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Research paper

Bond Pricing and the Term Structure of Spot and Forward Interest Rates: A Multi-factor Vasicek and Cir Model Approach

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Abstract:
This paper investigates multifactor affine models of the term structure of interest rates, focusing on those that admit closed-form solutions for zero-coupon bond prices. In particular, we study multifactor Vasicek and Cox–Ingersoll–Ross (CIR) models and their hybrid combinations, which integrate Gaussian and square-root dynamics within a single affine framework. By providing a unified analytical treatment, the paper clarifies the economic interpretation of model parameters and explores how they shape the spot and forward rate curves. The hybrid approach enhances the flexibility of term structure modeling, allowing one to capture level, slope, and curvature of the yield curve more accurately than single-class models. These results are directly applicable in practice for yield curve estimation, empirical calibration, risk management, and the pricing of interest rate derivatives.

Keywords: Combined multifactor Vasicek and CIR model; zero-coupon bond price; spot interest rate curve; forward interest rate curve.
Classification: 91G30

1 Introduction

Modeling the term structure of interest rates is central to asset pricing, risk management, and monetary policy analysis. Affine term structure models (ATSMs), particularly the Vasicek and Cox–Ingersoll–Ross (CIR) frameworks, are widely used because they provide closed-form expressions for zero-coupon bond prices while offering economically interpretable factor dynamics [11,16,17,20,35]. Both Gaussian (Vasicek-type) and square-root (CIR-type) specifications have been extended to multifactor settings, but the literature typically treats these two classes separately. As a result, a unified analytical treatment of hybrid multifactor models—combining Vasicek and CIR dynamics within a single affine structure—is largely absent.
This paper develops such a unified framework. We show that a multifactor system containing both Gaussian and square-root components remains affine and admits closed-form solutions for zero-coupon bond prices. Whereas previous work either focuses on purely Gaussian or purely square-root models, or provides general affine

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classifications without detailing hybrid configurations, this study offers a systematic derivation of the mathematical structure, pricing formulas, and economic implications of hybrid multifactor models.
Incorporating both Vasicek and CIR components within one system provides a number of theoretical and practical advantages. From a modeling perspective, multifactor structures naturally allow for richer yield curve dynamics: the number of factors governs the models ability to reproduce level, slope, and curvature movements, while the type of factor—Gaussian or square-root—affects mean-reversion properties, volatility behavior, and admissibility constraints. The hybrid approach therefore combines the flexibility of multifactor models with the interpretability and structural realism of Vasicek and CIR processes.
The paper also derives discount factors, spot rates, and forward rates in closed form, establishing their monotonicity, asymptotic limits, and sensitivity to each state variable. These analytical properties not only clarify how different factors influence the term structure but also support practical applications in calibration, simulation, and empirical yield curve fitting. The availability of closed-form expressions makes the hybrid model particularly attractive for pricing interest rate derivatives and for risk management applications where tractability and interpretability are essential.
Overall, this work provides a comprehensive and analytically tractable treatment of hybrid multifactor Vasicek–CIR models, filling a clear gap in the affine term structure literature and linking theoretical rigor with practical relevance for empirical finance and interest rate modelling.

2 Literature Review

Understanding the dynamics of the term structure of interest rates is crucial for accurately determining bond prices, valuing derivatives, and conducting risk analysis in financial markets. Early studies focused on one-factor models, which assumed that the risk-free short-term interest rate fully determines the yield curve movements for all maturities. Well-known examples include the Vasicek model (1977) [35], the CIR model (1985) [16], the Dothan model (1978) [18], the Courtadon model (1982) [14], and the exponential Vasicek (EV) model, differing in their treatment of volatility and interest rate dynamics. Later extensions, such as the Ho-Lee (1986) [24], Hull-White (Extended Vasicek and CIR, 1993) [25, 26, 36], Black-Karasinski (1991) [3], and Ait-Sahalia (1996) [1], introduced time-dependent coefficients to better align models with observed market data.
Although one-factor models provide closed-form solutions for bond pricing, they fail to capture changes in slope and curvature of the term structure. Multi-factor models were thus developed to incorporate additional sources of uncertainty. Two-factor models (Richard [33], Schaefer and Schwartz [34], Brennan and Schwartz [6,7], Longstaff and Schwartz [27]) and three-factor models (Chen [12]) include factors such as long-term and medium-term rates or spreads, providing a more realistic

description of the yield curve.

Empirical studies have shown that one-factor models do not fully match observed yield curves, while multi-factor approaches improve fit but increase estimation complexity (Canabarro [9]; Chen and Scott [13]). Statistical methods such as the Kalman filter and maximum likelihood have been used to estimate parameters in these models, although nonlinear dynamics can introduce biases (Duan and Simonato [19]; Lund [28]).

Research indicates that moving from a one-factor to a two-factor model significantly reduces measurement errors, while adding a third factor provides additional, though smaller, improvements (Babbs and Nowman [2]; Geyer and Pichler [22]). Models with more than three factors generally do not yield substantial gains in accuracy.

To complement analytical interest-rate modelling approaches, recent research has also focused on advanced numerical and computational techniques for pricing financial derivatives and interest-rate–related securities. Spectral collocation methods based on Chebyshev polynomials have been developed for simulating and pricing catastrophe (CAT) bonds in models with jump components [21], providing highly accurate approximations for integral–differential equations. Fractional quadratic interpolation has been proposed as an efficient time–discretization method for pricing options in the time-fractional Black–Scholes framework [31], offering high-order convergence for problems with memory effects. Moreover, convergence analyses for numerical schemes in tempered fractional Black–Scholes models have been established for double-barrier options [30], demonstrating stability and efficiency in fractional-order derivative pricing. Although these approaches operate in different modelling settings, they represent modern computational strategies that complement the analytical tractability of multi-factor Vasicek–CIR models and reflect current trends in quantitative finance.

Building upon this literature, the present paper focuses on multi-factor extensions of the Vasicek and CIR models, combining their features in a hybrid framework. This approach allows for analytical tractability while providing flexibility in modeling bond prices and the term structure. Moreover, simulations with one-, two-, and three-factor implementations are proposed to illustrate how the number of factors captures the level, slope, and curvature of the yield curve, highlighting the practical relevance of the theoretical framework.

