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Research paper

Stochastic-fractional optimal control problems and application in portfolio management

Saba Yaghobipour[1], **Majid Yarahmadi**[2]

[1] Department of Mathematics and Computer Science, Lorestan University, Lorestan, Iran. sabaphd88@gmail.com

[2] Department of Mathematics and Computer Science, Lorestan University, Lorestan 68151-44316, Iran.

yarahmadi.m@lu.ac.ir

Abstract:
The aim of this paper is to propose a new method for solving a calss of stochastic-fractional optimal control problems. To this end, we introduce an equivalent form for the presented stochastic-fractional optimal control problem and prove that these problems have the same solution. Therefore, the corresponding Hamilton–Jacobi–Bellman (HJB) equation to the equivalent stochastic-fractional optimal control problem is presented and then the Hamiltonian of the system is obtained. Finally, by considering Sharpe ratio as a performance index, Merton's portfolio selection problem is solved by the presented stochastic-fractional optimal control method. Moreover, for indicating the advantages of the proposed method, optimal pairs trading problem is simulated.

Keywords: Stochastic fractional function, Sharpe ratio, Optimal control, Portfolio management.
*Classification:*MSC2010 or JEL Classifications: 35F21, 35R60, 91G10.

1 Introduction

There are two main approaches for solving a stochastic optimal control problem, using Hamilton-Jacobi-Bellman (HJB) equation and using the Pontryagin Maximum Principle (PMP) which are based on the dynamic programming method and the calculus of variations, respectively [6]. In the dynamic programming method, first a value function is defined by considering sumation of the objective function values on a horizon. Then the corresponding HJB equation is defined. The value function have to be smooth that is not established in practical problems, necessarily. For avoiding this drawback, a suitable formulation in viscosity solutions for dynamic programming equations was introduced [7]. Solving the HJB equation for the dynamic programming method is difficult. In this case, although LQR approach is considered as a common approach but the path integral methods can solve a class of non-linear and non-quadratic control problems [8]. The path integral leads us to

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some methods such as non-linear Kalman filters method [9].

In finance, risk refers to the degree of uncertainty in an investment decision. The risk of a financial asset investment is one of the factors determining the rate of return considered by investors. Also, pricing is determined based on the rate of return [12] . For a portfolio that is comprised of financial assets, there are two types of risk, systematic and unsystematic risk [10]. The systematic risk is related to market risk that is inflexible and inevitable. The systematic risk is caused by economic factors [11]. Even, a variable portfolio has systematic risk. The unsystematic risk is related to assets of portfolio that is caused by condition of the asset firm. Unlike systematic risk, unsystematic risk can be adjusted. Indeed, the unsystematic risk can be decreased by selecting a variable portfolio [10]. Therefore, for obtaining total risk, the market risk must be estimated. Additionally, the market risk can be adjusted by considering some adequate financial indexes in formulating optimal portfolio allocation problems [12].

The Sharpe ratio is one of the financial indexes that provides adjusting the market risk to maximize the performance of an investment. Indeed, by using of the Sharpe ratio, investors can understand the return of an investment while it is compared to it's risk [13]. If the value of the Sharpe ratio be maximum then the risk-adjusted return will be minimum [14, 15]. Thereby, by considering the Sharpe ratio as the objective function, a stochastic optimization problem or a stochastic optimal control problem with fractional objective function is formulated.

In a fractional programming, the objective function is a ratio of two functions. The ratio to be optimized often describes some kind of efficiency of a system [18].

In this paper, we attempt to optimize of portfolio with respect to Sharpe ratio leads. For this purpose, we formulate a new stochastic-fractional optimal control(SFOC) problem and propose a new approach for solving SFOC problem. In this new approach, first, an equivalent form is introduced for the SFOC problem. It is proved that the mentioned problem and it's equivalent form have the same optimal solution. Then, the corresponding HJB equation to the equivalent problem is presented. Thereby, according to optimality conditions, the optimal strategy can be obtained. Finally, as an application, Merton's portfolio selection problem is solved by the presented SFOC method. Moreover, for indicating the advantages of the proposed method, optimal pairs trading problem is simulated.

The paper is organized as follows; the SFOC problem is stated in Section.2. The stochastic-fractional optimal control method and it's algorithm is presented in Section.3. Finally, solving Merton's portfolio selection problem and pairs trading problem are presented as the applications in Section.4. Also, the conclusion is presented in Section.5.

