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Research paper

# A mathematical model for deriving the optimal trajectory of life insurance demand

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**Abstract:**
This study explores the optimal trajectory of life insurance demand, a crucial financial tool for managing mortality risk and ensuring economic security for family. Various factors, including mortality risk, wealth growth, interest rates, and policyholder preferences, influence insurance decisions. To analyze these dynamics, the study develops two mathematical models. The first is a single-period, state-dependent model that maximizes expected utility under budget constraints, concluding that individuals optimally purchase only partial insurance. To obtain the optimal time path of life insurance coverage, a life-cycle model was solved using optimal control theory. By maximizing expected lifetime utility from consumption and bequests within the wealth accumulation process, the second model derives the trajectory of life insurance demand. The results indicate that individuals with higher risk tolerance experience a greater growth rate in life insurance demand. This growth rate is also positively influenced by mortality rates, loading factors, and interest rates. Conversely, life insurance demand declines as wealth increases, supporting the notion that wealth acts as a substitute for life insurance. Additionally, a higher rate of time preference negatively impacts the growth rate of life insurance demand. *Keywords:* Optimal Life Insurance Time-Path. Optimal

Control Theory. Mortality Risk. Risk Tolerance.
*JEL Classification:* C02, C61, D14.

# 1 Introduction

Throughout life, individuals must make key economic decisions regarding income management, investment allocation, and consumption. They choose between risky assets (e.g., stocks) and risk-free assets (e.g., government bonds, retirement benefits, and life insurance) while determining their optimal consumption level. Additionally, they face uncertainties such as fluctuating investment returns and labor income risk, including the possibility of premature death. To mitigate this risk, families opt to purchase life insurance.

Most theoretical studies on life insurance demand consider the breadwinner as the sole decision-maker. In this framework, the wage earners allocate a portion of their earnings to dependents, either during their lifetime or as a bequest, driven

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by altruism, moral obligations, or the value placed on their relationships with dependents. An alternative perspective views life insurance as a safeguard against the financial dependence of family members, ensuring support for dependents after the breadwinner's death. In this view, consumption, investment, and life insurance decisions are made collectively as a family. The economic basis for life insurance demand stems from the concept of family production, where the wage earner specializes in income generation while the other partner focuses on home production. This specialization increases overall family well-being compared to a scenario in which both partners divide their efforts between market work and household responsibilities. However, specialization also creates a financial vulnerability, necessitating compensation in the event of either partner's death. As a result, life insurance is sought as a means of financial protection against this risk.

Regardless of the underlying rationale, life insurance is demanded to manage the uncertainty related to human capital, which can create financial challenges for dependents. Existing literature primarily explores either an individual's or a family's response to the uncertainty of death age as the main driver for life insurance demand. However, this research focuses on the individual's decision-making process regarding the purchase of life insurance.

The objective of this study is to develop two mathematical models to analyze optimal life insurance demand. The first model maximizes expected utility subject to budget constraints, aiming to characterize the key properties of optimal life insurance demand. The second model adopts a dynamic life-cycle framework that maximizes expected lifetime utility from both consumption and bequest, subject to the wealth accumulation process. This model yields optimal trajectories for life insurance demand and consumption over time.

The structure of the research is as follows. Section 2 reviews the relevant literature, classifying theoretical studies on life insurance demand into two categories: individualistic optimal demand and family-based decision-making. The first group examines the wage earner as the sole decision-maker, where their preferences determine how earnings are allocated to beneficiaries via life insurance. The second group treats life insurance demand as a collective family decision.

Section 3 presents a state-dependent model that examines life insurance purchases as a decision-making process driven by utility maximization.

Section 4 introduces the life cycle model, that extends the state-dependent model. In this model, the lifetime utility function, consisting of the integral of utility from the consumption stream and a lump-sum utility from the legacy left at death, is maximized with respect to a non-stochastic wealth accumulation process. Applying optimal control theory solves this problem and derives the optimal time path for life insurance demand and consumption.