3 Multifactor Vasicek Model

Lemma 3.1 (Multifactor Vasicek Model Dynamics)**.** *The term structure of interest rates based on n state variables follows the multifactor Vasicek model, where the dynamics of each factor are governed by the following stochastic differential equations (SDEs):*

$$dr_1(t) = [k_1 (\theta_1 - r_1(t)) - \lambda(t, r_1)\sigma_1] \, dt + \sigma_1 dW_1(t),$$

$$\vdots$$

$$dr_n(t) = [k_n(\theta_n - r_n(t)) - \lambda(t, r_n)\sigma_n]\,dt + \sigma_n dW_n(t),$$

where $W_i(t)$ are independent standard Wiener processes.

Proof. Each factor $r_i(t)$ is mean-reverting with parameters k_i (speed of mean reversion), θ_i (long-term mean), and σ_i (volatility). The term $\lambda(t, r_i)$ represents the market price of risk. □

Lemma 3.2 (Linear Market Price of Risk)**.** *Assume $\lambda(t, r_i)$ is linear:*

$$\lambda(t, r_i) = a_i + b_i r_i(t).$$

Substituting into the SDEs yields:

$$dr_i(t) = \psi_i(\hat{\theta}_i - r_i(t))dt + \sigma_i dW_i(t), \quad i = 1, \dots, n,$$

with

$$\psi_i = k_i + b_i\sigma_i, \quad \hat{\theta}_i = \frac{k_i\theta_i - a_i\sigma_i}{\psi_i}.$$

Theorem 3.3 (PDE for the Multifactor Vasicek Model)**.** *The zero-coupon bond price $P(\tau, r_1, \dots, r_n)$ satisfies*

$$\sum_{i=1}^{n} \psi_i(\hat{\theta}_i - r_i)P_{r_i} + \frac{1}{2}\sum_{i=1}^{n} \sigma_i^2 P_{r_i r_i} - P_\tau - \left(\sum_{i=1}^{n} r_i\right) P = 0. \tag{1}$$

Proof. Applying Itô's lemma to $P(\tau, r_1, \dots, r_n)$ and using risk-neutral valuation gives the PDE above, linking the dynamics of the factors to the term structure of interest rates. □

Theorem 3.4 (Exponential-Affine Solution for Zero-Coupon Bond Price)**.** *Assuming the solution has exponential-affine form*

$$P(\tau, r_1, \dots, r_n) = \exp\Big[\sum_{i=1}^{n}(A_i(\tau) - B_i(\tau)r_i)\Big],$$

substitution into the PDE yields a system of ODEs for $A_i(\tau)$ and $B_i(\tau)$:

$$\frac{dB_i}{d\tau} = 1 - \psi_i B_i, \quad B_i(0) = 0,$$

$$\frac{dA_i}{d\tau} = \frac{1}{2}\sigma_i^2 B_i^2 - \psi_i\hat{\theta}_i B_i, \quad A_i(0) = 0.$$

Solving these ODEs gives:

$$B_i(\tau) = \frac{1 - e^{-\psi_i\tau}}{\psi_i}, \quad A_i(\tau) = \frac{\sigma_i^2 - 2\psi_i^2\hat{\theta}_i}{2\psi_i^2}\tau - \frac{\psi_i^2\hat{\theta}_i - \sigma_i^2}{\psi_i^3}(1 - e^{-\psi_i\tau}) - \frac{\sigma_i^2}{4\psi_i^3}(1 - e^{-2\psi_i\tau}).$$

Proof. The statement follows by substituting the exponential–affine ansatz into the risk-neutral PDE and matching coefficients of constant and linear terms in r_i. Since factors are independent, cross-derivatives vanish, yielding the ODEs for A_i and B_i. Solving these ODEs gives the stated expressions. $\square$

Remark 3.5. This derivation demonstrates rigorously that the multifactor Vasicek model admits closed-form solutions for zero-coupon bonds under the independence assumption. It also provides a clear chain of reasoning from SDE → PDE → ODE → analytical solution.

4 Multifactor CIR Model

Lemma 4.1 (Multifactor CIR Model Dynamics)**.** *The term structure of interest rates based on n state variables follows the multifactor CIR model, where each factor follows the stochastic differential equation (SDE):*

$$dr_i(t) = \Big[k_i\big(\theta_i - r_i(t)\big) - \lambda(t, r_i)\, \sigma_i \sqrt{r_i(t)}\Big] dt + \sigma_i \sqrt{r_i(t)}\, dW_i(t), \quad i = 1, 2, \ldots, n.$$

Proof. Each state variable $r_i(t)$ follows an SDE with mean-reverting dynamics characterized by the parameters k_i, θ_i, and σ_i, with Wiener processes $W_i(t)$ assumed to be independent. The term $\lambda(t, r_i)$ represents the market price of risk function. $\square$

Lemma 4.2 (Rewritten Dynamics with Risk Adjustment)**.** *Assume the market price of risk* $\lambda(t, r_i)$ *takes the form:*

$$\lambda(t, r_i) = \frac{\lambda_i^* \sqrt{r_i(t)}}{\sigma_i} \qquad \textit{[15].}$$

Then, the SDEs simplify to:

$$dr_i(t) = \psi_i \left(\hat{\theta}_i - r_i(t)\right) dt + \sigma_i \sqrt{r_i(t)}\, dW_i(t), \quad i = 1, 2, \ldots, n,$$

where

$$\psi_i = k_i + \lambda_i^*, \quad \hat{\theta}_i = \frac{k_i \theta_i}{\psi_i}.$$

Proof. Substituting the risk adjustment into the original SDE yields:

$$dr_i(t) = \left[k_i(\theta_i - r_i) - \frac{\lambda_i^* \sqrt{r_i}}{\sigma_i}\, \sigma_i \sqrt{r_i}\right] dt + \sigma_i \sqrt{r_i}\, dW_i(t) = \psi_i(\hat{\theta}_i - r_i) dt + \sigma_i \sqrt{r_i}\, dW_i(t).$$

$\square$

Theorem 4.3 (General PDE for the Multifactor CIR Model)**.** *Under the assumption of independent Wiener processes, the partial differential equation governing the price of a zero-coupon bond in the multifactor CIR model is:*

$$\sum_{i=1}^{n} \Big[\psi_i(\hat{\theta}_i - r_i) P_{r_i} + \frac{1}{2}\sigma_i^2 r_i P_{r_i r_i}\Big] - P_\tau - \Big(\sum_{i=1}^{n} r_i\Big) P = 0. \tag{2}$$

