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On the Existence and Uniqueness of Solutions to Rough Fractional Stochastic Differential Equations

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Abstract:
Precise modeling of financial asset volatility is significant for robust risk management and derivative pricing. Recent scholarly investigations have demonstrated a significant interest in employing stochastic processes with short-term memory for this purpose. Consequently, rigorous examination of the existence and uniqueness of solutions for these processes assumes critical importance. This study commences with the precise definition of a fractional operator for $H \in (0, \frac{1}{2})$. Subsequently, the finiteness of the second-order moment of the Itô-Skorokhod integral is meticulously investigated, utilizing the aforementioned operator, specifically within the range of $H \in (0, \frac{1}{2})$. Ultimately, leveraging this moment and rigorously applying Lipschitz and linear growth conditions, and through the application of Gronwall's inequality, the existence and uniqueness of solutions for stochastic differential equations with short-term memory are definitively established.

Keywords: Fractional Brownian motion, Wiener integral, Stochastic differential equations, Lipschitzian condition,Gronwall's inequality.
Classification: 60G22, 91B70.

1 Introduction

The study of stochastic differential equations (SDEs) driven by fractional Brownian motion (FBM) has become a prominent area of research in stochastic analysis. This interest stems from FBM's capacity to model phenomena exhibiting long-range and short-range dependence, a characteristic absent in standard Brownian motion. While early SDE research predominantly focused on Brownian motion as the driving process, leading to a well-developed theory (e.g. [17]; [20]), the non-Markovian nature of FBM necessitates specialized techniques due to the inherent dependence between its increments.

A foundational element in this field is the characterization of the regularity properties of stochastic integrals driven by FBM. Yan et al. [22] contributed significantly

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by investigating the p-variation of integral functionals, providing crucial insights into the behavior of solutions to FBM-driven SDEs. A key challenge arises from the fact that the classical Itô's formula, a cornerstone of stochastic calculus, is not directly applicable to FBM (except in the trivial case where the Hurst parameter $H = \frac{1}{2}$, where FBM reduces to standard Brownian motion). This limitation spurred the development of novel stochastic calculus tools. Hu et al. [11] addressed the intricacies of singular SDEs driven by FBM, likely contributing to the development of specialized solution methodologies for these equations, characterized by mathematical irregularities requiring careful treatment.

The scope of SDE research has broadened to encompass more complex classes, including backward stochastic differential equations (BSDEs). Hu and Peng [12] explored BSDEs driven by FBM, confronting the added complexity of "backward" evolution in time within the context of long-range dependence. This work holds particular relevance for applications in areas such as financial modeling, where past information significantly influences present behavior. Nualart [19] provides a comprehensive and influential treatise on FBM, offering a detailed exposition of the process, developing the necessary stochastic calculus (including various integration theories), and thoroughly investigating SDEs driven by FBM. The Hurst parameter (H) plays a critical role in determining the properties of FBM. Deya and Tindel [6] focused their research on the case where $H > \frac{1}{2}$. This condition often simplifies the analysis, as FBM exhibits "less roughness" in this regime. Their work likely provides refined existence and uniqueness results tailored to this specific range of H values, leveraging the smoother nature of the driving process.

Further generalizations have considered the inclusion of "neutral" terms in SDEs. Boufoussi and Hajji [3] studied neutral stochastic functional differential equations (NSFDEs) driven by FBM in a Hilbert space. NSFDEs incorporate memory effects, making them suitable for modeling systems where the past evolution of the system influences its present state. The use of a Hilbert space framework in this work indicates a focus on infinite-dimensional systems, where solutions are often functions.

Beyond the consideration of pure randomness, the literature has also explored the impact of other forms of uncertainty. Fei [9] examined fuzzy SDEs driven by local martingales under a non-Lipschitzian condition. SDEs incorporate fuzzy sets, allowing for the modeling of systems where parameters are not known precisely but are instead represented by fuzzy sets. The relaxation of the Lipschitz condition in this work is also noteworthy, as it broadens the class of equations that can be analyzed.