Section 5 presents the conclusions and offers suggestions for future research.

# 2 Literature Review

Since Yaari's (1965) seminal work, various theoretical models have been developed to describe the behavior of a risk-averse, rational economic individual facing human capital uncertainty. These models can be broadly classified into two main categories.

The first group of studies examines the policyholder's decision-making, focusing on the household's optimal lifetime portfolio and consumption strategies. Here, the breadwinner's preferences define the objective function, with life insurance purchased to maximize their expected utility. This individualistic perspective views the wage earner as the sole decision-maker, allocating a portion of earnings to life insurance for dependents. Under this approach, the wage earner's bequest motive is the primary driver of life insurance demand.

The second group of studies approaches the problem from a family decision-making perspective, emphasizing that the family, as a collective utility-maximizing entity, is the relevant unit of analysis. In these models, utility is measured at the household level rather than the individual level. As a result, the individual's bequest motive may not be the primary driver for purchasing life insurance. Instead, life insurance is viewed as a means to optimize the expected lifetime utility of the beneficiaries.

## 2.1 Individualistic Optimal Life Insurance Demand

The problem of uncertain lifetime and life insurance in its individualistic form began with Yaari (1965). He has studied the problem of uncertain lifetime and life insurance demand in the context of the expected utility hypothesis using a continuous time model.

Chuma (1997) attempts to find a feasible method for measuring an individual's intended bequest motives and their effects on savings and life insurance purchases. For this purpose, a specific Yaarian life-cycle model is constructed that determines how an individual judges the demand for life insurance. The main distinction of this model with Yaari's model is the explicit appearance of life insurance demand in the model.

Pliska and Ye (2007) study optimal insurance and consumption strategies for a wage earner with a random lifetime. The wage earner has initial wealth and a continuous income stream, which may terminate upon premature death. Applying dynamic programming, they derive optimal insurance and consumption rules, providing explicit solutions for the Constant Reletive Risk Aversion (CRRA) utility functions.

Duarte, Pinheiro, Pinto, and Pliska (2012) extend Merton's continuous-time model of optimal consumption and investment by introducing a wage earner with a random lifetime. The wage earner allocates part of their income to life insurance and invests the rest in a market with one risk-free asset and multiple risky assets driven by multi-dimensional Brownian motion. They analyze optimal strategies for consumption, investment, and insurance, aiming to maximize expected utility from family consumption, estate size in case of premature death, and estate size at retirement. Using dynamic programming, they provide explicit solutions for discounted CRRA utility functions.

Hana and Hung (2017) derive the optimal life insurance, consumption, and portfolio decisions for a wage earner before retirement, considering risks from interest rates and inflation. The wage earner's preferences are modeled using stochastic differential utility and the demand for life insurance is influenced by fluctuations in interest rates and inflation. Specifically, the optimal life insurance demand decreases as nominal interest rates rise. Under the assumption of deterministic nominal income, inflation has no effect on life insurance demand. Additionally, when investment opportunities have greater volatility, wage earners who optimally allocate their wealth to the financial market tend to benefit more from these investments, reducing their demand for life insurance. An analysis of risk aversion reveals that life insurance demand over the planning horizon increases with higher relative risk aversion.

Guambe and Kufakunesu (2018) solve the optimal investment, consumption, and life insurance problem for an investor subject to a capital guarantee constraint. They model the market as incomplete, using a jump-diffusion process with stochastic volatility. Applying the martingale approach, they prove the existence of the optimal strategy and the optimal martingale measure, and derive explicit solutions for power utility functions.

Mousa, Pinheiro, Pinheiro and Pinto (2022) examine optimal consumption, investment, and life insurance strategies for a wage earner with an uncertain lifetime in a financial market consisting of a risk-free asset and a risky asset. The life insurance market is modeled with a fixed number of providers offering distinct contracts. Using dynamic programming, they derive solutions for a general class of utility functions, with particular emphasis on discounted CRRA utilities.