Proof. Applying Itô's lemma to $P(t, r_1, \ldots, r_n, T)$ with independent CIR factors, and using $(dr_i)^2 = \sigma_i^2 r_i dt$, leads to the PDE (2). □

Theorem 4.4 (Solution for the Zero-Coupon Bond Price in the CIR Model)**.** *For n independent CIR state variables, the zero-coupon bond price admits an exponential-affine representation:*

$$P(\tau, r_1, \ldots, r_n) = \exp\left[\sum_{j=1}^{n} \big(A_i)\tau) - B_i(\tau)r_j\big)\right].$$

Substitution into the corresponding PDE yields the following system of differentail equations:

$$\frac{dB_i}{d\tau} = -\frac{\sigma_i^2}{2} B_i^2(\tau) - \psi_i B_i(\tau) + 1, \qquad B_i(0) = 0, \tag{3}$$

$$\frac{dA_i}{d\tau} = -\psi_i \hat{\theta}_i\, B_i(\tau), \qquad A_i(0) = 0. \tag{4}$$

Solving this system gives:

$$\begin{aligned}
B_i(\tau) &= \frac{2\big(e^{\xi_i \tau} - 1\big)}{(\psi_i + \xi_i)\big(e^{\xi_i \tau} - 1\big) + 2\xi_i}, \\
A_i(\tau) &= \frac{2\psi_i \hat{\theta}_i}{\sigma_i^2} \ln\left(\frac{2\xi_i\, e^{(\psi_i + \xi_i)\tau/2}}{(\psi_i + \xi_i)(e^{\xi_i \tau} - 1) + 2\xi_i}\right), \\
\xi_i &= \sqrt{\psi_i^2 + 2\sigma_i^2}.
\end{aligned}$$

Proof. Substituting the exponential–affine form into the risk-neutral PDE and using $(dr_i)^2 = \sigma_i'^2 r_i dt$ produces terms that are at most linear in r_i plus a quadratic term arising from the diffusion contribution. Collecting coefficients of r_i and the constant term yields the Riccati equation (3) for $B_i(\tau)$ and the linear ODE (4) for $A_i(\tau)$. The Riccati ODE for B_i is standard and admits the closed-form solution displayed above (see [16] or standard ATSM references). Once $B_i(\tau)$ is known, $A_i(\tau)$ follows by direct integration of (4) with the given initial condition. □

5 Combination of Multifactor Vasicek and CIR Models

The instantaneous short-term interest rate is expressed as a linear combination of n Vasicek factors y_i and m CIR factors z_j:

$$r(t) = \sum_{i=1}^{n} y_i(t) + \sum_{j=1}^{m} z_j(t).$$

Definition 5.1 (Multifactor Vasicek-CIR Model)**.** The state variables evolve according to:

$$
\begin{aligned}
dy_i(t) &= \psi_i(\hat{\theta}_i - y_i(t))\,dt + \sigma_i\,dW_i(t), \quad i = 1, \dots, n, \\
dz_j(t) &= \psi'_j(\hat{\theta}'_j - z_j(t))\,dt + \sigma'_j\sqrt{z_j(t)}\,dW'_j(t), \quad j = 1, \dots, m,
\end{aligned}
$$

where $W_i(t)$ and $W'_j(t)$ are independent Brownian motions, and

$$
\psi_i = k_i + b_i\sigma_i, \quad \hat{\theta}_i = \frac{k_i\theta_i - a_i\sigma_i}{\psi_i}, \quad \psi'_j = k'_j + \lambda'^*_j, \quad \hat{\theta}'_j = \frac{k'_j\theta'_j}{\psi'_j}.
$$

Theorem 5.2 (General PDE for the Combined Model)**.** *The zero-coupon bond price $P(\tau, r)$ satisfies the following PDE under the risk-neutral measure:*

$$
\sum_{i=1}^{n}\left[\psi_i\left(\hat{\theta}_i - y_i(t)\right)\right]P_{y_i} + \sum_{j=1}^{m}\left[\psi'_j\left(\hat{\theta}'_j - z_j(t)\right)\right]P_{z_i}
$$

$$
+\frac{1}{2}\sum_{i=1}^{n}\sigma_i^2 P_{y_iy_i} + \frac{1}{2}\sum_{j=1}^{m}\sigma_j'^2 z_j P_{z_jz_j} - P_\tau - \left(\sum_{i=1}^{n} y_i + \sum_{j=1}^{m} z_j\right)P(\tau, r) = 0 \qquad (5)
$$

Proof. Let $P(t, y_1, \dots, y_n, z_1, \dots, z_m, T)$ denote the zero-coupon bond price. Applying Itô's Lemma gives:

$$
dP = P_t dt + \sum_{i=1}^{n} P_{y_i} dy_i + \sum_{j=1}^{m} P_{z_j} dz_j + \frac{1}{2}\sum_{i=1}^{n} P_{y_iy_i}(dy_i)^2 + \frac{1}{2}\sum_{j=1}^{m} P_{z_jz_j}(dz_j)^2.
$$

Substitute the SDEs for y_i and z_j:

$$
dy_i = \psi_i(\hat{\theta}_i - y_i)dt + \sigma_i dW_i, \quad dz_j = \psi'_j(\hat{\theta}'_j - z_j)dt + \sigma'_j\sqrt{z_j}dW'_j.
$$

Quadratic variations:

$$
(dy_i)^2 = \sigma_i^2 dt, \quad (dz_j)^2 = \sigma_j'^2 z_j dt.
$$

Then

$$
\begin{aligned}
dP = \Bigg[&P_t + \sum_{i=1}^{n}\psi_i(\hat{\theta}_i - y_i)P_{y_i} + \sum_{j=1}^{m}\psi'_j(\hat{\theta}'_j - z_j)P_{z_j} \\
&+ \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2 P_{y_iy_i} + \frac{1}{2}\sum_{j=1}^{m}\sigma_j'^2 z_j P_{z_jz_j}\Bigg]dt + \sum_{i=1}^{n}\sigma_i P_{y_i} dW_i + \sum_{j=1}^{m}\sigma'_j\sqrt{z_j}P_{z_j}dW'_j.
\end{aligned}
$$

Under risk-neutral measure, the drift must equal $rPdt = (\sum_i y_i + \sum_j z_j)Pdt$, giving PDE (5) after changing variables to $\tau = T - t$. $\square$