More recently, research has shifted towards fractional stochastic differential equations which generalize SDEs by employing fractional derivatives. Ahmadova and Mahmudov [1] studied certain fractional stochastic neutral differential equations, introducing the additional complexity of considering derivatives of the unknown function at both present and past times. Their work contributes to the broader understanding of fractional stochastic differential equations by considering the "neutral" aspect, which involves derivatives of the unknown function at both present and past times. This added complexity makes the analysis more challenging and relevant for systems where the rate of change depends not only on the current and past states but also on the history of the derivative itself. Their focus on the existence and uniqueness provides a rigorous foundation for the applicability of these equations in modeling real-world phenomena. Complementing this, Zhang et al. [23] focused on the existence and uniqueness analysis of solutions for fractional stochastic differential equations where the fractional order q is greater than 1. This work specifically addresses a more specialized area within fractional calculus, where the order of the derivative exceeds one. This is particularly relevant for modeling processes exhibiting super-diffusion or other anomalous behaviors. The inclusion of finite delays further acknowledges the importance of memory effects in these systems. By focusing on the case where q is greater than 1, the authors contribute to a deeper understanding of a specific class of fractional stochastic differential equations with significant applications. The detailed analysis of existence and uniqueness provides essential theoretical guarantees for the validity and interpretability of solutions obtained from these models.

For the attainment of efficient risk management and the precise valuation of derivatives instruments, accurate modeling of financial market volatility is of paramount significance. Recent research endeavors have increasingly demonstrated the utilization of stochastic processes with finite memory for this purpose (see [10]; [4]; [13]; [14]; [15]; [16] for more information). Consequently, the rigorous and meticulous examination of the existence and uniqueness of solutions derived from these processes is deemed essential. In this paper, a fractional operator is initially defined within the range of $H \in (0, \frac{1}{2})$. Subsequently, employing this operator, the boundedness of the second-order moment of the Wick-Itô-Skorohod (WIS) integral is investigated within the same range. Finally, relying on this moment and through the precise application of Lipschitz and linear growth conditions, as well as the utilization of Gronwall's inequality, the existence and uniqueness of solutions for stochastic differential equations with short-term memory are definitively established. These theoretical findings provide a foundational framework for the development and application of more sophisticated and reliable financial models.

The subsequent structure of this paper is delineated as follows: Section 2 is devoted to the presentation of the RiemannLiouville fractional derivative. In Section 3, the proposed fractional operator for values within the range of $H \in (0, \frac{1}{2})$ is introduced. Section 4 undertakes the analysis of the boundedness of the second-order moment of the WIS integral for $H \in (0, \frac{1}{2})$. Section 5 addresses the existence and uniqueness of solutions for fractional stochastic differential equations within the interval $H \in (0, \frac{1}{2})$. Finally, Section 6 provides a comprehensive summary and conclusion.

2 RiemannLiouville fractional derivative and its properties

[18] For a function f in the space $L^1([a,b])$ and a fractional order $d>0$, the left RiemannLiouville integral of f of order d on (a,b) is defined almost everywhere by the following expression

$$I^d_{a^+}f(x)=\frac{1}{\Gamma(d)}\int_a^x (x-y)^{d-1}f(y)dy,$$

with Γ signifying the Euler function, this integral provides an extension of the conventional n-order iterated integrals of f for $d=n\in\mathbb{N}$. Subsequently, we present the initial composition formula

$$I^d_{a^+}(I^k_{a^+}f)=I^{d+k}_{a^+}f.$$

Fractional derivatives can be thought of as the opposite of fractional integrals. We assume that $0<d<1$ and $p>1$.

We define $I^d_{a^+}(L^p)$ as the image of $L^p([a,b])$ under the operator $I^d_{a^+}$. For any $f\in I^d_{a^+}(L^p)$, there is a unique function ψ within L^p such that $f=I^d_{a^+}\psi$. This ψ is equivalent to the left RiemannLiouville derivative of f of order d, which is given by

$$D^d_{a^+}f(x)=\frac{1}{\Gamma(1-d)}\frac{d}{dx}\int_a^x \frac{f(y)}{(x-y)}dy.$$

The derivative of f has the following Weil representation

$$D^d_{a^+}f(x)=\frac{1}{\Gamma(1-d)}\left(\frac{f(x)}{(x-a)^d}+d\int_a^x\frac{f(x)-f(y)}{(x-y)^{d+1}}dy\right)1_{(a,b)}(x).$$

The integrals converge at the singularity $x=y$ in the L^p sense. If $dp>1$, then any function in $I^d_{a^+}(L^p)$ is $(d-\frac{1}{p})$-Holder continuous. Moreover, any Holder continuous function with order $k>d$ has a fractional derivative of order d, implying that $C^k([a,b])$ is contained within $I^d_{a^+}(L^p)$ for all $p>1$.