2 Statement of the problem

Consider a stochastic control system where dynamics of the state variable is the following stochastic differential equation(SDE) with control-dependent diffusion and drift coefficients:

$$dX_s = b(X_s, u_s)dt + \sigma(X_s, u_s)dW_s, \tag{1}$$

where W is a d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_s)_{s\geq 0}, P)$. The control $u = \{u_s\}_{s\geq 0}$ is an adapted-$\mathbb{F}$ process in admissible control set $\mathcal{A} \subseteq \mathbb{R}^m$. The functions $b : \mathbb{R}^n \times \mathcal{A} \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times \mathcal{A} \to \mathbb{R}^{n\times d}$ satisfy a uniform Lipschitz condition in $\mathcal{A}$:
$\exists K > 0, \forall X, Y \in \mathbb{R}^n, \forall u \in \mathcal{A},$

$$|b(X, u) - b(Y, u)| + |\sigma(X, u) - \sigma(Y, u)| \leq K|X - Y|. \tag{2}$$

Also, for a finite horizon $0 < t < T < \infty$, $\mathcal{A}$ is a set of control processes u satisfying the following condition:

$$E\Big[\int_0^T (|b(0, u_t)|^2 + |\sigma(0, u_t)|^2)dt\Big] < \infty. \tag{3}$$

The conditions (2) and (3) ensure the existence and uniqueness of the solution of SDE (1) for all $u \in \mathcal{A}$ and for any initial condition $(t, x) \in [0, T] \times \mathbb{R}^n$, starting from x at $s = t$. This solution is denoted by $\{X_s^{t,x},\ 0 \leq t \leq s \leq T\}$.

Now, Consider the following fractional objective function on time interval $[0, t_f]$ where t_f is final time :

$$J(u_s) = \frac{E[f(X_s^{t,x}, u_s)]}{\sqrt{Var(f(X_s^{t,x}, u_s))}} \tag{4}$$

where $f(.,.)$ is an integrable function on $\mathbb{R}^m \times [0, t_f]$. The objective is to find a stochastic control $u = \{u_s\}_{s\geq 0}$ that satisfies the SDE (1) and maximizes the objective fractional function (4). To this end, the following Mayer type stochastic-fractional optimal control problem with initial condition is to be solved [18].

$$\begin{aligned} &\max_{u_{t_f}\in\mathcal{A}} J(u_s) = \frac{E[f(X_s^{t,x}, u_s)]}{\sqrt{Var(f(X_s^{t,x}, u_s))}} \\ &\text{subject to :} \\ &\quad dX_s = b(X_s, u_s)dt + \sigma(X_s, u_s)dW_s, \\ &\quad X(0) = X_0, \end{aligned} \tag{5}$$

3 Solving stochastic-fractional optimal control problems

In this section, first an equivalent form of the Mayer type stochastic-fractional optimal control problem(5) is introduced and then by considering the corresponding (HJB) equation, the equivalent problem is solved.

Theorem 3.1. *Consider the stochastic-fractional optimal control problem(5). Then there exists optimal value $\hat{J}$ for which the optimal solution of problem(5) is the same as the optimal solution of the following stochastic-fractional optimal control problem:*

$$\begin{aligned} &\max_{u_{t_f}\in\mathcal{A}} Z(u_{t_f}) = E[f(X_{t_f}^{t,x}, u_{t_f})] - \hat{J}Var(f(X_{t_f}^{t,x}, u_{t_f})), \\ &\textit{subject to}: \\ &\quad dX_s = b(X_s, u_s)dt + \sigma(X_s, u_s)dW_s, \\ &\quad X(0) = X_0, \end{aligned} \tag{6}$$

where $\hat{J}$ is optimal value of the functional $J(.)$.

Proof. According to the constraints of the problem(5), for $s \in [0, t_f]$ define the function $H : \mathbb{R}^n \longrightarrow \mathbb{R}^n$:

$$H(X_s) = X_s - X_0 - \int_0^s b(X_t, u_t)dt - \int_0^s \sigma(X_t, u_t)dW_t, \tag{7}$$

differentiating, we observe that $H(X_t) = 0$. Therefore, the two problems (5) and (6) are converted to the following optimization problems, respectively:

$$\begin{cases} \max_{u_{t_f}\in\mathcal{A}} J(u_{t_f}) = \dfrac{E[f(X_{t_f}^{t,x}, u_{t_f})]}{\sqrt{Var(f(X_{t_f}^{t,x}, u_{t_f}))}} \\ \text{subject to}: \\ \qquad H(X_s) = 0 \end{cases} \tag{8}$$

and,

$$\begin{cases} \max_{u_{t_f}\in\mathcal{A}} Z(u_{t_f}) = E[f(X_{t_f}^{t,x}, u_{t_f})] - \hat{J}Var(f(X_{t_f}^{t,x}, u_{t_f})), \\ \text{subject to}: \\ \qquad H(X_s) = 0. \end{cases} \tag{9}$$

Let, $X_{t_f}^*$ be an optimal solution of the problem (8). According to Karush-Kuhn-Tucker (KKT) necessary and sufficiency conditions [17], for the optimal solution $X_{t_f}^*$, there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that :

$$\frac{d}{dX}\Big(\frac{E(f(X_{t_f}^*, u_{t_f}^*)}{\sqrt{Var(f(X_{t_f}^*, u_{t_f}^*))}}\Big) + \lambda\frac{d}{dX}H(X_{t_f}^*) = 0. \tag{10}$$

Therefore, we have:

$$\frac{1}{\sqrt{Var(f(X^*_{t_f},u^*_{t_f}))}}\Big[\frac{dE[f(X^*_{t_f},u^*_{t_f})]}{dX}-\frac{1}{2}\frac{dVar(f(X^*_{t_f},u^*_{t_f}))}{dX}\frac{E(f(X^*_{t_f},u^*_{t_f}))}{Var(f(X^*_{t_f},u^*_{t_f}))}\Big]+ \lambda\frac{d}{dX}H(X^*_{t_f})=0. \tag{11}$$