Theorem 5.3 (Solution for the Zero-Coupon Bond Price). *Assuming independence among state variables, the bond price can be factorized:*

$$P(\tau, r) = \prod_{i=1}^{n} P_i(\tau, y_i) \prod_{j=1}^{m} P'_j(\tau, z_j),$$

where P_i satisfies the single-factor Vasicek PDE and P'_j satisfies the single-factor CIR PDE:

$$P_i(\tau, y_i) = \exp(A_i(\tau) - B_i(\tau) y_i),$$
$$P'_j(\tau, z_j) = \exp(A'_j(\tau) - B'_j(\tau) z_j).$$

The functions $B_i(\tau)$, $A_i(\tau)$ for Vasicek factors are:

$$B_i(\tau) = \frac{1 - e^{-\psi_i \tau}}{\psi_i}, \quad A_i(\tau) = \frac{\sigma_i^2}{4\psi_i}(1 - e^{-2\psi_i \tau}) - \hat{\theta}_i (1 - e^{-\psi_i \tau})^2.$$

The functions $B'_j(\tau)$, $A'_j(\tau)$ for CIR factors are:

$$B'_j(\tau) = \frac{2(e^{\xi_j \tau} - 1)}{(\psi'_j + \xi_j)(e^{\xi_j \tau} - 1) + 2\xi_j}, \quad A'_j(\tau) = \frac{2\psi'_j \hat{\theta}'_j}{\sigma'^2_j} \ln\left[\frac{2\xi_j e^{(\psi'_j + \xi_j)\tau/2}}{(\psi'_j + \xi_j)(e^{\xi_j \tau} - 1) + 2\xi_j}\right],$$

where $\xi_j = \sqrt{\psi'^2_j + 2\sigma'^2_j}$.
Substituting these into the factorized form confirms that $P(\tau, r)$ satisfies PDE (5).

Remark 5.4. This derivation explicitly demonstrates that, under the independence assumption, the multifactor Vasicek-CIR model admits a closed-form analytical solution for zero-coupon bonds. The derivation follows the chain: SDE → PDE → ODE → solution, and the factorization satisfies the combined PDE by construction.

6 Numerical Illustration and Simulation of the Yield Curve

This section provides a numerical illustration of the yield curve dynamics within the combined Vasicek–CIR framework. The purpose of this simulation is *not* to calibrate the model to market data, but to demonstrate how including additional factors increases the models flexibility and allows it to produce more realistic shapes of the yield curve.
We compare three configurations:

(i) A one-factor Vasicek model

(ii) A two-factor Vasicek model (long-term + short-term)

(iii) A three-factor model (two Vasicek factors + one CIR factor)

The comparison is performed by computing zero-coupon yields for a set of maturities:

$$T \in \{0.5, 1, 2, 3, 5, 7, 10, 20, 30\} \text{ years.}$$

6.1 Simulation Setup

Table 1 summarizes the parameters of all factors used in the simulations. This table is placed immediately here so that readers can see the parameter values before interpreting the yield curves.

Table 1: Model Factors and Parameters

Factor	Type	r_0	k	θ	σ	Term Characteristic
1	Vasicek	0.03	0.10	0.04	0.01	Slow, long-term factor
2	Vasicek	0.015	0.70	0.02	0.05	Fast, short-term factor
3	CIR	0.02	0.40	0.06	0.03	Medium-term, state-dependent volatility

Explanation of Factors:

- **Factor 1 (long-term Vasicek):** Slow mean reversion anchors the long end of the curve. Low volatility ensures smooth movement. This factor sets the overall level of the yield curve.
- **Factor 2 (short-term Vasicek):** Fast mean reversion allows short maturities to move independently of the long end. High volatility introduces curvature or humps in the short- and medium-term portion of the curve.
- **Factor 3 (CIR factor):** State-dependent volatility mostly affects medium maturities. Moderate mean reversion and volatility allow the curve to display realistic medium-term humps while preserving a smooth long-term slope.

6.2 Yield Curve Simulation

To illustrate the effect of factor number, we compute the combined yield curve using:

- **One-factor model:** Only the long-term Vasicek factor.
- **Two-factor model:** Long-term Vasicek + short-term Vasicek.
- **Three-factor model:** Two Vasicek factors + CIR factor.

All yields are plotted on the same axes for direct comparison.

6.3 Interpretation of the Results

One-Factor Model: The yield curve is smooth and monotonic. Its slope depends on the long-term mean θ of the Vasicek factor. The single-factor model cannot generate humps or realistic short-term curvature, demonstrating the limitation of one-factor affine models.

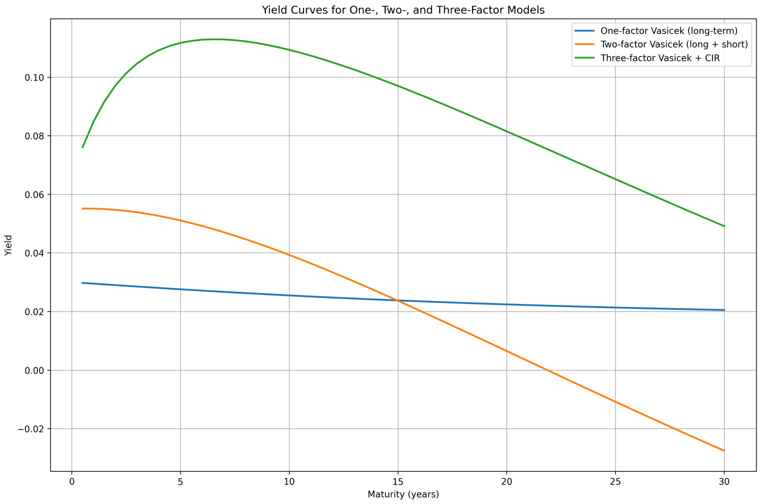

Figure 1: Simulated yield curves for one-factor, two-factor, and three-factor models.

Two-Factor Model: Adding the short-term Vasicek factor introduces curvature, especially at short maturities. The short end of the curve can rise or fall independently of the long end. This shows the interaction between slow long-term and fast short-term factors, producing a more flexible term structure.
Three-Factor Model: Incorporating the CIR factor adds state-dependent volatility, primarily influencing medium-term maturities. The resulting yield curve may display:

- A mild hump around 2–5 years
- A smooth long end dominated by the slow Vasicek factor
- Enhanced short-term flexibility from the fast Vasicek factor

This demonstrates how three-factor models can capture level, slope, and curvature effects simultaneously.