Recall that by construction for $f\in I^d_{a^+}(L^p)$

$$I^d_{a^+}(D^d_{a^+}f)=f,$$

and for general $f\in L^1([a,b])$ we have

$$D^d_{a^+}(I^d_{a^+}f)=f.$$

If $f\in I^{d+k}_{a^+}(L^1)$, $d\geq 0$, $k\geq 0$, $d+k\leq 1$ we have the second composition formula

$$D^d_{a^+}(D^k_{a^+}f)=D^{d+k}_{a^+}f.$$

3 The fractional operator M

This section defines the fractional operator M for the range $0 < H < 1$.

Definition 3.1. [8] The FBM $B^H = \{B^H(t), t \in \mathbb{R}\}$, with Hurst parameter H in the range $(0,1)$, is a Gaussian process with the following expected value

$$\mathbb{E}[B^H(t)] = 0,$$

and covariance

$$\mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2}\left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\}.$$

We take $B^H(0) = 0$. For $H = \frac{1}{2}$, $B^{\frac{1}{2}}$ is the standard Brownian motion.

Define $\mathcal{S}(\mathbb{R})$ as the Schwartz space, consisting of smooth functions on the real line with rapid decay. Suppose $\Omega := \mathcal{S}'(\mathbb{R})$, be the topological dual of $\mathcal{S}(\mathbb{R})$, representing the space of tempered distributions. Let $\mathcal{B}$ denote the Borel σ-algebra defined on the set Ω . The white noise probability measure $\mathbb{P}$ on $\mathcal{B}(\mathcal{S}'(\mathbb{R}))$ is defined by the following characteristic functional

$$\int\limits_{\mathcal{S}'(\mathbb{R})} \exp\left(i\left\langle \omega, f\right\rangle\right) d\mathbb{P}(\omega) = \exp\left(-\frac{1}{2}\|f\|^2_{L^2(\mathbb{R})}\right), \quad f \in \mathcal{S}(\mathbb{R}),$$

where $i^2 = -1$ and $\langle \omega, f\rangle = \omega(f)$ is the action of $\omega \in \Omega = \mathcal{S}'(\mathbb{R})$ on $f \in \mathcal{S}(\mathbb{R})$.

The central idea here is to connect FBM $B^{(H)}(t)$ with Hurst parameter $H \in (0,1)$ to standard Brownian motion $B(t)$ (which has $H = \frac{1}{2}$) using a transformation operator M.

Definition 3.2. Biagini et al. [2] Let $0 < H < 1$. The operator $M = M_H$ is defined on function $f \in \mathcal{S}(\mathbb{R})$ by

$$\widehat{M}f(\beta) = |\beta|^{\frac{1}{2}-H}\hat{f}(\beta), \quad \beta \in \mathbb{R},$$

where

$$\hat{f}(\beta) := \int\limits_{\mathbb{R}} e^{-i\alpha\beta} f(\alpha) d\alpha,$$

denotes the Fourier transform.

It is noted that in the subsequent discussion, we generally employ the notation M in lieu of M_H, except when explicit reference to the corresponding Hurst parameter H is required. Alternatively, for any Hurst parameter H within the interval $(0,1)$, the operator M can be equivalently defined as (Biagini et al. [2]):

$$Mf(\alpha) = -\frac{d}{d\alpha} C_H \int\limits_{\mathbb{R}} (\beta - \alpha)|\beta - \alpha|^{H-3/2} f(\beta) d\beta,$$

where $f \in \mathcal{S}(\mathbb{R})$, and

$$C_H = \left(2\Gamma(H - \frac{1}{2}) \cos(\frac{\pi}{2}(H - \frac{1}{2}))\right)^{-1}.$$

For $H = \frac{1}{2}$ we have

$$Mf(\alpha) = f(\alpha).$$

For $0 < H < \frac{1}{2}$ we have

$$Mf(\alpha) = C_H \int\limits_{\mathbb{R}} \frac{f(\alpha - \beta) - f(\beta)}{|\beta|^{3/2-H}} d\beta,$$

For $\frac{1}{2} < H < 1$ we have

$$Mf(\alpha) = C_H \int\limits_{\mathbb{R}} \frac{f(\beta)}{|\beta - \alpha|^{3/2-H}} d\beta.$$

Definition 3.3. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function by bounded growth, and let $x_0 \in \mathbb{R}$ be a singular point. The principal value (p.v.) of the singular integral at x_0 is defined as follows [5, 21]:

$$\text{p.v.} \int\limits_{\mathbb{R}} \frac{g(y)}{y - x_0} dy := \lim_{\epsilon \to 0} \left(\int_{-\infty}^{x_0 - \epsilon} \frac{g(y)}{y - x_0} dy + \int_{x_0 + \epsilon}^{\infty} \frac{g(y)}{y - x_0} dy \right).$$