Now, let $a=\frac{1}{\sqrt{Var(f(X^*_{t_f},u^*_{t_f}))}}$ and $b=\frac{1}{2}a\frac{E(f(X^*_{t_f},u^*_{t_f}))}{Var(f(X^*_{t_f},u^*_{t_f}))}$. Thus, we have:

$$a\frac{dE[f(X^*_{t_f},u^*_{t_f})]}{dX}-b\frac{dVar(f(X^*_{t_f},u^*_{t_f}))}{dX}+\lambda\frac{d}{dX}H(X^*_{t_f})=0. \tag{12}$$

By taking $\Lambda=\frac{\lambda}{a}$ and $\hat{J}=\frac{b}{a}$, we obtain:

$$\frac{dE[f(X^*_{t_f},u^*_{t_f})]}{dX}-\hat{J}\frac{dVar(f(X^*_{t_f},u^*_{t_f}))}{dX}+\Lambda\frac{d}{dX}H(X^*_{t_f})=0. \tag{13}$$

Therefore, $X^*_{t_f}$ satisfies the KKT necessary and sufficienct conditions for the equivalent stochastic-fractional optimal control problem(9), this completes proof. □

According to theorem (3.1), the problems (5) and (6) have the same optimal solution. Therefore, by solving the problem(6), the optimal solution of the problem(5) is obtained, as well. For this purpose, consider the value function:

$$v(t,x)=\max_{u\in\mathcal{A}}\Big[E[f(X^{t,x}_{t_f},u_{t_f})]-\hat{J}Var(f(X^{t,x}_{t_f},u_{t_f}))\Big] \tag{14}$$

In the following theorem, the corresponding HJB equation to function (14) is presented.

Theorem 3.2. *Consider the stochastic-fractional optimal control problem(6), the corresponding HJB equation to function (14) is:*

$$\begin{aligned}\frac{\partial v(t,x)}{\partial t}+\max\nolimits_{u\in\mathcal{A}}\Big\{&F(X^{t,x}_s,u_s)+b(x_t,u_t)\frac{\partial v(t,x)}{\partial x}\\ &+\frac{1}{2}tr(\sigma(x_t,u_t)\sigma(x_t,u_t)^{\dagger})\frac{\partial^2 v(t,x)}{\partial x^2}\Big\}=0,\end{aligned} \tag{15}$$

where $F(X^{t,x}_s,u_s)=\frac{df(X^{t,x}_s,u_s)}{ds}-\hat{J}\frac{d(f-E(f))^2(X^{t,x}_s,u_s)}{ds}$.

Proof. The Mayer type stochastic-fractional optimal control problem(6) is converted to a Lagrange one, as follows:
Let,

$$\max_{u\in\mathcal{A}}\Phi(t_f)=E[f(X^{t,x}_{t_f},u_{t_f})]-\hat{J}Var(f(X^{t,x}_{t_f},u_{t_f})),$$

where $\Phi(t) = 0$ for $t < t_f$. One can obtain,

$$\frac{dE[f(X_t^{t,x}, u_t)]}{dt} - \hat{J}\frac{dVar(f(X_{t_f}^{t,x}, u_{t_f}))}{dt} = \frac{d\Phi}{dt}, \tag{16}$$

since $Var(f) = (f - E(f))^2$ so, according to the linearity of $E(.)$, by integrating the second equality in (16) on interval $[0, t_f]$, we have:

$$\Phi(t_f) = E\Big[\int_t^{t_f} F(X_s^{t,x}, u_s)ds\Big]. \tag{17}$$

where,

$$F(X_s^{t,x}, u_s) = \frac{df(X_s^{t,x}, u_s)}{ds} - \hat{J}\frac{d(f - E(f))^2(X_s^{t,x}, u_s)}{ds}. \tag{18}$$

Therefore, by considering the objective function (17), a Lagrange type equivalent stochastic-fractional optimal control problem is obtained whose the corresponding value function is:

$$v(t, x) = \max_{u \in \mathcal{A}} E\Big[\int_t^{t_f} F(X_s^{t,x}, u_s)ds\Big]. \tag{19}$$

Now, suppose $0 \leq t \leq t + h \leq t_f$, for a fixed control $\{u_t\}_{t\geq 0} = u$ we have:

$$v(t, x) \geq E\Big[\int_t^{t+h} F(X_s^{t,x}, u_s)ds\Big] + E\Big[\int_{t+h}^{t_f} F(X_s^{t,x}, u_s)ds\Big], \tag{20}$$

According to the definition of the value function(19), we have:

$$v(t, x) \geq E\Big[\int_t^{t+h} F(X_s^{t,x}, u_s)ds\Big] + v(t + h, X_{t+h}^{t,x}). \tag{21}$$