6.4 Summary of Findings

The numerical simulation highlights the role of additional factors:

- Single-factor models control the overall yield level but cannot reproduce realistic term-structure shapes.
- Two-factor models introduce curvature, improving flexibility at short and medium maturities.
- Three-factor models produce realistic humped shapes and allow state-dependent volatility effects to appear.

Overall, this simulation confirms the theoretical motivation for multifactor models: adding factors increases the dimensionality of yield movements, allowing the model to simultaneously capture level, slope, and curvature of the term structure.

7 Analysis of the Zero-Coupon Bond Price in the Combined Multifactor Vasicek–CIR Model

7.1 Model Framework and Reduction to Special Cases

This section presents an analysis of zero-coupon bond pricing and the term structure of spot and forward interest rates within a combined multifactor Vasicek and Cox-Ingersoll-Ross (CIR) model. The model offers a comprehensive framework for capturing interest rate dynamics by incorporating both mean-reverting Gaussian processes (Vasicek) and square-root diffusions (CIR).
The model allows natural reductions:

- Setting $m = 0$ yields a multifactor Vasicek model.
- Setting $n = 0$ yields a multifactor CIR model.
- One-dimensional special cases are included: Vasicek $(n = 1, m = 0)$ and CIR $(n = 0, m = 1)$.

Given these reductions, separate analyses of the multifactor Vasicek and CIR models are omitted, as their properties follow directly from the general model.

7.2 Mathematical Properties of the Discount Function

Let $P(\tau, r)$ denote the price at time t of a zero-coupon bond maturing at $T = t + \tau$, where τ is the time to maturity. This function is commonly referred to as the discount function and is assumed to be free of default risk.

Definition 7.1 (Discount Function)**.** In the combined multifactor Vasicek-CIR model, the zero-coupon bond price is given by

$$P(\tau, r) = \exp\left(\sum_{i=1}^{n} (A_i(\tau) - B_i(\tau)y_i)\right) \exp\left(\sum_{j=1}^{m} (A_j(\tau) - B_j(\tau)z_j)\right)$$

where y_i and z_j are state variables driven by Vasicek and CIR processes, respectively.

Definition 7.2 (Model Parameters)**.** The state variables are mean-reverting, with parameters:

- Mean-reversion speeds $k_i, k_j > 0$, ensuring the rates revert to their long-term means.

- Diffusion coefficients $\sigma_i, \sigma_j > 0$ controlling the volatility.
- Long-term means $\theta_i, \theta_j > 0$, representing equilibrium levels.

Proposition 7.3. *The discount function is strictly positive for all $\tau > 0$:*

$$P(\tau, r) > 0.$$

Proof. Since $P(\tau, r)$ is a product of exponentials of real numbers, and $A_i(\tau), B_i(\tau), A_j(\tau), B_j(\tau)$ are real-valued, it follows directly that $P(\tau, r) > 0$ for all $\tau > 0$. □

Proposition 7.4. *The zero-coupon bond price at maturity equals one:*

$$P(0, r) = \lim_{\tau \to 0} P(\tau, r) = 1.$$

Proof. The zero-coupon bond price in the multifactor Vasicek–CIR model is a product of exponentials involving functions $A_i(\tau), B_i(\tau), A_j(\tau), B_j(\tau)$. Evaluating the limit as $\tau \to 0$ gives:

$$\lim_{\tau \to 0} A_i(\tau) = 0, \quad \lim_{\tau \to 0} B_i(\tau) = 0, \quad \lim_{\tau \to 0} A_j(\tau) = 0, \quad \lim_{\tau \to 0} B_j(\tau) = 0.$$

Therefore, the exponent in the bond pricing formula tends to zero:

$$\lim_{\tau \to 0} P(\tau, r) = \exp(0) = 1.$$

□

Economic interpretation: At maturity, a zero-coupon bond pays its face value, ensuring no-arbitrage consistency.

Proposition 7.5. *The discount function converges to zero as the state variables approach infinity:*

$$\lim_{y_i \to \infty} P(\tau, r) = 0, \quad \lim_{z_j \to \infty} P(\tau, r) = 0.$$

Proof. From the definition, the discount function depends exponentially on linear combinations of the state variables with negative coefficients $-B_i(\tau)$ and $-B_j(\tau)$, where $B_i(\tau), B_j(\tau) > 0$ for all $\tau > 0$.
As $y_i \to \infty$ or $z_j \to \infty$, the corresponding exponent tends to $-\infty$. Hence, the exponential term—and thus the discount function—converges to zero:

$$\lim_{y_i \to \infty} P(\tau, r) = 0, \quad \lim_{z_j \to \infty} P(\tau, r) = 0.$$

□

Economic interpretation: Extremely high interest rates (state variables) drastically reduce the present value of future cash flows. This reflects the intuition that when the opportunity cost of capital becomes very large, bonds are worth almost nothing today.

Proposition 7.6. *The discount function strictly increases as any state variable approaches zero:*

$$\lim_{y_i \to 0} P(\tau, r) > P(\tau, r), \quad \lim_{z_j \to 0} P(\tau, r) > P(\tau, r).$$

Proof. The discount function depends exponentially on linear combinations of the state variables with negative coefficients, $-B_i(\tau)y_i$ and $-B_j(\tau)z_j$, where $B_i(\tau), B_j(\tau) > 0$ for all $\tau > 0$.

$$\lim_{y_i \to 0} P(\tau, r) = \exp\left(\sum_{i=1}^{n} A_i(\tau)\right) \cdot \exp\left(\sum_{j=1}^{m} (A_j(\tau) - B_j(\tau)z_j)\right) > P(\tau, r),$$

$$\lim_{z_j \to 0} P(\tau, r) = \exp\left(\sum_{i=1}^{n} (A_i(\tau) - B_i(\tau)y_i)\right) \cdot \exp\left(\sum_{j=1}^{m} A_j(\tau)\right) > P(\tau, r). \quad \square$$

Economic interpretation: Lower interest rates (state variables) increase bond prices, reflecting the higher present value of future cash flows when the opportunity cost of capital is small.