Ferreira, Pinheiro, and Pinheiro (2023) analyze the problem faced by a wage earner with an uncertain lifetime, who has access to a BlackScholes-type financial market consisting of one risk-free security and one risky asset. The wage earner's preferences regarding consumption, investment, and life insurance are modeled using robust expected utility. They reformulate this problem as a two-player zero-sum stochastic differential game and derive the optimal strategies for the wage earner, considering a general class of utility functions. They focus particularly on the case of discounted CRRA utility functions.

Maggistro, Marino, and Martire (2024) analyze a multi-agent portfolio optimization model with life insurance for two players with random lifetimes using a dynamic game model. Each player, acting as a price taker, invests to maximize utility from consumption and bequest in a complete market with assets, where $(n-1)$ risky assets following Geometric Brownian motion and one risk-free asset. They explore both non-cooperative and cooperative scenarios and, assuming CRRA utility functions, derive closed-form solutions for optimal consumption, investment, and life insurance.

## 2.2 Life Insurance Demand as a Family Decision

The second group of theoretical research on life insurance demand focuses on household preferences as the primary factor influencing policy decisions. In this approach, life insurance becomes a family decision, often involving the spouse or offspring. The objective function in these models does not solely reflect the behavior of the household head but also includes the preferences of other family members. This perspective suggests that an individual's bequest motive is not necessarily the basis for life insurance demand.

Fitzgerald (1989) delivers a model of life insurance demand by married couples. He derives the demand function for term insurance for a two-earner family. The two earners face a joint mortality distribution with their non-survival assumed independent. They face four possible states over the period: both may live through the period, one may live while the other one dies, or both may die. In order to find the explicit demand functions, the commonly used CRRA utility functions are maximized and explicit annuitized consumption streams for wife and husband are found. The paper didn't find the life insurance demand functions and their determinants explicitly.

Lewis' treatment (1989) differs from prior studies by setting the households' goal as maximization of the beneficiaries' expected lifetime utility. In Lewis' analysis, life insurance is demanded to satisfy the needs of heirs. His model doesn't include bequest motive, although the preferences of survivors are the basis for life insurance demand of households. So, the demand of life insurance is viewed from the beneficiaries' perspective.

Kwak, Shin, and Choi (2011) examine the optimal portfolio and consumption decisions of a family, including life insurance for parents who receive deterministic labor income up to a fixed time, $T$. The study separately models the utility functions of parents and children, assuming that parents' lifetimes are uncertain. If the parents die before time $T$, the children, with no labor income, must optimize their consumption and portfolio choices based on the remaining wealth and life insurance benefits. The family's objective is to maximize the weighted average of the parents' and children's utilities. The authors derive analytical solutions for both the value function and the optimal policies.

Oluwaseyitan, Hashim, Yusof and Kok (2023) highlight the importance of values in family decision-making, particularly regarding investment and life insurance demand between husbands and wives. The study draws on the Theory of Basic Human Values to explore how personal values influence decision-making power within the family. Using this framework, a multiple-category typology of family types is proposed: Traditional Valued Couple, Power Valued Couple, and Security Valued

Couple. The research offers valuable insights for family purchase decisions, particularly in the areas of segmentation, targeting, advertising, and personal selling. It also highlights how deeply held values shape decision-making between husbands and wives. The study adds to marketing literature by exploring how families make decisions based on the role of values, an area that has not been studied much.

In the next section, we develop a state-dependent model that departs from the existing literature by maximizing standard expected utility-including utility from both consumption and bequest-subject to a budget constraint. This model contributes to the literature by demonstrating the existence of partial life insurance demand at the optimal point.