By applying Ito's formula for function v on $[t, t + h]$, we have:

$$\begin{aligned} v(t + h, X_{t+h}^{t,x}) = v(t, x) + \int_t^{t+h} \frac{\partial v(s, X_s^{t,x})}{\partial s}ds + \int_t^{t+h} \frac{\partial v(s, X_s^{t,x})}{\partial x}dX_s \\ +\frac{1}{2}\int_t^{t+h} \frac{\partial^2 v(s, X_s^{t,x})}{\partial x^2}d\langle X_s\rangle. \end{aligned} \tag{22}$$

where $d\langle X_s\rangle = (dX_s)^2$. According to (1), by substituting dX in (22) and the fact that the stochastic process W is a Standard Brownian Motion so, $d\langle W_t\rangle = (dW_t)^2 = dt$, we have:

$$\begin{aligned} v(t + h, X_{t+h}^{t,x}) \geq v(t, x) + \int_t^{t+h} \frac{\partial v(s, X_s^{t,x})}{\partial s}ds + \int_t^{t+h} \frac{\partial v(s, X_s^{t,x})}{\partial x}b(x, u)ds \\ +\frac{1}{2}\int_t^{t+h} tr(\sigma(x, u)\sigma^\dagger(x, u))\frac{\partial^2 v(s, X_s^{t,x})}{\partial x^2}ds. \end{aligned} \tag{23}$$

According to inequality (23), by substituting value of $v(t + h, X_{t+h}^{t,x})$ in (21), we have:

$$\begin{aligned} 0 \geq E\Big[\int_t^{t+h} F(X_s^{t,x}, u_s)ds\Big] + \int_t^{t+h}\Big(\frac{\partial v(s, X_s^{t,x})}{\partial s} + b(x, u)\frac{\partial v(s, X_s^{t,x})}{\partial x} \\ +\frac{1}{2}tr(\sigma(x, u)\sigma^\dagger(x, u))\frac{\partial^2 v(s, X_s^{t,x})}{\partial x^2}\Big)ds \end{aligned} \tag{24}$$

According to the mean-value theorem, one can infer:

$$0 \geq \lim_{h\to 0} \frac{1}{h} E\Big[\int_t^{t+h} F(X_s^{t,x}, u_s) ds \Big] + \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \Big(\frac{\partial v}{\partial s} + b(x,u) \frac{\partial v}{\partial x} + \frac{1}{2} tr(\sigma(X_s, u_s)\sigma^{\dagger}(x,u)) \frac{\partial^2 v}{\partial x^2} \Big) ds. \tag{25}$$

Since, the inequality (25) holds for any fixed $u \in \mathcal{A}$, for all $(t,x) \in [0,t_f) \times \mathbb{R}^n$,

$$\frac{\partial v(t,x)}{\partial t} + \max_{u\in\mathcal{A}} \Big\{ F(X_s^{t,x}, u_s) + b(x_t, u_t) \frac{\partial v(t,x)}{\partial x} + \frac{1}{2} tr(\sigma(x_t,u_t)\sigma(x_t,u_t)^{\dagger}) \frac{\partial^2 v(t,x)}{\partial x^2} \Big\} = 0. \tag{26}$$

□

This completes proof □.

According to theorem(3.2), the following stochastic control process maximizes the objective function of the equivalent stochastic-fractional optimal control problem(6):

$$u_t^* \in arg\Big(\max_{u\in\mathcal{A}} \Big\{ F(X_s^{t,x}, u_s) + b(x_t, u_t) \frac{\partial v(t,x)}{\partial x} + \frac{1}{2} tr(\sigma(x_t,u_t)\sigma(x_t,u_t)^{\dagger}) \frac{\partial^2 v(t,x)}{\partial x^2} \Big\} \Big). \tag{27}$$

Algorithm of Solving SFOC problems

In the application section of this paper, this algorithm is programmed and implemented with MATLAB software.

Input : Stochastic-fractional optimal control problem(5).
Output : Stochastic optimal pair (X^*, u^*).

Step 1: Convert the stochastic-fractional optimal control problem to the equivalent form(6).
Step 2: Describe HJB equation(15) corresponding to the equivalent SFOC problem
Step 3: Compute stochastic process $\{u_t\}$ by solving the HJB equation.
Step 4: Compute the response of the control process, i.e compute the solution $\mathbb{F}$-adapted process $\{X_t\}$ of the SDE (1).

4 Applications

In this section, two examples, Merton portfolio allocation problem and pairs trading problem are considered for indicating the advantages and applicable capacity of the presented method.

4.1 Sharp Ratio

Sharp Ratio measures return per unit of deviation (risk) in an investment asset or a trading strategy. For better understanding of how this ratio works, its formula is presented as follows [2]

$$S(X) = \frac{E(X)}{StdD(X)} \tag{28}$$

where:

- $E(X)$ is the return of a portfolio.

- $StdD(X)$ is the standard deviation of X.

The formula (28) is compatible with the definition Sharp ratio in page 3, completely. Because, $E(X)$ denotes return and $StdD(X)$ denotes risk. If the value of the Sharpe ratio be maximum then the risk-adjusted return will be minimum.This index expresses the return of investing in a portfolio for a level of market risk(systematic risk). The higher Sharp Ratio, the lower risk of portfolio [2].