Proposition 7.7. *The zero-coupon bond price function $P(\tau, r)$ is strictly decreasing and convex with respect to each state variable y_i and z_j. Specifically:*

- *First-order partial derivatives are negative:*

$$\frac{\partial P(\tau, r)}{\partial y_i} = -B_i(\tau)P(\tau, r) < 0, \quad i = 1, \ldots, n,$$

$$\frac{\partial P(\tau, r)}{\partial z_j} = -B_j(\tau)P(\tau, r) < 0, \quad j = 1, \ldots, m.$$

- *Second-order partial derivatives are positive:*

$$\frac{\partial^2 P(\tau, r)}{\partial y_i^2} = B_i^2(\tau)P(\tau, r) > 0, \quad \frac{\partial^2 P(\tau, r)}{\partial z_j^2} = B_j^2(\tau)P(\tau, r) > 0,$$

$$\frac{\partial^2 P(\tau, r)}{\partial y_i \partial y_k} = B_i(\tau)B_k(\tau)P(\tau, r) > 0, \quad i \neq k,$$

$$\frac{\partial^2 P(\tau, r)}{\partial y_i \partial z_j} = B_i(\tau)B_j(\tau)P(\tau, r) > 0,$$

$$\frac{\partial^2 P(\tau, r)}{\partial z_j \partial z_v} = B_j(\tau)B_v(\tau)P(\tau, r) > 0, \quad j \neq v.$$

Proof. Since $P(\tau, r)$ is exponential–affine with negative coefficients $-B_i(\tau)$ and $-B_j(\tau)$, and with $B_i(\tau), B_j(\tau) > 0$, the first derivatives follow immediately from differentiating the exponential and are strictly negative. Differentiating once more yields expressions proportional to $P(\tau, r)$ multiplied by squared or cross-products of $B_i(\tau)$ and $B_j(\tau)$, all of which are positive. Thus monotonicity and convexity hold as stated. $\square$

Economic interpretation: Bond prices decrease as interest rates increase, but the rate of decrease diminishes for higher rates. Convexity reflects this non-linear sensitivity, which is important for interest rate risk management.

Proposition 7.8. *The zero-coupon bond price $P(\tau, r)$ converges to zero as the time to maturity τ tends to infinity:*

$$\lim_{\tau\to\infty} P(\tau, r) = 0.$$

Proof. The bond price in the multifactor Vasicek–CIR model is

$$P(\tau, r) = \exp\left(\sum_{i=1}^{n} \big(A_i(\tau) - B_i(\tau)y_i\big)\right) \exp\left(\sum_{j=1}^{m} \big(A_j(\tau) - B_j(\tau)z_j\big)\right).$$

For the Vasicek factors (i=1,…,n), under the standard assumption $2\psi_i^2\hat{\theta}_i > \sigma_i^2$:

$$\lim_{\tau\to\infty} A_i(\tau) = -\infty, \quad \lim_{\tau\to\infty} B_i(\tau) = \frac{1}{\psi_i} > 0.$$

Thus,

$$\lim_{\tau\to\infty} \big(A_i(\tau) - B_i(\tau)y_i\big) = -\infty.$$

Similarly, for the CIR factors $j = 1, \dots, m$, under the assumption $2\psi_j'\hat{\theta}_j' > \sigma_j'^2$ (so that $\psi_j' - \xi_j < 0$):

$$\lim_{\tau\to\infty} A_j(\tau) = -\infty, \quad \lim_{\tau\to\infty} B_j(\tau) = \frac{2}{\psi_j' + \xi_j} > 0,$$

implying

$$\lim_{\tau\to\infty} \big(A_j(\tau) - B_j(\tau)z_j\big) = -\infty.$$

Combining these results, the total exponent in the bond price formula tends to $-\infty$:

$$\lim_{\tau\to\infty} \sum_{i=1}^{n} \big(A_i(\tau) - B_i(\tau)y_i\big) + \sum_{j=1}^{m} \big(A_j(\tau) - B_j(\tau)z_j\big) = -\infty.$$

Consequently, the bond price itself converges to zero:

$$\lim_{\tau\to\infty} P(\tau, r) = \exp(-\infty) = 0. \qquad \square$$

Economic interpretation: As the maturity becomes very large, future cash flows are heavily discounted, leading the present value of the bond to approach zero.

Proposition 7.9. *The zero-coupon bond price $P(\tau, r)$ is strictly decreasing in the time to maturity τ. Moreover, it satisfies the boundary conditions:*

$$\lim_{\tau\to 0} P(\tau, r) = 1, \quad \lim_{\tau\to\infty} P(\tau, r) = 0.$$

Proof. The bond price in the multifactor Vasicek–CIR model is

$$P(\tau, r) = \exp\left(\sum_{i=1}^{n} \left(A_i(\tau) - B_i(\tau) y_i\right)\right) \exp\left(\sum_{j=1}^{m} \left(A_j(\tau) - B_j(\tau) z_j\right)\right).$$

Differentiating with respect to τ:

$$\frac{\partial P(\tau, r)}{\partial \tau} = \left(\sum_{i=1}^{n} \left(A_i'(\tau) - B_i'(\tau) y_i\right) + \sum_{j=1}^{m} \left(A_j'(\tau) - B_j'(\tau) z_j\right)\right) P(\tau, r).$$

It suffices to show that each term in the parentheses is negative:

- For Vasicek factors ($i = 1, \ldots, n$):

  $$B_i'(\tau) = e^{-\psi_i \tau} > 0, \quad A_i'(\tau) = -\hat{\theta}_i (1 - e^{-\psi_i \tau}) + \frac{\sigma_i^2}{2\psi_i^2} (1 - e^{-\psi_i \tau})^2 < 0,$$

  under the standard assumption $2\psi_i \hat{\theta}_i > \sigma_i^2$.

- For CIR factors ($j = 1, \ldots, m$):

  $$B_j'(\tau) > 0, \quad A_j'(\tau) = \frac{\psi_j' \hat{\theta}_j' (\psi_j'^2 - \xi_j^2)(e^{\tau \xi_j} - 1)}{\sigma_j'^2 ((\psi_j' + \xi_j)(e^{\tau \xi_j} - 1) + 2\xi_j)} < 0,$$

  since $\xi_j^2 > \psi_j'^2$.