Definition 3.4. According to Definition 3.3, the fractional kernel $|\alpha - \beta|^{(\frac{3}{2} - H)}$ has a singularity at $x = 0$ for $0 < H < \frac{1}{2}$. Hence, the operator M for $0 < H < \frac{1}{2}$ can be defined as follows:

$$Mf(\alpha) = C_H \text{p.v.} \int\limits_{\mathbb{R}} \frac{f(\beta)}{|\alpha - \beta|^{\frac{3}{2} - H}} d\beta := C_H \lim_{\epsilon \to 0} \int_{|\alpha - \beta| > \epsilon} \frac{f(\beta)}{|\alpha - \beta|^{\frac{3}{2} - H}} d\beta.$$

Lemma 3.5. *Let $H \in (0, \frac{1}{2})$. Then the following attribute is true for operator M*

$$M^2 f(x) = C_H^2 p.v. \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} |\beta\gamma|^{H-3/2} f(\alpha - \beta - \gamma) d\beta d\gamma.$$

Proof. According to Definition 3.4 and by employing a suitable change of variables $(\beta = \alpha - \beta)$, we obtain

$$Mf(\alpha) = C_H \text{p.v.} \int\limits_{\mathbb{R}} |\beta|^{H-3/2} f(\alpha - \beta) d\beta.$$

Then

$$\begin{aligned} M^2 f(\alpha) &= M(Mf)(\alpha) = C_H \text{p.v.} \int\limits_{\mathbb{R}} |\gamma|^{H-3/2} Mf(\alpha - \gamma) d\gamma \\ &= C_H \text{p.v.} \int\limits_{\mathbb{R}} |\gamma|^{H-3/2} C_H \text{p.v.} \int\limits_{\mathbb{R}} |\beta|^{H-3/2} f(\alpha - \gamma - \beta) d\beta d\gamma \\ &= C_H^2 \text{p.v.} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} |\beta\gamma|^{H-3/2} f(\alpha - \beta - \gamma) d\beta d\gamma. \end{aligned}$$

□

4 Second-order moment of WIS integrals

This section first introduces the Wick product and subsequently analyzes the second-order moment of WIS integrals for $H \in (0, \frac{1}{2})$.
Consider the following Hermite polynomials as

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}, n = 0, 1, 2, ...,$$

and let $\mathcal{J}$ denote the set of all finite multi-indices of non-negative integers as follows

$$\eta = (\eta_1, ..., \eta_n) \in \mathcal{J}.$$

Additionally, suppose $(e_n)_{n \in N}$ be an orthonormal basis in $L^2(\mathbb{R})$ and $\omega \in \Omega$. Then $\mathcal{H}_\eta(\omega)$ is defined as

$$\mathcal{H}_\eta(\omega) = h_{\eta_1}(\langle \omega, e_1 \rangle) h_{\eta_2}(\langle \omega, e_2 \rangle) ... h_{\eta_n}(\langle \omega, e_n \rangle).$$

Definition 4.1. [8] Let $G \in L^2(\mathbb{R})$ and $c_\eta \in \mathbb{R}$. Then the set of all expansions of $G(\omega) = \sum\limits_{\eta \in \mathcal{J}} c_\eta \mathcal{H}_\eta(\omega)$ such that for $\delta \in \mathbb{N}$ we have

$$\|G\|^2_{H,-\delta} = \sum_{\eta \in \mathcal{J}} \eta! c_\eta^2 (2\mathbb{N})^{-\delta\eta},$$

is called fractional Hida distribution space and denoted by $\mathcal{S}^*$.

Definition 4.2. [8] (Wick Product) If $F(\omega) = \sum\limits_{\eta \in \mathcal{J}} a_\eta \mathcal{H}_\eta(\omega) \in \mathcal{S}^*$ and $G(\omega) = \sum\limits_{\xi \in \mathcal{J}} b_\xi \mathcal{H}_\xi(\omega) \in \mathcal{S}^*$, then the Wick product of $F$and G is defined as follows

$$F \diamondsuit G(\omega) := \sum_{\eta \in \mathcal{J}} a_\eta b_\xi \mathcal{H}_{\eta+\xi}(\omega).$$

Definition 4.3. Dordevic and Øksendal [7] Given a function ϕ mapping the real numbers to the space of generalized distributions $\mathcal{S}^*$, if the product $\phi(s)\diamondsuit W^H(s)$ is integrable with respect to ds within the space $\mathcal{S}^*$, then ϕ is deemed dB^H-integrable. The WIS integral of this function is then defined as follows

$$\int_{\mathbb{R}} \phi(s)dB^H(s) = \int_{\mathbb{R}} \phi(s)\diamondsuit W^H(s)ds.$$