4.2 Merton portfolio allocation problem

Suppose that a portfolio is constructed on time interval $[0, T]$ such that for any $t \in [0, T]$ a fraction of wealth $u(t)$ is invested in stock with interest rate μ and fixed volatility σ whereas the remaining fraction $1 - u(t)$ invested in a bond with interest rate r. According to Black-Scholes model, if the dynamics of the rate of return of bond and stock are respectively expressed as follows:

$$\frac{dS_0(t)}{S_0} = rdt, \qquad \frac{dS(t)}{S} = \mu dt + \sigma d\omega_t, \tag{29}$$

where r, μ and σ are positive real numbers and also, ω is Brownian motion then the dynamics of variable of wealth obtained from investing in the portfolio, is expressed as follows:

$$\begin{aligned} \frac{dW_t}{W_t} &= u_t \frac{dS(t)}{S} + (1 - u_t)\frac{dS_0(t)}{S_0} \\ &= u_t(\mu dt + \sigma d\omega_t) + (1 - u_t)rdt \\ &= \Big[u_t(\mu - r) + r\Big]dt + u_t \sigma d\omega_t. \end{aligned} \tag{30}$$

Let, $dX_t = \frac{dW_t}{W_t}$ then X_t is the rate of return of the portfolio.

The objective is that the return of the investing in this portfolio be maximized in a finite horizon $T < \infty$. According to the modern portfolio theory: for a given level of market risk, portfolios can be constructed to maximize expected return. The relationship between risk level and return can be shown by a curve called efficient frontier [1]. Every efficient portfolio corresponds to a point on curve.

Note that the optimal portfolios have maximum return for a given level of market risk. Therefore, in this paper Merton portfolio allocation problem is solved by the proposed method such that the resulting portfolio on the efficient frontier.

According to section 4.1 and assumptions mentioned above, the aim of solving the investing problem in portfolio constructed based on Merton portfolio allocation is to design an optimal strategy such that Sharp Ratio of portfolio maximizes. Therefore, the following stochastic-fractional optimal control must be solved:

$$\begin{aligned} &\max_{u\in\mathcal{A}} J(u_{t_f}) = \frac{E(X_{u_f})}{\sqrt{Var(X_{u_f})}}, \\ &\text{subject to :} \\ &\qquad dX_t = \Big[u_t(\mu - r) + r\Big]dt + u_t\sigma d\omega_t, \\ &\qquad X(0) = X_0, \end{aligned} \tag{31}$$

where r, σ, μ are as (29).

According to algorithm of solving stochastic-fractional optimal control problems and specially theorem(3.1), the problem(31) is converted to a Lagrange type equivalent SFOC problem and then the following value function considered:

$$v(t,x) = \max_{u\in\mathcal{A}} \Big[E(X_{u_f}) - \hat{J}Var(X_{u_f})\Big] \tag{32}$$

According to theorem(3.2), the corresponding HJB equation to the value function(32) is:

$$\begin{aligned} &\frac{\partial v(t,x)}{\partial t} + \max_{u\in\mathcal{A}}\{F(X_s^{t,x}, u_s) + x_t(u_t(\mu - r) + r)\frac{\partial v}{\partial x} \\ &\quad + \frac{1}{2}\sigma^2 u^2 x_t^2 \frac{\partial^2 v(t,x)}{\partial x^2}\} = 0, \end{aligned} \tag{33}$$

where

$$F(X_s^{t,x}, u_s) = \frac{dX}{ds} - \hat{J}\frac{d(X - E(X))^2}{ds}.$$

Now, suppose that (x^*, u^*) is the stochastic optimal solution of the equivalent form of the problem (31). Therefore we have:

$$\begin{cases} \dfrac{\partial v}{\partial t}(t, x^*) = 0, \\ F(X_s^{t,x}, u_s) = 0 \\ r\dfrac{\partial v}{\partial x}(x^*, t) = 0 \\ x_t^*(u_t^*(\mu - r))\dfrac{\partial v}{\partial x} + \dfrac{1}{2}\sigma^2 u^{*2} x_t^{*2} \dfrac{\partial^2 v}{\partial x^2} = 0 \end{cases} \tag{34}$$

From fourth equation in (34):

$$u^* = \frac{-2(\mu - r)}{x_t^* \sigma^2} \frac{\dfrac{\partial v}{\partial x}}{\dfrac{\partial^2 v}{\partial x^2}} \tag{35}$$

where,

$$\frac{\partial v}{\partial x} = \frac{\partial E\Big[\int_t^{t_f} F(X_s^{t,x}, u_s)ds\Big]}{\partial x}, \\ = \frac{\partial E\Big[\int_t^{t_f} \Big(\frac{df(X_s^{t,x}, u_s)}{ds} - \hat{J}\frac{d(f-E(f))^2(X_s^{t,x}, u_s)}{ds}\Big)ds\Big]}{\partial x}, \tag{36}$$

here, $f(X_s^{t,x}, u_s) = X_s^{t,x}$. Therefore, we have:

$$\frac{\partial v}{\partial x} = \frac{\partial E\Big[\int_t^{t_f} \Big(dX_s^{t,x} - \hat{J}d(X_s^{t,x} - E(X))^2\Big)\Big]}{\partial x}, \\ = \frac{\partial E\Big[\Big(X_{t_f}^{t,x} - X_t^{t,x} - \hat{J}(X_{t_f}^{t,x} - E(X))^2 + \hat{J}(X_t^{t,x} - E(X))^2\Big)\Big]}{\partial x}, \\ = \frac{\partial E\Big(-x + \hat{J}(x - E(X))^2\Big)}{\partial x}, \\ = -1 + 2\hat{J}E(x - \bar{X}) \tag{37}$$

where $\bar{X} = E(X)$. Also

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\Big(\frac{\partial v}{\partial x}\Big) = 2\hat{J}. \tag{38}$$