# 3 State-Dependent Optimal Life Insurance Demand

Consumer behavior is analyzed within a single-period model for simplicity. Given that we are considering an economically rational individual whose behavior remains consistent across all life periods, the key results can be extended to multiple periods or a continuous-time model. Suppose the individual purchases a single-period term insurance policy with insurance coverage $I(t)$ and premium $P(t)$. The total premium charged by the firm for providing coverage $I(t)$ is determined by:

$$P(t) = \phi(t)(1+\ell)I(t) \tag{1}$$

Here, $\phi(t)$ represents the probability of death during the period, and $\ell$ denotes the loading factor. Due to the law of large numbers, insurers can obtain an unbiased estimation of the probability of loss. Consequently, the loading factor is incorporated into the premium to cover administrative costs and contingencies.

The policyholder pays $P(t)$ to the insurer to secure coverage $I(t)$ for their family as financial protection in the event of death. After purchasing the policy, two possible outcomes exist:

(i) If mortality occurs, the family receives the indemnity with probability $\phi(t)$, and the total family assets will be:

$$A^D = W(t) + I(t) \tag{2}$$

(ii) In the case of survival, with probability $(1-\phi(t))$, the family will not receive any indemnity, and the total family assets will be:

$$A^L = W(t) + Y(t) - \phi(t)(1+\ell)I(t) \tag{3}$$

Here, $W(t)$ and $Y(t)$ represent wealth and labor income during the period, respectively. Therefore, the expected utility for the family can be expressed as:

$$E(U) = (1-\phi(t))U(A^L) + \phi(t)B(A^D) \tag{4}$$

The state-dependent family preference is modeled using:

- a utility function, $U(\cdot)$ when the household head survives, and
- A bequest function $B(\cdot)$ in the event of the household head's death.

Since individuals are assumed to be risk-averse, the first derivatives of $U(\cdot)$ and $B(\cdot)$ are positive, while the second derivatives are negative. Furthermore, both functions are continuous and satisfy twice-differentiability conditions.

If we set the $E(U)$ equal to a constant value, it will represent an indifference curve. The slope of the indifference curve is determined as follows:

$$\frac{dA^D}{dA^L} = -\frac{(1-\phi(t))U_L'}{\phi(t)B_D'} < 0 \tag{5}$$

The consumer allocates a fraction of $A^L$ as a premium to ensure coverage for their heirs in the event of death. In other words, they forgo this amount to provide financial protection for their family. They pay $P(t) = [\phi(t)(1+\ell)I(t)]$ to the insurer, and their survivors will receive $I(t)$ if mortality occurs. Thus, the slope of their budget line is:

$$\frac{dA^D}{dA^L} = \frac{[A_0^D + I(t)] - A_0^D}{[A_0^L - \phi(t)(1+\ell)I(t)] - A_0^L} = -\frac{1}{\phi(t)(1+\ell)} \tag{6}$$

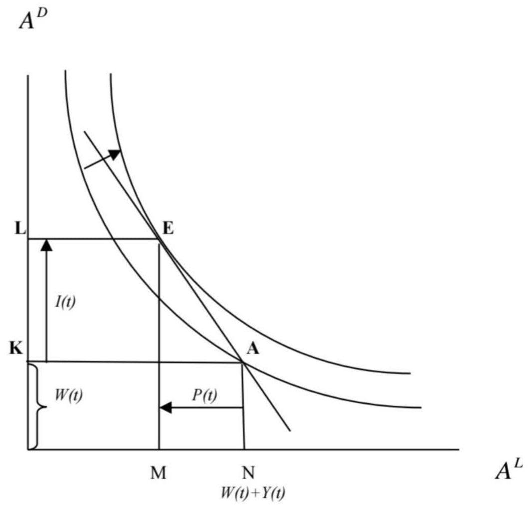


Figure 1: Individuals maximize their utility by paying a premium $P(t)$ at point $A(A_0^L, A_0^D)$ and receiving an indemnity $I(t)$ at point $E(A_0^L - P(t), A_0^D + I(t))$.