Hence, the derivative $\frac{\partial P}{\partial \tau} < 0$ for all $\tau > 0$, confirming that the bond price is strictly decreasing in τ.
The boundary conditions follow from standard properties of zero-coupon bonds:

$$\lim_{\tau \to 0} P(\tau, r) = 1, \quad \lim_{\tau \to \infty} P(\tau, r) = 0. \qquad \square$$

Economic interpretation: As maturity increases, the present value of the bond decreases due to discounting of future cash flows. Convexity in τ ensures that the rate of decrease diminishes as maturity becomes very large, reflecting typical interest rate behavior.

Based on the analysis of the price of the zero-coupon bond in the combined multifactor Vasicek and CIR model, the following conclusions can be drawn:

(i) The discount function $P(\tau, r)$ is a positive decreasing function, with values ranging from 1 to 0:

$$P(0, r) = 1, \quad P(\infty, r) = 0.$$

(ii) The behavior of the components of the discount function — $A_i(\tau)$, $B_i(\tau)$, $A_j(\tau)$, and $B_j(\tau)$ — as functions of maturity τ, is summarized in Table 2.

(iii) The first derivatives of these components with respect to τ are presented in Table 3, which summarizes their behavior.

Table 2: Values of the functions $A_i(\tau)$, $B_i(\tau)$, $A_j(\tau)$ and $B_j(\tau)$ in the combined multifactor Vasicek and CIR model

	$\tau > 0$	$\tau \to 0$	$\tau \to \infty$
$A_i(\tau)$	$A_i(\tau) > 0$	0	$-\infty$
$B_i(\tau)$	$B_i(\tau) > 0$	0	$\frac{1}{\psi_i} > 0$
$A_j(\tau)$	$A_j(\tau) > 0$	0	$-\infty$
$B_j(\tau)$	$B_j(\tau) > 0$	0	$\frac{2}{\psi_j' + \xi_j} > 0$

Table 3: Values of the first derivatives by τ of functions $A_i(\tau)$, $B_i(\tau)$, $A_j(\tau)$ and $B_j(\tau)$ in the combined multifactor Vasicek and CIR model

	$\tau > 0$	$\tau \to 0$	$\tau \to \infty$
$A_i'(\tau)$	$A_i'(\tau) < 0$	0	$\frac{\sigma_i^2}{2\psi_i^2} - \hat{\theta}_i$
$B_i'(\tau)$	$B_i'(\tau) > 0$	1	0
$A_j'(\tau)$	$A_j'(\tau) < 0$	0	$\frac{\psi_j' \hat{\theta}_j'}{\sigma_j'^2} \left(\psi_j' - \xi_j\right)$
$B_j'(\tau)$	$B_j'(\tau) > 0$	1	0

7.3 Spot Interest Rate Curve Analysis within the Combined Multifactor Vasicek and CIR Model

The term structure of interest rates represents the relationship between the interest rates and the maturities of debt instruments, such as zero-coupon bonds. It can be expressed equivalently in terms of bond prices, spot interest rates, or forward interest rates. These quantities are mathematically linked through the fundamental identity between the bond price, the spot rate, and the forward rate, as shown below.

To analyze the spot interest rate curve within this framework, we adopt a combined multifactor Vasicek–CIR model. The following assumptions ensure mathematical tractability while preserving economic realism: (i) the short rate is driven by n Vasicek factors and m CIR factors; (ii) all factors are independent, mean-reverting stochastic processes; (iii) the model parameters $\psi_i, \hat{\theta}_i, \sigma_i$ for Vasicek and $\psi_j', \hat{\theta}_j', \sigma_j', \xi_j$ for CIR are constant, strictly positive, and economically interpretable in terms of mean-reversion speed, long-term mean, and volatility; (iv) state variables $y_i, z_j \geq 0$, ensuring non-negativity for CIR components.

These assumptions provide mathematical tractability and economic plausibility, and allow the derivation of closed-form expressions for zero-coupon bond prices.

Definition 7.10. Let $P(\tau, r)$ denote the price at time t of a zero-coupon bond maturing at $T = t + \tau$, where r is the short rate. Then the bond price satisfies the identity:

$$P(\tau, r) = \exp\left[-\int_t^T f(t, r, u), du\right] = \exp\left[-R(t, r, T)(T-t)\right]. \tag{6}$$

where $f(t, r, u)$ is the instantaneous forward rate and $R(t, r, T)$ is the spot interest rate.

Proposition 7.11 (Spot Rate as Average Forward Rate)**.** *The spot interest rate $R(t, r, T)$ is given by the average of the instantaneous forward rate over the maturity interval:*

$$R(t, r, T) = \frac{1}{T-t}\int_t^T f(t, r, u), du. \tag{7}$$

This representation ensures consistency between bond pricing and interest rate definitions in the hybrid Vasicek–CIR framework.

Proof. Taking the natural logarithm of both sides of Equation (6) gives

$$\ln P(\tau, r) = -\int_t^T f(t, r, u), du = -R(t, r, T)(T-t), \tag{8}$$

from which isolating $R(t, r, T)$ yields the stated result. This derivation demonstrates the internal consistency of the pricing framework and links the spot rate to forward rates explicitly. □

Definition 7.12 (Spot Rate from Bond Price)**.** Assuming that the spot rate depends only on time to maturity τ, an alternative expression is

$$R(\tau, r) = -\frac{1}{\tau}\ln P(\tau, r).$$

This form is particularly useful for deriving closed-form expressions in the multifactor Vasicek–CIR model.

Theorem 7.13 (Spot Rate under Combined Vasicek and CIR Factors)**.** *Consider a short rate model composed of n Vasicek factors y_i and m CIR factors z_j. The zero-coupon bond price in affine form is*

$$P(\tau, r) = \exp\left(\sum_{i=1}^{n}(A_i(\tau) - B_i(\tau)y_i)\right)\exp\left(\sum_{j=1}^{m}(A_j(\tau) - B_j(\tau)z_j)\right), \tag{9}$$

where $A_i(\tau), B_i(\tau)$ correspond to Vasicek factors and $A_j(\tau), B_j(\tau)$ to CIR factors, capturing mean-reversion, volatility, and long-term behavior of the short rate.
Then the spot interest rate is given by

$$\begin{aligned} R(\tau, r) &= -\frac{1}{\tau}\ln P(\tau, r) \\ &= -\frac{1}{\tau}\sum_{i=1}^{n}A_i(\tau) + \frac{1}{\tau}\sum_{i=1}^{n}B_i(\tau)y_i - \frac{1}{\tau}\sum_{j=1}^{m}A_j(\tau) + \frac{1}{\tau}\sum_{j=1}^{m}B_j(\tau)z_j. \end{aligned}$$

This expression clearly demonstrates how the spot rate integrates the effects of both Vasicek and CIR factors, with each component contributing linearly through the affine functions $B_i(\tau), B_j(\tau)$*.*

Proof. The bond price is expressed as the exponential of a sum of affine functions of the state variables. Applying the logarithm and dividing by $-\tau$ directly yields the stated formula for $R(\tau, r)$.
Economically, this shows that the spot rate is a weighted combination of the short-rate factors, with the weights determined by the functions $B_i(\tau), B_j(\tau)$. Mathematically, the affine structure ensures closed-form solutions, providing both tractability and consistency of the hybrid Vasicek–CIR pricing framework. □

Based on the results in Tables 2 and 3, the following properties of the spot rate curve are derived in the multifactor Vasicek–CIR framework.