Lemma 4.4. *Dordevic and Øksendal [7] (Product rule) Consider a measurable process η, such that $\mathbb{E}\left[\int_0^t |\eta(s)|\,ds\right] < \infty$ for all $t \geq 0$. Furthermore, suppose λ be an $\mathcal{F}$-adapted stochastic process exhibiting cadlag paths and satisfying the criteria for the WIS integrability. Let*

$$dY_i = \eta_i(t)dt + \lambda_i(t)dB^H(t), \quad i = 1, 2,$$

with $Y_i(0) = y_i$. Then the following holds

$$\mathbb{E}\left[Y_1(t)Y_2(t)\right] = y_1y_2 + \mathbb{E}\left[\int_0^t \left(Y_1(s)\eta_2(s) + Y_2(s)\eta_1(s)\right)ds + \int_0^t \left(\lambda_1(s)M^2(\lambda_{2\chi[0,t]})(s) + \lambda_2(s)M^2(\lambda_{1\chi[0,t]})(s)\right)ds\right].$$

Theorem 4.5. *Let $0 < H < \frac{1}{2}$ and let $\phi(t) = \phi(t,\omega)$ be an $\mathcal{F}$-adapted stochastic process such that*

$$\mathbb{E}\left[\int_0^T \phi^2(s)ds\right] < \infty.$$

Define

$$Y(t) = \int_0^t \phi(s)dB^H(s), \quad 0 \leq t \leq T,$$

where the integration is performed using the WIS method. Then

$$\mathbb{E}\left[Y^2(t)\right] \leq K_H\mathbb{E}\left[\int_0^t \phi^2(s)ds\right],$$

where

$$K_H = 2C_H^2\bar{C},$$

and C_H and $\bar{C}$ is constant.

Proof. We can, without loss of generality, consider ϕ to be a non-negative function. This is achieved by separating ϕ into positive and negative portions and adjusting the estimate by a factor of two. Subsequently, application of Lemma 4.4 yields

$$\Psi(t) := \mathbb{E}\left[Y^2(t)\right] = \mathbb{E}\left[\int_0^t 2\phi(s)M^2(\phi\chi_{[0,t]})(s)ds\right].$$

Also, by Lemma 3.5

$$M^2 f(\alpha) = C_H^2 \text{p.v.} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} |\beta\gamma|^{H-3/2} f(\alpha-\beta-\gamma) d\beta d\gamma.$$

Therefore, by employing the Cauchy-Schwarz inequality in two consecutive instances, we deduce

$$\begin{aligned}
\Psi(t) &= 2C_H^2 \text{p.v.}\, \mathbb{E}\left[\int_0^t \left(\text{p.v.} \iint_{\mathbb{R}^2} |\beta\gamma|^{H-\frac{3}{2}} \phi(s)\phi(s-\beta-\gamma)\right.\right. \\
&\quad \left.\left. \times \chi_{[0,t]}(s-\beta-\gamma)\, d\beta\, d\gamma\right) ds\right] \\
&= 2C_H^2 \left[\text{p.v.} \iint_{\mathbb{R}^2} \left(\int_0^t \mathbb{E}\left[\phi(s)\phi(s-\beta-\gamma)\right]\right.\right. \\
&\quad \left.\left. \times \chi_{[0,t]}(s-\beta-\gamma)\, ds\right) |\beta\gamma|^{H-\frac{3}{2}} d\beta\, d\gamma\right] \\
&\leq 2C_H^2 \left[\text{p.v.} \iint_{\mathbb{R}^2} \left(\int_0^t \mathbb{E}\left[\phi^2(s)\right] \chi_{[0,t]}(s-\beta-\gamma)\, ds\right)^{\frac{1}{2}}\right. \\
&\quad \times \left(\int_0^t \mathbb{E}\left[\phi^2(s-\beta-\gamma)\right] \chi_{[0,t]}(s-\beta-\gamma)\, ds\right)^{\frac{1}{2}} \\
&\quad \left. \times |\beta\gamma|^{H-\frac{3}{2}} d\beta\, d\gamma\right] \\
&\leq 2C_H^2 \left\{\left[\text{p.v.} \iiint_{\mathbb{R}^2\times[0,t]} \mathbb{E}\left[\phi^2(s)\right] |\beta\gamma|^{H-\frac{3}{2}}\right.\right. \\
&\quad \left. \times \chi_{[0,t]}(s-\beta-\gamma)\, ds\, d\beta\, d\gamma\right]^{\frac{1}{2}} \\
&\quad \times \left[\text{p.v.} \iint_{\mathbb{R}^2} \left(\int_0^t \mathbb{E}\left[\phi^2(s-\beta-\gamma)\right]\right.\right. \\
&\quad \left.\left.\left. \times \chi_{[0,t]}(s-\beta-\gamma)\, ds\right) |\beta\gamma|^{H-\frac{3}{2}} d\beta\, d\gamma\right]^{\frac{1}{2}}\right\} \\
&= 2C_H^2 u_1^{\frac{1}{2}} u_2^{\frac{1}{2}},
\end{aligned}$$