4.3 Optimal pairs trading by using SFOC approach

Now, the optimal pair-trading problem is formulated as a SFOC Problem. According to section 4.1 and dynamics of paired stock prices, spread and wealth value [5], the objective of investing in portfolio constructed based on pair-trading strategy is to design an optimal strategy such that Sharp Ratio of portfolio be maximized. Therefore, the following stochastic-fractional optimal control is to solved:

$$\begin{cases} \max_{u\in\mathcal{A}} J(u_{t_f}) = \dfrac{E(X_{u_f})}{\sqrt{Var(X_{u_f})}} \\ \text{subject to :} \\ \quad dX_t = u(t)\Big(k(\theta - Spr(t)) + \dfrac{1}{2}\lambda^2 + \rho\sigma\lambda + r\Big)dt + \lambda u(t)d\omega(t) \\ \quad dSpr_t = k(\theta - Spr(t))dt + \lambda d\omega(t) \\ \quad X(0) = X_0, \qquad Spr(0) = Spr_0, \end{cases} , \tag{39}$$

where, the first and second constraints describe the wealth and spread dynamics, respectively. The third constraint specifies the initial wealth of the portfolio and the spread.

Solving Optimal Pairs Trading Problem by SFOC Method

Consequently, the optimal pairs trading problem (39) is solved by the proposed stochastic-fractional optimal control method.

According to algorithm of solving stochastic-fractional optimal control problems and specially theorem(3.1), the problem(39) is converted to a Lagrange type equivalent SFOC problem and then according to (19), the following value function considered:

$$v(t,x) = \max_{u\in\mathcal{A}} E\Big[\int_t^{t_f} F(X_s^{t,x}, u_s)ds\Big]. \tag{40}$$

where $F(.,.)$ is as (18). According to theorem(3.2), the corresponding HJB equation to the value function(40) is:

$$\begin{aligned}&\frac{\partial v(t,x)}{\partial t} + \max_{u\in\mathcal{A}}\{F(X_s^{t,x}, u_s)\\ &\quad + x_t u_t\Big(k(\theta - Spr(t)) + \frac{1}{2}\lambda^2 + \rho\sigma\lambda + r\Big)\frac{\partial v}{\partial x} + \frac{1}{2}\lambda^2 u_t^2 x_t^2 \frac{\partial^2 v}{\partial x^2}\} = 0.\end{aligned} \tag{41}$$

in which optimal control Hamiltonian is

$$\begin{aligned}H(u) = &\max_{u\in\mathcal{A}}\{F(X_s^{t,x}, u_s) + x_t u_t\Big(k(\theta - Spr(t)) + \frac{1}{2}\lambda^2 + \rho\sigma\lambda + r\Big)\frac{\partial v}{\partial x}\\ &+ \frac{1}{2}\lambda^2 u_t^2 x_t^2 \frac{\partial^2 v}{\partial x^2}\}.\end{aligned}$$

Since, the optimal pairs-trading strategy u^* satisfies the equation $\frac{\partial H}{\partial u}(u^*) = 0$ therefore, according to the theorem(3.2):

$$u_t^* = \frac{2\Big(k(\theta - Spr(t)) + \frac{1}{2}\lambda^2 + \rho\sigma\lambda + r\Big)}{x_t^*\lambda^2} \frac{\frac{\partial v}{\partial x}}{\frac{\partial^2 v}{\partial x^2}} \tag{42}$$

where $\frac{\partial v}{\partial x}$ and $\frac{\partial^2 v}{\partial x^2}$ are determined by Eqs (37) and (38), respectively.

Simulation Results

In this section, a stochastic-fractional optimal pairs-trading strategy for a given pair stocks price, S_1 and S_2, is obtained. Also, all of the parameters of this simulation are chosen same as [5] such that the results of this paper can be compared with the presented results of method in [5].

According to [5], the price process of stock 2, $S_2(t)$ is simulated by taking a series of 1-day time period. We have assumed 251 trading days in a year. So, stock 2 is simulated where, $\mu = 0.3$ and $\sigma = 0.1$. Moreover, since the price variable of stock 2 and the spread are correlated by ρ, the processes $B = \{B(t)\}$ and $\omega = \{\omega(t)\}$ are set to be two standard normal random processes such that $\rho = 0.19$.

Additionally, according to [5], the stock price of stock 1 can be obtained as:

$$S_1(t) = S_2(t)e^{Spr(t)}. \tag{43}$$

For obtaining the stock price paths for stock 1 and 2, the above simulations are implemented for one trading year(251 days). The results has been shown in the Figure 1.

