_l is determined by

$$M_l = 2\epsilon^{-2} \sqrt{V_{l}{}_{l}} \left(\sum_{i=0}^{L} \sqrt{V_i/_i} \right),$$

where V_l is the variance of $P_l - P_{l-1}$ and ϵ represents the chosen Root Mean Square error (RMSE) [8]. This optimization minimizes the computational effort $\sum_{l=0}^{L} M_l/_l$ under the condition that the variance of the MLMC estimator is less than $\varepsilon^2/2$.

The advantage of the MLMC approach lies in its flexibility to adjust the number of paths required for estimating the expectation based on the variance of $P_l - P_{l-1}$. As the step size decreases, so does the variance. Thus, we aim to allocate a significant number of paths only when each path is inexpensive, i.e., when we perform few time steps per path. Conversely, we aim to limit the number of paths when they are costly, i.e., when many time steps are required.

Giles' general theorem [8] states the following.

Theorem 2.1. *Let P denote a functional of the solution of a SDE, and let P_l denote the corresponding approximation using step size h_l. If there exists independent esti-*

mators θ_l based on M_l Monte-Carlo paths and positive constants $\alpha > 0.5$, β, c_1, c_2, and c_3 such that

(i) $|\mathbb{E}[P_l - P]| \leq_l^\alpha$,

(ii) $[\theta_l] = \begin{cases} [P_0], & l = 0 \\ [P_l] - [P_{l-1}], & l > 0, \end{cases}$

(iii) $(\theta_l) \leq c_2 M_l^{-1}{}_l^{\beta}$,

(iv) C_l*, the computational complexity on each level, is bounded by* $C_l \leq c_3 M_l{}_l^{-1}$,

then there exists a positive constant c_4 such that for any $\epsilon < e^{-1}$, there are values L and M_l, for which the multilevel estimator

$$\theta = \sum_{l=0}^{L} \theta_l$$

has a Mean Squared Error (MSE) with bound

$$MSE := \mathbb{E}[(\theta - \mathbb{E}(P))^2] \leq \epsilon^2$$

with a computational complexity C with bound

$$C \leq \begin{cases} c_4 \epsilon^{-2}, & \beta > 1 \\ c_4 \epsilon^{-2} (\log(\epsilon^{-1}))^2, & \beta = 1, \\ c_4 \epsilon^{-2-(1-\beta)/\alpha}, & 0 < \beta < 1. \end{cases}$$

Out of the four requirements, the third one is the most challenging to fulfill. The first requirement implies weak convergence of order α for the underlying method used to approximate X_t based on the step size h_l. Constructing estimators with properties (2) and (4) is relatively straightforward, as the former ensures θ_l is unbiased, and the latter concerns bounding the computational cost of each level. The classical approach to fulfilling requirement (3) involves using strong order of consistency, wherein $2p_{strong} = \beta$ for Lipschitz continuous functions.

Weak MLMC method

In [3], Belomestny and Nagapetyan introduced the Weak MLMC method, demonstrating that MLMC based on the weak Euler scheme maintains $C = (\epsilon^{-2}(\log \epsilon)^2)$ with proper level coupling. Specifically, for an arbitrary level l, let $\xi_{l,i}^f$ and $\xi_{l,i}^c$, $i = 1, \ldots, 2^l$, be, possibly approximate, Wiener increments with variance Δ_l, utilized in the approximation of $\hat{P}_l^f$ and $\hat{P}_l^c$, respectively. Then, we impose the condition

$$\mathcal{R} = \xi_{l-1,i}^c - \xi_{l,2i-1}^f - \xi_{l,2i}^f, \quad \xi_{l-1,i}^c \stackrel{D}{=} \xi_{l-1,i}^f \tag{5}$$

with $\mathcal{R}$ sufficiently small. Here, we will only consider $\mathcal{R} = 0$.