where

$$u_1 := \text{p.v.} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \int_0^t \mathbb{E}[\phi^2(s)] \chi_{[0,t]}(s-\beta-\gamma) |\beta\gamma|^{H-3/2} ds d\beta d\gamma,$$

$$u_2 := \text{p.v.} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \int_0^t \mathbb{E}[\phi^2(s-\beta-\gamma)] \chi_{[0,t]}(s-\beta-\gamma) ds |\beta\gamma|^{H-3/2} d\beta d\gamma.$$

To facilitate the estimation of the $dsd\beta d\gamma$-integral u_1, we decompose it into sequential integrals: a ds-integral followed by a $d\beta d\gamma$-integral, as demonstrated below.

Observe that given $0 \leq s \leq t$ and $s-\beta-\gamma \in [0,t]$ it necessarily implies that

$-\beta-\gamma \in [-t,t]$. Consequently

$$u_1 \leq \int_0^t \mathbb{E}[\phi^2(s)]ds \text{ p.v.} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \chi_{[-t,t]}(-\beta-\gamma)|\beta\gamma|^{H-3/2} d\beta d\gamma.$$

The integral $d\beta\, d\gamma$ is convergent according to the definition of power-type integral convergence (since $H-\frac{3}{2}<-1$). Therefore,

$$\text{p.v.} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \chi_{[-t,t]}(-\beta-\gamma)\,|\beta\gamma|^{\,H-3/2}\, d\beta\, d\gamma \leq \tilde{C}.$$

To facilitate the estimation of u_2, we employ a variable substitution

$$\begin{cases} z = s-\beta-\gamma \\ \upsilon = -\beta \\ \varpi = -\gamma \end{cases} \Leftrightarrow \begin{cases} s = z-\upsilon-\varpi =: \Theta(z,\upsilon,\varpi) \\ \beta = -\upsilon =: \Lambda(z,\upsilon,\varpi) \\ \gamma = -\varpi =: \Phi(z,\upsilon,\varpi). \end{cases}$$

Then the Jacobian is

$$\begin{vmatrix} \frac{\partial\Theta}{\partial z} & \frac{\partial\Theta}{\partial \upsilon} & \frac{\partial\Theta}{\partial \varpi} \\ \frac{\partial\Lambda}{\partial z} & \frac{\partial\Lambda}{\partial \upsilon} & \frac{\partial\Lambda}{\partial \varpi} \\ \frac{\partial\Phi}{\partial z} & \frac{\partial\Phi}{\partial \upsilon} & \frac{\partial\Phi}{\partial \varpi} \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 1.$$

So we get

$$u_2 = \text{p.v.} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \int_0^t \mathbb{E}\left[\phi^2(z)\right] \chi_{[0,t]}(z) dz |\upsilon\varpi|^{H-3/2} d\upsilon d\varpi.$$

Given that this integral exhibits a structure identical to u_1, we have

$$\int_0^t \mathbb{E}\left[\phi^2(z)\right] dz \text{p.v.} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \chi_{[0,t]}(z)|\upsilon\varpi|^{H-3/2} d\upsilon d\varpi.$$

Then

$$\text{p.v.} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \chi_{[0,t]}(z)|\upsilon\varpi|^{H-3/2} d\upsilon d\varpi \leq \hat{C}.$$

As result

$$\mathbb{E}[Y^2(t)] \leq 2C_H^2 \bar{C} \mathbb{E}\left[\int_0^T \phi^2(s)ds\right], \quad t \leq T,$$

where $\bar{C} = \hat{C}\tilde{C}$.

□

5 Existence and uniqueness

This section provides a rigorous proof of the existence and uniqueness of a solution to the fractional stochastic differential equation for $H \in (0, \frac{1}{2})$.