His budget constraint can be expressed as:

$$A^D = -\frac{1}{\phi(t)(1+\ell)I(t)}A^L + \alpha_0, \qquad \alpha_0 \textit{ is the } Y - \textit{intercept} \tag{7}$$

The individual's problem is to maximize the objective function (4) subject to the budget constraint (7).

$$\max \bar{U} = (1-\phi(t))U(A^L) + \phi(t)B(A^D) \tag{8}$$

Subject to:

$$A^D = -\frac{1}{\phi(t)(1+\ell)}A^L + \alpha_0$$

We define the Lagrangian function as follows:

$$\mathcal{L} = (1-\phi(t))U(A^L) + \phi(t)B(A^D) - \lambda\left[A^D + \frac{1}{\phi(t)(1+\ell)}A^L - \alpha_0\right]$$

The first-order conditions are given by:

$$\frac{\partial \mathcal{L}}{\partial A^D} = \phi(t)B'_D - \lambda = 0 \tag{9}$$

$$\frac{\partial \mathcal{L}}{\partial A^L} = (1-\phi(t))U'_L - \lambda\frac{1}{\phi(t)(1+\ell)} = 0 \tag{10}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = A^D + \frac{1}{\phi(t)(1+\ell)}A^L - \alpha_0 = 0 \tag{11}$$

From the first-order conditions (9) and (10), we obtain:

$$\frac{(1-\phi(t))U'_L}{\phi(t)B'_D} = \frac{1}{\phi(t)(1+\ell)} \tag{12}$$

Equation (12) represents the optimality condition for the demand for term life insurance. Specifically, the point of tangency between the indifference curve and the budget line determines the optimal choice. This optimality condition can be rewritten as:

$$(1-\phi(t))U'_L = \frac{1}{1+\ell}B'_D. \tag{13}$$

Since $(1-\phi(t))$ represents the probability of survival and $\frac{1}{1+\ell} < 1$, the optimality condition for life insurance demand states that life insurance is purchased up to the point where the marginal utility of family asset, weighted by the probability of survival, equals a fraction of the marginal utility of bequest.

When the probability of death $\phi(t)$ is low, as is typically the case for individuals eligible to purchase life insurance, the optimality condition can be approximated as:

$$U'_L = \frac{1}{1+\ell}B'_D \tag{14}$$

From (14), we conclude that:

$$U'_L < B'_D.$$

If the utility functions $U(\cdot)$ and $B(\cdot)$ have the same functional form, then:

$$A^{*D} < A^{*L}.$$

This implies that individuals typically purchase less than full insurance coverage. According to this model, people buy life insurance to partially compensate for the loss of human capital. This phenomenon occurs due to the presence of a loading factor in the insurance market, which appears in the optimality condition and makes full insurance coverage suboptimal.

# 4 Life-Cycle Model of Life Insurance Demand

An individual with an uncertain lifetime derives utility from both consumption streams and the legacy left to survivors. They maximize expected lifetime utility by optimally choosing consumption $C(t)$ and insurance indemnity $I(t)$ to manage uncertainty regarding the age of death, subject to the wealth accumulation constraint.
The utility function consists of two components:

(i) The integral of lifetime utility derived from consumption,

(ii) A lump-sum utility from the legacy left at death.

The legacy comprises both the remaining wealth at death and the life insurance payout to beneficiaries. Thus, the problem is to maximize:

$$E\left\{\int_0^T U[C(t)]e^{-\rho t}dt + B[W(T) + I(T)]e^{-\rho T}\right\} \tag{15}$$

where:
$T$ represents the individual's lifetime.
$E$ is the expectation operator.
$\rho$ denotes the rate of time preference.
$U(\cdot)$ and $B(\cdot)$ represent the utility from consumption and bequest, respectively
Since the individual is assumed to be risk-averse, the marginal utility of both consumption and legacy must be positive but diminishing, i.e.,
$U' > 0, U'' < 0$ and $B' > 0, B'' < 0$.

The term $U[C(t)]$ is multiplied by $e^{-\rho t}$ because, given an uncertain lifetime and the risk of death, present consumption is valued more than future consumption. Similarly, the utility from bequests is also subject to discounting, as individuals tend to be myopic, prioritizing current consumption over future consumption, even though the bequest is left as a lump sum. Thus, the bequest utility is discounted by $e^{-\rho T}$.