Equation (5) couples the random variables on a single level together and to the lower levels. One selects an initial distribution for the random variables $\xi^f_{L,i}$, which, by equation (5), generates the distributions of the random variables on the lower levels. For instance, the choice $\xi^f_{L,i} \sim \mathrm{N}(0, \sqrt{\Delta_L})^m$ yields $\xi^f_{l,i} \sim \mathrm{N}(0, \sqrt{\Delta_l})^m$, representing the classical MLMC. Meanwhile, for the two-point distribution approximation with $(\xi^f_{L,i,j} = \pm\sqrt{\Delta_L}) = \frac{1}{2}$, $j = 1, \ldots, m$, we obtain

$$\xi^f_{l,i} \sim \left(\mathrm{Bin}\left(2^{L-l}, 0.5\right) - 2^{L-l-1}\right) \cdot 2\sqrt{\Delta_L}. \tag{6}$$

The efficient implementation of generating binomial random numbers is discussed in [3, Section 4.1].

3 Numerical Results

In this section, using the idea of coupling levels, we test combinations of the MLMC method with two weak numerical schemes and discuss the numerical results.

Case 1

The scheme we examine first is chosen from [2], comprises a drift-implicit Milstein scheme for the volatility and an Euler scheme for the log-price. The discritization of $[0, T]$ is $0 = t_0 < t_1 < \cdots < t_N = T$.

$$\begin{aligned}
x_{n+1} =& x_n + \left(r - \frac{1}{2}v_n\right)(t_{n+1} - t_n) + \sqrt{v_n}\left(\rho\left(W_{t_{n+1}} - W_{t_n}\right) + \sqrt{1-\rho^2}\left(B_{t_{n+1}} - B_{t_n}\right)\right), \\
v_{n+1} =& v_n + \kappa\left(\theta - v_{n+1}\right)(t_{n+1} - t_n) + \sigma\sqrt{v_n}\left(W_{t_{n+1}} - W_{t_n}\right) \\
&+ \frac{\sigma^2}{4}\left(\left(W_{t_{n+1}} - W_{t_n}\right)^2 - (t_{n+1} - t_n)\right),
\end{aligned}$$

with $x_0 = X_0$ and $v_0 = V_0$. This scheme can be rewritten as

$$v_{n+1} = \frac{1}{1 + \kappa\left(t_{n+1} - t_n\right)}\left(\left(\sqrt{v_n} + \frac{\theta}{2}\left(W_{t_{n+1}} - W_{t_n}\right)\right)^2 + \left(\kappa\lambda - \frac{\theta^2}{4}\right)(t_{n+1} - t_n)\right), \tag{7}$$

and thus $v_n > 0, n = 1, \ldots$ iff $\frac{4\kappa\theta}{\sigma^2} \geq 1$. Using the normal distributed increments, this method has weak order of convergence near 1 for the European put payoff [2]. We combined this method with MLMC using both normal and binomial increments, but we did not observe any significance of computational efficacy. The numerical order of variance reduction β is around 0.5 and so based on theorem 2.1, we cannot expect much complexity reduction over Monte-Carlo method, as seen in practice.

Case 2

The second method applied is consisted of Euler scheme for the price and Lamperti backward Euler method [12] for the variance process for equation (2). In this equation, if $v_0 > 0$ and $2\kappa\theta > \sigma^2$ is imposed, then v_t is strictly positive. First, by using the Lamperti transform $X_t = F(\nu_t)$, where $F(z) = \int_0^z \frac{1}{\sqrt{y}}, dy = 2\sqrt{z}$, the positivity of the numerical approximation of the transformed stochastic differential equation (SDE) is guaranteed. Applying the Itô formula, we have

$$dX_t = \kappa\left(\left(\theta - \frac{\sigma^2}{4\kappa}\right)\frac{2}{X(t)} - \frac{X(t)}{2}\right)dt + \sigma dW_t,$$

which is an SDE with additive noise. Then, we use backward Euler approximation

$$X_{n+1} = X_n + \kappa\left(\left(\theta - \frac{\sigma^2}{4\kappa}\right)\frac{2}{X_{n+1}} - \frac{X_{n+1}}{2}\right)h + \sigma\Delta W_t.$$

Here, we combine the MLMC algorithm together with this numerical method using the binomial increments (coupling idea). The parameters are chosen close to [14] to ensure the Feller condition satisfies for lower variance assets. Also, picking root mean square errors $RMSE = [5e-3, 1e-3, 5e-4, 1e-4]$, the simulations for the European call option payoff $f(S) = (S-K)^+$ (standard for Lipschitz payoff), are shown below.