Theorem 5.1. *Let $T > 0$ and $b(.,.) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma(.,.) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable functions satisfying*

$$|b(t,y)| + |\sigma(t,y)| \leq D(1 + |y|), \quad y \in \mathbb{R}^n, t \in [0,T],$$

and

$$|b(t,y) - b(t,\bar{y})| + |\sigma(t,y) - \sigma(t,\bar{y})| \leq K\,|y - \bar{y}|\,, \quad y, \bar{y} \in \mathbb{R}^n, t \in [0,T].$$

Let Y_0 be a random variable which is independent of both $\mathcal{F}$ and the σ-field $\mathbb{F} = \mathcal{F}_\infty$ generated by $B^H_s(.), s \geq 0$ such that

$$\mathbb{E}\left[|Y_0|^2\right] < \infty.$$

Then the stochastic differential equation

$$\begin{aligned} dY_t &= b(t,Y_t)dt + \sigma(t,Y_t)dB^H_t, \quad 0 \leq t \leq T, \\ Y(0) &= Y_0, \end{aligned}$$

for $H \in (0, \frac{1}{2})$ has a unique solution $Y(t)$ with the property that

$$\mathbb{E}\left[\int_0^T |Y_t|^2 dt\right] < \infty.$$

Proof. In order to demonstrate uniqueness, we posit the existence of two solutions, Y_t and $\bar{Y}_t$, with corresponding initial values Y_0 and $\bar{Y}_0$. Then by Lyapunov inequality we have

$$\begin{aligned} \mathbb{E}\left[\left|Y_t - \bar{Y}_t\right|^2\right] &= \mathbb{E}\Bigg[\Bigg|Y_0 - \bar{Y}_0 + \int_0^t \left(b(s,Y_s) - b(s,\bar{Y}_s)\right) ds \\ &\qquad + \int_0^t \left(\sigma(s,Y_s) - \sigma(s,\bar{Y}_s)\right) dB^H_s\Bigg|^2\Bigg] \\ &\leq 3\mathbb{E}\left[\left|Y_0 - \bar{Y}_0\right|^2\right] + 3\mathbb{E}\left[\left|\int_0^t \left(b(s,Y_s) - b(s,\bar{Y}_s)\right) ds\right|^2\right] \\ &\qquad + 3\mathbb{E}\left[\left|\int_0^t \left(\sigma(s,Y_s) - \sigma(s,\bar{Y}_s)\right) dB^H_s\right|^2\right]. \end{aligned}$$

Applying Lipschitz condition, Jensens inequality and Theorem 4.5 we have

$$\begin{aligned}\mathbb{E}\left[\left|Y_t-\bar{Y}_t\right|^2\right] \leq 3\mathbb{E}\left[\left|Y_0-\bar{Y}_0\right|^2\right] + 3tK^2\int_0^t \mathbb{E}\left[\left|Y_s-\bar{Y}_s\right|^2\right] ds \\ + 3K_H\int_0^t \mathbb{E}\left[\left|Y_s-\bar{Y}_s\right|^2\right] ds \\ \leq 3\mathbb{E}\left[\left|Y_0-\bar{Y}_0\right|^2\right] + 3\left(tK^2+K_H\right)\int_0^t \mathbb{E}\left[\left|Y_s-\bar{Y}_s\right|^2\right] ds.\end{aligned}$$

Suppose $F = 3\mathbb{E}\left[\left|Y_0-\bar{Y}_0\right|^2\right]$ and $P_t = \mathbb{E}\left[\left|Y_s-\bar{Y}_s\right|^2\right]$. Using Gronwall's inequality, we have

$$P_t = F + A\int_0^t P_s ds \leq Fe^{At},$$

where $A = 3\left(tK^2+K_H\right)$. The uniqueness of solution can be proved using $Y_0 = \bar{Y}_0$. In order to establish the existence of a solution, we define an increasing family of σ-fields $\mathcal{F}_t(0 \geq t \geq T)$, such that Y_0 is measurable with respect to $\mathcal{F}_0$ and B_t^H is measurable with respect to $\mathcal{F}_t$ for all $t \geq 0$. We may choose $\mathcal{F}_t$ to be the σ-fields generated by Y_0 and $\mathcal{F}(B_s^H, 0 \geq s \geq T)$, relying on the premise that Y_0 is independent of $\mathcal{F}(B_s^H, 0 \geq s \geq T)$. Define $Y_t^{(0)} = Y_0$ and

$$Y_t^{(m+1)} = Y_0 + \int_0^t b(Y_s^{(m)}, s)ds + \int_0^t \sigma(Y_s^{(m)}, s)dB_s^H.$$