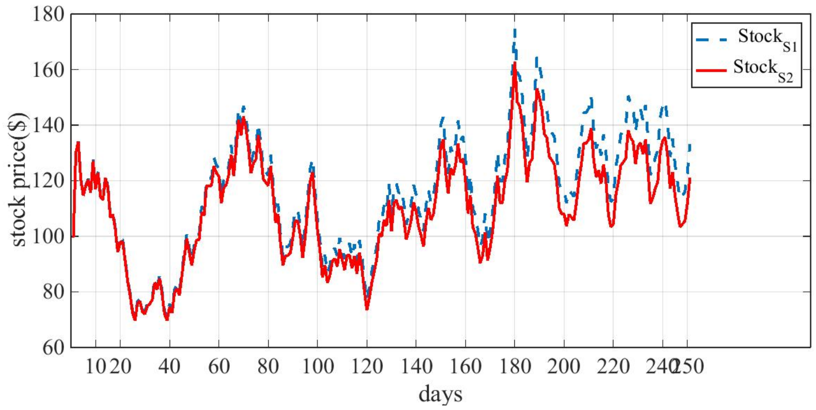

Figure 1: Time series of stock 1 and 2 by SFOC method.

Figure 1 depicts the time series of stock 1 and 2. Therefore, we can identify the path dynamics of the two stocks and adopt it for describing the wealth value dynamics.

Now, according to Equation (42), the optimal pairs-trading strategy can be computed via the above identified data for stock 1, stock 2, the spread between the pair of stocks and the corresponding parameter values. The optimal pairs-trading strategy is simulated in Figure (2).

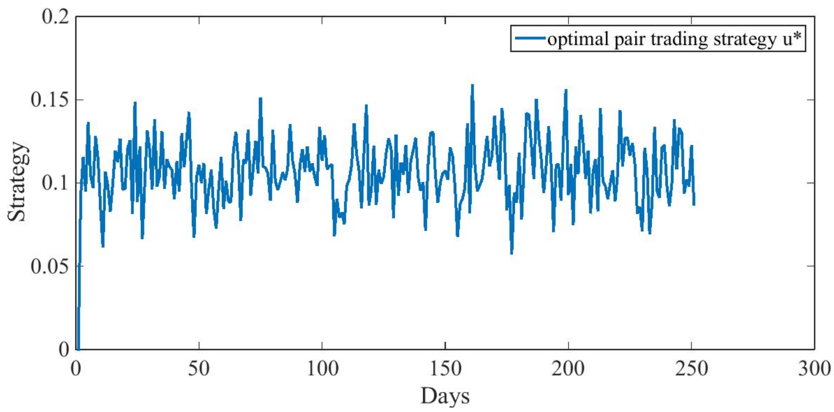

Figure 2: optimal pairs-trading strategy by SFOC method.

Figure.2 shows the obtained stochastic-fractional Optimal control signal. According to the control curve, the proportion of portfolio that is allocated to investing in stock is determined at any time. Since an optimal strategy maximizes Sharp Ratio of a portfolio and consequently, the resulting optimal portfolio is on the efficient frontier curve, therefore by adopting the optimal pairs-trading strategy shown in figure.2 , we can obtain an efficient portfolio.

Now, according to the above simulated data and [5] the wealth values simulation is illustrated in the Figure 3. Figure.3 shows that, the value of the portfolio

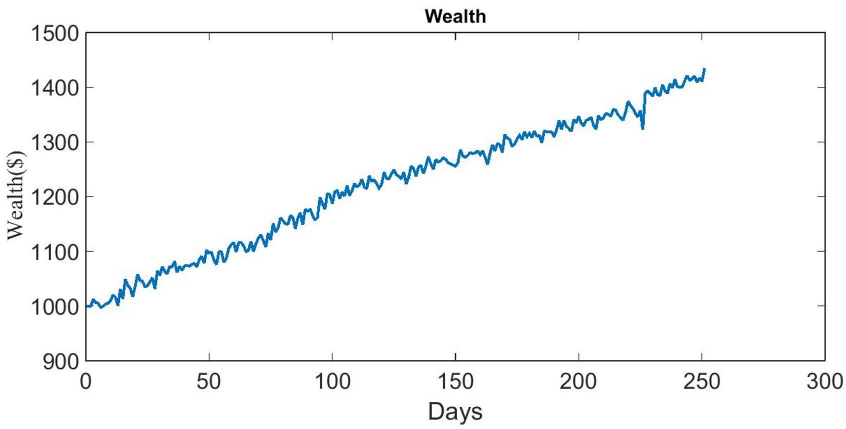

Figure 3: Wealth evolution by SFOC method.

increases from 1000 to 1432 in one year. Indeed, this represents a yearly return of over 42% for the portfolio based on SFOC pairs-trading strategy. Figure.3 shows increasing behaviour of the wealth value of the portfolio at any time. For the given level of market risk, the obtained portfolio have maximum return at any time.