Suppose the time of death in an uncertain lifetime follows a stochastic process with a probability density function $\phi(t)$, representing the likelihood of death at time $t$. Let $\eta(t)$ denote the probability of survival to age $t$. Then, the relationship must hold as:

$$-d\eta(t) = \phi(t)dt,$$

or equivalently

$$\eta(t) = \int_t^T \phi(\tau)d\tau. \tag{16}$$

The expected utility function (15) can be written as:

$$\int_0^T \int_\tau^T \phi(\tau)U[C(t)]e^{-\rho t}dtd\tau + \int_0^T \phi(t)B[W(t) + I(t)]e^{-\rho t}dt. \tag{17}$$

According to definition (16), the expected utility function can be further simplified as:

$$\int_0^T \{U[C(t)]\eta(t) + \phi(t)B[W(t) + I(t)]\} e^{-\rho t}dt. \tag{18}$$

The non-stochastic wealth accumulation process is defined as:

$$W(t) = W_0 + \int_0^T Y(t)dt + \int_0^T rW(t)dt - \int_0^T C(t)dt - \int_0^T P(t)dt. \tag{19}$$

In this equation:
$r$ represents the instantaneous risk-free rate of return.
$Y(t)$ denotes income.
$P(t)$ represents the insurance premium.
By differentiating equation (19) with respect to time, we derive the wealth accumulation process:

$$dW(t) = Y(t)dt + rW(t)dt - C(t)dt - P(t)dt. \tag{20}$$

Dividing both sides of (20) by "$dt$" results Equation of Motion of Wealth:

$$\dot{W} = Y(t) + rW(t) - C(t) - P(t), \tag{21}$$

where a dot above represents the derivative with respect to time.

As in the single-period model, the total premium charged by the insurance company for insuring is determined by:

$$P(t) = \phi(t)(1 + \ell)I(t).$$

Thus, substituting this into (21), the Equation of Motion of Wealth can be rewritten as:

$$\dot{W} = Y(t) + rW(t) - C(t) - \phi(t)(1 + \ell)I(t) \tag{22}$$

## Optimal Control Problem

Formally, we aim to solve the following optimal control problem:

$$\max_{C(t),I(t)} \int_0^T \{U[C(t)]\eta(t) + \phi(t)B[W(t) + I(t)]\} e^{-\rho t} dt \tag{23}$$

subject to:

$$\dot{W} = Y(t) + rW(t) - C(t) - \phi(t)(1+\ell)I(t)$$

We apply the current-value Hamiltonian to derive the optimal paths of the control variables $C(t)$ and $I(t)$ (Chiang, 1999). The current-value Hamiltonian is given by:

$$\begin{aligned} H_{cv} =& \eta(t)U[C(t)] + \phi(t)B[W(t) + I(t)] \\ &+ \lambda(t)[Y(t) + rW(t) - C(t) - \phi(t)(1+\ell)I(t)] \end{aligned}$$

## First-Order Conditions

Taking the partial derivatives of $H_{cv}$ with respect to the control variables and the state variable, we obtain:

$$\frac{\partial H_{cv}}{\partial C(t)} = \eta(t)U'[C(t)] - \lambda(t) = 0. \tag{24}$$

$$\frac{\partial H_{cv}}{\partial I(t)} = \phi(t)B'[W(t) + I(t)] - \lambda(t)\phi(t)(1+\ell) = 0. \tag{25}$$

$$\dot{\lambda}(t) = -\frac{\partial H_{cv}}{\partial W} + \rho\lambda(t) = -\phi(t)B'[W(t) + I(t)] - \lambda(t)r + \rho\lambda(t). \tag{26}$$

$$\dot{W}(t) = \frac{\partial H_{cv}}{\partial \lambda(t)} = Y(t) + rW(t) - C(t) - \phi(t)(1+\ell)I(t). \tag{27}$$

## Optimality Condition

From equations (24) and (25), we derive:

$$\eta(t)U'[C(t)] = \frac{1}{1+\ell}B'[W(t) + I(t)] \tag{28}$$

Equation (28) represents the marginal utility condition for the optimization problem. This optimality condition corresponds with the condition obtained for the state-dependent model (13). It states that, at the optimal point, the marginal utility of consumption, weighted by the probability of survival, equals a fraction of the marginal utility of bequest, as $\dfrac{1}{1+\ell}$ is less than unity.