	1	2	3	4
k	5	2.6	1.6	6.2
θ	0.09	0.04	0.04	0.02
σ	0.35	0.25	0.3	0.2
ρ	-0.3	-0.9	-0.5	-0.7

In all cases, $T = 1, v_0 = \theta, r = 0.05$, and $S_0 = K = 1$

The variance reduction order β and weak order of convergence α are calculated in figures. The results for all set of parameters agree on $\beta = 1$ and $\alpha = 1$. More precisely, we expect $\beta = 1$ (strong order =1/2) and $\alpha = 1$ from the Euler method and $\beta = 1$ and $\alpha = 1$ for the backward Euler method [7]. We, therefore, anticipate the same for the combined method, as is proven by the numerical results. The significance of using weak MLMC here lies in not relying on a strong method. Despite this, we achieve $\beta = 1$, resulting in a reduced computational cost compared to the classical Monte-Carlo method. The expected results are confirmed across all sets of parameters.

The figure 5 demonstrates the computational cost of the method. As expected from theorem 2.1 for $\beta = 1$, we see the order of cost is slightly less than 2.

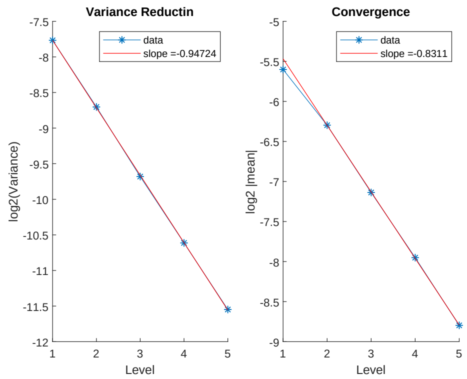

Figure 1: Data column 1, CPU time= 86.0488 seconds.

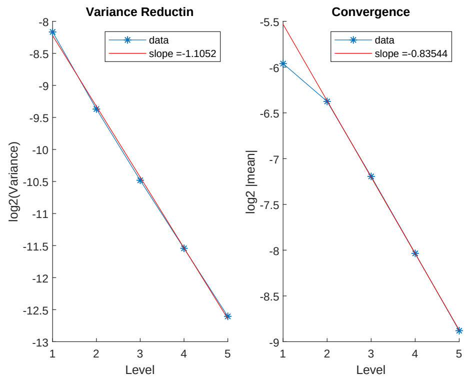

Figure 2: Data column 2, CPU time= 52.5207 seconds.

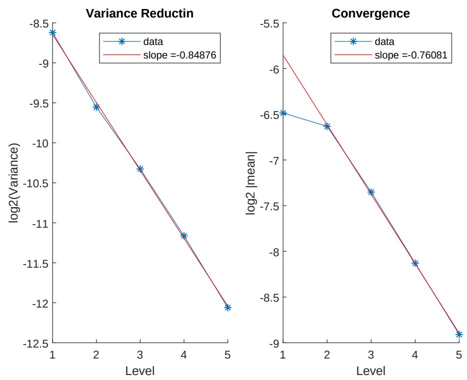

Figure 3: Data column 3, CPU time= 46.2955 seconds.

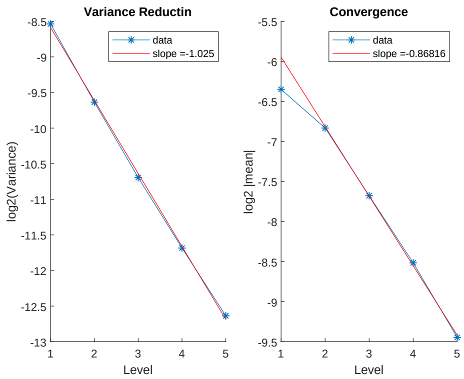

Figure 4: Data column 4, CPU time= 45.7574 seconds.