Now, the stochastic control approach [5] is implemented for two stocks 1 and 2 and results are compared with the proposed SFOC method results. For this purpose, the simulation of the stock price paths for stocks 1 and 2 are run in the one year (251 trading days) for corresponding parameters values which are mentioned above. Figure 4 shows the logarithm values of data for stock 1 and 2.
Additionally, the optimal pairs-trading strategy can be computed by using the above identified data. The simulated values of the optimal pairs-trading are shown in Figure 5.
Additionally, according to [5] and pairs-trading strategies, the wealth values are obtained and the results of simulation are shown in Figure 6.
Figure 6 shows that value of the portfolio increases from 1000$ to 1380$ in one year. Figure 6 shows that the wealth value of the portfolio deviates from increasing procedure. Also, since the risk index is not considered therefore the obtained portfolio is not efficient one, necessarily.

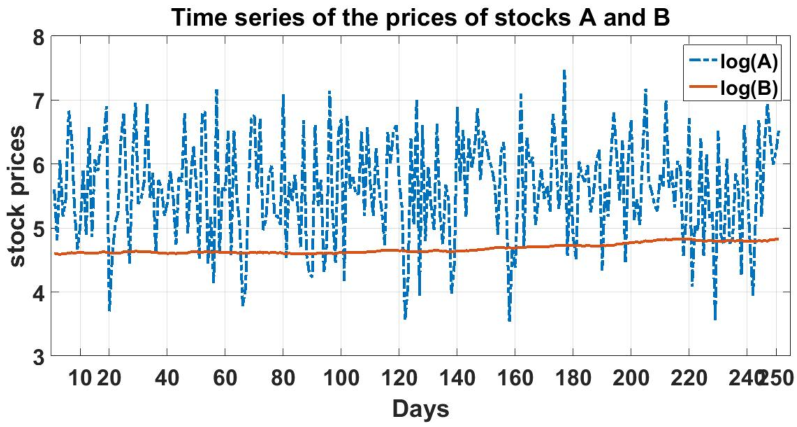

Figure 4: Time series of stock 1 and 2 by [5] .

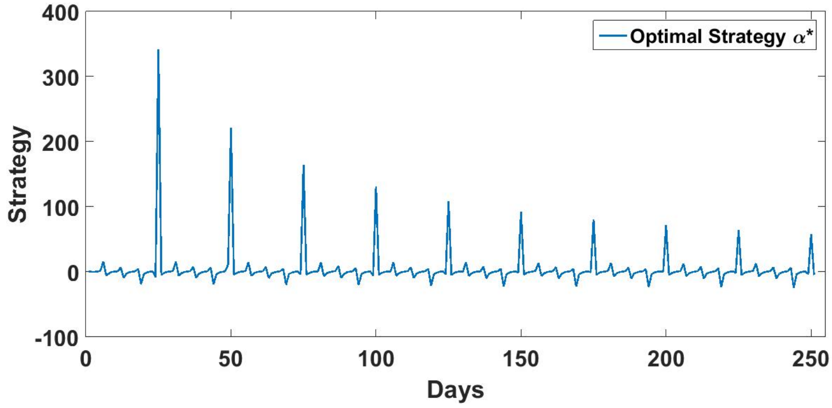

Figure 5: Optimal Stochastic Pairs-Trading by [5].

5 Conclusion

In this paper, the stochastic–fractional optimal control problem was formulated and a new algorithm was proposed for solving it. According to the proposed algorithm, first, the equivalent form of the problem was introduced. It was proved that optimal solutions of the two problems are the same. Therefore, by solving the equivalent stochastic–fractional optimal control problem based on (HJB) equation method, the optimal control was obtained. Indeed, the corresponding value function to the equivalent stochastic–fractional optimal control problem allowed to calculate the optimal control rule based on applying the optimality conditions to the Hamiltonian system. For facilitating the stochastic–fractional optimal control method, two theorems were proved. For highlighting the application of proposed

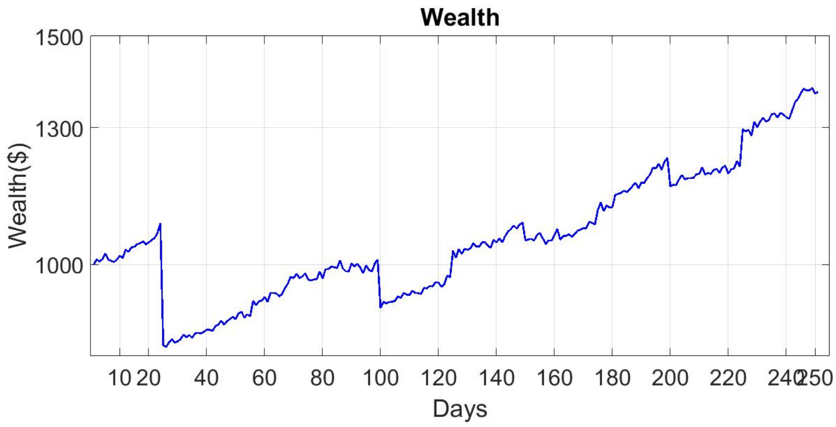

Figure 6: Wealth evolution by [5].

method, by considering Sharpe ratio as performance index, Merton's portfolio selection problem was solved and optimal pairs trading problem was simulated.

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