When the probability of survival is close to one, the marginal utility of consumption becomes smaller than the marginal utility of bequest. In such cases, if the utility and bequest functions follow the same functional form, the family's consumption

will exceed the amount allocated for legacy in terms of both wealth and life insurance. This result supports the state-dependent model's conclusion, which suggests that partial insurance is typically obtained. Consequently, both the loading charge and the probability of survival play crucial roles in determining the optimal amount of legacy left for heirs.
From equation (24), we have:

$$\eta(t)U'[C(t)] = \lambda(t).$$

Differentiating both sides with respect to time, we obtain

$$\dot{\eta}(t)U'[C(t)] + \eta(t)U''[C(t)]\dot{C} = \dot{\lambda}(t) \tag{29}$$

Substituting the results of equations (24), (26), and (28) into (29), and applying appropriate mathematical manipulations, we derive the optimal consumption path as:

$$\dot{C} = -\frac{U'(\cdot)}{U''(\cdot)}\left[\frac{\dot{\eta}(t)}{\eta(t)} + (1+\ell)\phi(t) + r - \rho\right] \tag{30}$$

For a risk-averse individual, we have $U' > 0$, and $U'' < 0$. Additionally, it is evident that $\ell > 0, \phi(t) > 0$, and $\eta(t) > 0$, while the probability of survival decreases over time, i.e., $\dot{\eta}(t) < 0$.

The inverse measure of absolute risk aversion index (ARAI), given by $\frac{-U'(\cdot)}{U''(\cdot)}$ represents an investor's willingness to take risks, commonly referred to as Risk Tolerance.

The derived optimal consumption path (30) shows that consumption over time is positively influenced by risk tolerance $\frac{-U'(\cdot)}{U''(\cdot)}$, the dynamics of the survival rate $\frac{\dot{\eta}(t)}{\eta(t)}$, the life insurance loading factor $\ell$, the probability of death $\phi(t)$, and the interest rate $r$. Conversely, as expected, the path is negatively affected by the rate of time preference $\rho$.

From equation (25), the co-state variable $\lambda$ is given by:

$$\lambda = \frac{1}{1+\ell}B'[W(t) + I(t)] \tag{31}$$

Differentiating both sides of Equation (31) with respect to time yields:

$$\dot{\lambda} = \frac{1}{1+\ell}B''[W(t) + I(t)](\dot{W} + \dot{I}) \tag{32}$$

By substituting the co-state variable from Equation (25) into Equation (26), equating (26) and (32), and solving for the time path of life insurance demand, the optimal life insurance coverage path is derived as:

$$\dot{I}(t) = -\frac{B'(\cdot)}{B''(\cdot)}[\phi(t)(1+\ell) + r - \rho] - \dot{W} \tag{33}$$

The bracketed term in the optimal life insurance coverage path is positive:

$$[\phi(t)(1+\ell)+r-\rho]>0.$$

Since $\phi(t)>0, \ell>0$, and the condition $r>\rho$ holds, this ensures positivity. Otherwise, if $r<\rho$, consumption would be preferred over investment, as the interest rate ($r$) would be lower than the rate of time preference ($\rho$).
In other words, the risk-free interest rate is typically higher than the rate of time preference because any rational investor considers not only their time preference but also the expected profit rate, additional costs, and potential risks when making investment decisions. If the interest rate were lower than the rate of time preference, no investment would take place.
Therefore, it is evident that the term $[\phi(t)(1+\ell)+r-\rho]$ will always be unambiguously positive under standard economic conditions.
For a risk-averse individual, $B'>0, B''<0$, it follows that