The inductive assumption is that $Y^m \in L^2$ norm and that

$$\mathbb{E}\left|Y_t^{(k+1)} - Y_t^{(k)}\right|^2 \leq \frac{(\mathcal{A}t)^{(k+1)}}{(k+1)!} \quad for \quad 0 \leq k \leq m-1, \tag{1}$$

where $\mathcal{A}$ is some positive constant (depending only on D, K, T). Since $Y_0 \in \mathcal{F}_0$, $Y_t^{(m+1)}$ is well defined if $m = 0$. Further

$$\left|Y_t^{(1)} - Y^{(0)}\right|^2 \leq 2\left|\int_0^t b(Y_0, s)ds\right|^2 + 2\left|\int_0^t \sigma(Y_0, s)dB_s^H\right|^2.$$

Taking the expectation, we get

$$\left|Y_t^{(1)} - Y^{(0)}\right| \leq 2D^2t^2\left(1+\mathbb{E}|Y_0|^2\right) + 2D^2K_H\left(1+\mathbb{E}|Y_0|^2\right) \leq \mathcal{A}t.$$

This implies that $Y^{(1)} \in L^2$ norm and Eq. (1) holds for $m = 0$. We postulate the inductive hypothesis, which holds for all $m \geq 0$, and then establish its truth for the successor, $m+1$. Then

$$\begin{aligned}\left|Y_t^{(m+1)} - Y_t^{(m)}\right|^2 \leq 2\left|\int_0^t \left(b(Y_s^{(m)}, s) - b(Y_s^{(m-1)}, s)\right) ds\right|^2 \\ + 2\left|\int_0^t \left(\sigma(Y_s^{(m)}, s) - \sigma(Y_s^{(m-1)}, s)\right) dB_s^H\right|^2.\end{aligned}$$

Taking the epectation and using Theorem 4.5, we have

$$\mathbb{E}\left[\left|Y_s^{(m+1)} - Y_s^{(m)}\right|^2\right] \leq 2tK^2\mathbb{E}\left[\int_0^t \left|Y_t^{(m)} - Y_t^{(m-1)}\right|^2 ds\right] + 2K_H K^2\mathbb{E}\left[\int_0^t \left|Y_s^{(m)} - Y_s^{(m-1)}\right|^2 ds\right].$$

Thus

$$\mathbb{E}\left[\left|Y_t^{(m+1)} - Y_t^{(m)}\right|^2\right] \leq C\int_0^t \left|Y_s^{(m)} - Y_s^{(m-1)}\right|^2 ds,$$

if $C \geq 2K^2(t + K_H)$. Substituting (1) with $k = m - 1$ into the right-hand side, we get

$$\mathbb{E}\left[\left|Y_t^{(m+1)} - Y_t^{(m)}\right|^2\right] \leq C\int_0^t \frac{(\mathcal{A}s)^m}{m!} ds = \frac{(\mathcal{A}s)^{m+1}}{(m+1)!}.$$

Thus Eq. (1) holds for $k = m$. As this result demonstrates that $Y^{(m+1)}$ belongs to the L^2 norm, the inductive step for $m + 1$ is thereby established.

$\square$

6 Conclusion

The dynamic fluctuations of financial markets, characterized by their complex and memory-driven nature, necessitate the development of rigorous and efficient modeling paradigms. In this context, stochastic processes exhibiting short-term memory assume particular significance due to their capacity to capture and represent temporal dependencies of limited yet substantial magnitude. Consequently, fractional stochastic differential equations with short-term memory are recognized as a potent analytical tool for modeling financial market volatility. This paper is dedicated to the investigation of the existence of solutions for fractional stochastic differential equations characterized by short-term memory. Initially, the boundedness of the WIS integral is established for values of the Hurst parameter within the interval $H \in (0, \frac{1}{2})$, utilizing the formal definition of the integral and the inherent properties of fractional processes. Subsequently, by imposing conditions of Lipschitz continuity and linear growth on the coefficients of the aforementioned stochastic differential equation, and by employing the Gronwall's inequality, the existence of solutions is analytically demonstrated. The findings of this investigation reveal that fractional stochastic differential equations with short-term memory, under the specified conditions, admit solutions. The boundedness of the WIS integral is rigorously established as a necessary prerequisite for solution existence, while the conditions of Lipschitz continuity and linear growth are presented as sufficient criteria. This paper contributes to the advancement of theoretical understanding in this domain by providing a rigorous analytical framework for examining the existence of solutions for fractional stochastic differential equations characterized by short-term memory.

Declarations

Conflict of interest There are no Conflict of interest to declare.

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