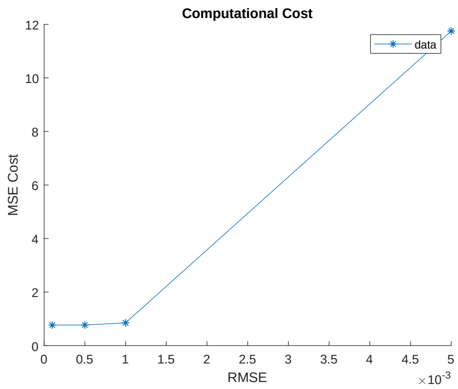

Figure 5: Computational Complexity.

In comparison to this method, the recent MLMC Heston scheme proposed by [14] with $\beta = 2$ for path-independent polynomial payoffs employed MLMC with a strong discretization method for the log-price and normally distributed increments. Additionally, they simulated the variance process exactly, making it trivial to achieve a higher order of variance reduction. It is also worth mentioning that our initial simulations of the weak antithetic MLMC method [5] yielded acceptable results, suggesting its potential for future investigations. Another additional field of research could be, how to improve this method to adjust for the more volatile assets.

Bibliography

[1] Martin Altmayer and Andreas Neuenkirch, *Multilevel monte carlo quadrature of discontinuous payoffs in the generalized heston model using malliavin integration by parts*, SIAM Journal on Financial Mathematics **6** (2015), no. 1, 22–52.

[2] , *Discretising the heston model: an analysis of the weak convergence rate*, IMA Journal of Numerical Analysis **37** (2017), no. 4, 1930–1960.

[3] Denis Belomestny and Tigran Nagapetyan, *Multilevel path simulation for weak approximation schemes with application to Lévy-driven SDEs*, Bernoulli **23** (2017), no. 2, 927–950.

[4] Mark Broadie and Özgür Kaya, *Exact simulation of stochastic volatility and other affine jump diffusion processes*, Operations research **54** (2006), no. 2, 217–231.

[5] Kristian Debrabant, Azadeh Ghasemifard, and Nicky C Mattsson, *Weak antithetic mlmc estimation of sdes with the milstein scheme for low-dimensional wiener processes*, Applied Mathematics Letters **91** (2019), 22–27.

[6] Azadeh Ghasemifard and Mohammad Taghi Jahandideh, *Weak multilevel path simulation for jump-diffusion assets*, Mathematical Researches **7** (2021), no. 2, 353–370.

[7] Azadeh Ghasemifard and Mahdieh Tahmasebi, *Multilevel path simulation to jump-diffusion process with superlinear drift*, Applied Numerical Mathematics **144** (2019), 176–189.

[8] Michael B Giles, *Multilevel monte carlo path simulation*, Operations research **56** (2008), no. 3, 607–617.

[9] Michael B Giles and Lukasz Szpruch, *Antithetic multilevel monte carlo estimation for multidimensional sdes*, Monte Carlo and Quasi-Monte Carlo Methods 2012, Springer, 2013, pp. 367–384.

[10] Paul Glasserman, *Monte carlo methods in financial engineering*, vol. 53, Springer, 2004.

[11] Peter E Kloeden and Eckhard Platen, *Stochastic differential equations*, Springer, 1992.

[12] L Szpruch and A Neuenkirch, *First order strong approximations of scalar sdes with values in a domain*, Numerische Mathematik **128** (2014), no. 1, 103–136.

[13] Chao Zheng, *Weak convergence rate of a time-discrete scheme for the heston stochastic volatility model*, SIAM Journal on Numerical Analysis **55** (2017), no. 3, 1243–1263.

[14] , *Multilevel monte carlo simulation for the heston stochastic volatility model*, Advances in Computational Mathematics **49** (2023), no. 6, 81.

How to Cite: Azadeh Ghasemifard[1], Ali Valinejad[2], *On the numerical performance of the weak multilevel Monte-Carlo method for the Heston Model*, Journal of Mathematics and Modeling in Finance (JMMF), Vol. 4, No. 1, Pages:57–66, (2024).