$$-\frac{B'(\cdot)}{B''(\cdot)}>0.$$

The inverse of the Absolute Risk Aversion Index (ARAI), given by $-B'/B''$, is positively related to the growth rate of life insurance demand. As mentioned earlier, this inverse measure represents an individual's risk tolerance-a key determinant in financial decision-making.
This finding clearly indicates that the more risk-tolerant consumers are, the higher the growth rate of life insurance demand will be.
The time path of life insurance coverage is clearly influenced by factors such as risk tolerance, mortality rate, loading factor, interest rate, rate of time preference, and the growth rate of wealth. As indicated in Equation (33), the growth rate of life insurance demand is positively correlated with risk tolerance, represented by $-B'/B''$, and negatively correlated with the rate of time preference and the growth rate of wealth ($\dot{W}$). The negative relationship between the rate of time preference ($\rho$) and life insurance demand is clear: an increase in the rate of time preference ($\rho$) leads to higher current consumption and lower demand for life insurance. In other words, as $\rho$ rises, individuals prioritize consumption over purchasing life insurance.
Additionally, the negative correlation between the growth rate of life insurance demand and the growth rate of wealth is justified by the fact that life insurance and wealth can act as substitutes within an individual's portfolio. As wealth increases, individuals may self-insure, reducing the need for life insurance coverage.
The optimal life insurance coverage path (33) clearly shows that the growth rate of life insurance demand is positively correlated with the mortality rate, loading factor, and interest rate.
These relationships suggest that higher mortality risk, higher insurance costs, and higher returns on investment all incentivize greater life insurance demand.

# 5 Conclusions

This study develops two life insurance models to analyze optimal insurance purchasing behavior.

The first model is a single-period, state-dependent model, which demonstrates that, at the optimal point, the discounted marginal utility of bequest equals the marginal utility of consumption. The model concludes that, due to the presence of a loading factor, individuals optimally purchase only partial insurance. This finding supports the hypothesis of optimal under-insurance, suggesting that individuals generally acquire less than full insurance coverage.

According to this model, life insurance serves as a mechanism to partially offset the financial loss associated with human capital uncertainty. However, in cases of extreme altruism, individuals would optimally choose full insurance coverage, as they prioritize securing their dependents' financial well-being.

The second model is a life-cycle model of life insurance demand. By optimizing expected lifetime utility from consumption and bequest, the time path of life insurance coverage is derived. This optimal trajectory of life insurance demand is influenced by key factors such as:

- Risk tolerance
- Mortality rate
- Loading factor
- Interest rate
- Rate of time preference
- Growth rate of wealth

The model concludes that risk tolerance positively influences the growth rate of life insurance demand, meaning that more risk tolerant consumers tend to have a higher growth rate for life insurance demand.

The optimal time path of life insurance coverage also highlights the following key relationships:

- Positive correlation with mortality rate: This conclusion supports the theory of adverse selection, as individuals facing higher mortality risks are more likely to purchase life insurance.
- Positive relationship with the loading factor: Higher insurance costs encourage individuals to secure coverage earlier to hedge against uncertainty.
- Positive relationship with interest rate: Higher returns on insurance-linked investments make life insurance more attractive.

- Inverse relationship with time preference rate: Individuals with higher time preference rates prioritize current consumption over life insurance, reducing their demand for coverage.

- Negative correlation with wealth growth: Supports the hypothesis that wealth acts as a substitute for life insurance, reducing the need for coverage as financial security increases.

These findings provide valuable insights into consumer behavior in the life insurance market, highlighting how economic and psychological factors shape optimal insurance purchasing decisions.

This research can be extended by incorporating stochastic mortality or a stochastic wealth accumulation process. Another suggestion for future research is to quantify the analysis through simulations and assign numerical values to the model parameters.

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