Theorem 7.14 (Short-Maturity Limit of Spot Rate)**.** *As* $\tau \to 0$*, the spot rate converges to the instantaneous short rate:*

$$\lim_{\tau \to 0} R(\tau, r) = r(t) = \sum_{i=1}^{n} y_i + \sum_{j=1}^{m} z_j. \tag{10}$$

This reflects that at very short maturities, the spot rate is dominated by the current values of the underlying short-rate factors.

Proof. Using the affine expression for $R(\tau, r)$ and applying L'Hôpital's rule:

$$\lim_{\tau \to 0} R(\tau, r) = \lim_{\tau \to 0} \left(-\sum_{i=1}^{n} A_i'(\tau) + \sum_{i=1}^{n} B_i'(\tau) y_i - \sum_{j=1}^{m} A_j'(\tau) + \sum_{j=1}^{m} B_j'(\tau) z_j \right). \tag{11}$$

From Table 3, the derivatives at zero satisfy

$$A_i'(0) = 0, \quad B_i'(0) = 1, \quad A_j'(0) = 0, \quad B_j'(0) = 1. \tag{12}$$

Substituting these values gives

$$\lim_{\tau \to 0} R(\tau, r) = \sum_{i=1}^{n} y_i + \sum_{j=1}^{m} z_j = r(t), \tag{13}$$

confirming consistency with the instantaneous short rate. □

Theorem 7.15 (Long-Maturity Limit of Spot Rate)**.** *As* $\tau \to \infty$*, the spot rate converges to a deterministic value that depends only on the long-term parameters of the Vasicek and CIR factors:*

$$\lim_{\tau \to \infty} R(\tau, r) = \sum_{i=1}^{n} \left(\hat{\theta}_i - \frac{\sigma_i^2}{2\psi_i^2} \right) + \sum_{j=1}^{m} \frac{\psi_j' \hat{\theta}_j'}{\sigma_j'^2} (\xi_j - \psi_j'). \tag{14}$$

This illustrates that the term structure flattens at long maturities, reflecting only the long-term model parameters.

Proof. Using the asymptotic behavior of the affine functions as $\tau \to \infty$:

$$\lim_{\tau\to\infty} B_i(\tau) = 0, \quad \lim_{\tau\to\infty} B_j(\tau) = 0, \tag{15}$$

$$\lim_{\tau\to\infty} A_i'(\tau) = -\left(\hat{\theta}_i - \frac{\sigma_i^2}{2\psi_i^2}\right), \quad \lim_{\tau\to\infty} A_j'(\tau) = -\frac{\psi_j'\hat{\theta}_j'}{\sigma_j'^2}(\xi_j - \psi_j'). \tag{16}$$

Substituting these limits into the expression for $R(\tau, r)$ confirms convergence to a deterministic long-term spot rate. □

Corollary 7.16. *The spot rate curve becomes flat at long maturities, reflecting only long-term parameters of the model, which is consistent with the economic interpretation of risk-neutral expectations.*

Lemma 7.17 (Monotonicity in State Variables)**.** *The spot rate is a strictly increasing linear function of each state variable:*

$$\frac{\partial R(\tau, r)}{\partial y_i} = \frac{B_i(\tau)}{\tau} > 0, \quad \frac{\partial^2 R(\tau, r)}{\partial y_i^2} = 0, \quad \frac{\partial R(\tau, r)}{\partial z_j} = \frac{B_j(\tau)}{\tau} > 0, \quad \frac{\partial^2 R(\tau, r)}{\partial z_j^2} = 0. \tag{17}$$

This ensures that the spot rate responds positively to increases in the underlying short-rate factors.

Proof. Differentiating the affine expression for $R(\tau, r)$ with respect to y_i and z_j yields the stated results. Since $B_i(\tau), B_j(\tau) > 0$ by model construction, the spot rate is monotonically increasing in all state variables. □

Corollary 7.18. *The sensitivity of the spot rate to short-rate factors is time-dependent but always positive, preserving monotonicity with respect to economic or monetary shocks.*

Forward Interest Rate Curve Analysis within the Combined Multifactor Vasicek and CIR Model

In this section, we analyze the forward interest rate curve implied by the multifactor hybrid Vasicek–CIR model, emphasizing both mathematical consistency and economic interpretation.

Definition 7.19 (Forward Interest Rate)**.** Let $P(\tau, r)$ denote the price at time t of a zero-coupon bond maturing at time $t + \tau$. The forward interest rate $f(\tau, r)$ is defined as:

$$f(\tau, r) = -\frac{\partial_\tau P(\tau, r)}{P(\tau, r)}. \tag{18}$$

This definition assumes a continuously compounded rate and reflects the instantaneous cost of borrowing for a short period starting at maturity τ.

Lemma 7.20 (Affine Representation of Forward Rate)**.** *If the zero-coupon bond price admits the affine form*

$$P(\tau, r) = \exp\left(\sum_{i=1}^{n} A_i(\tau) - \sum_{i=1}^{n} B_i(\tau) y_i + \sum_{j=1}^{m} A_j(\tau) - \sum_{j=1}^{m} B_j(\tau) z_j\right), \tag{19}$$

then the forward interest rate can be expressed as

$$f(\tau, r) = -\sum_{i=1}^{n} A_i'(\tau) + \sum_{i=1}^{n} B_i'(\tau) y_i - \sum_{j=1}^{m} A_j'(\tau) + \sum_{j=1}^{m} B_j'(\tau) z_j. \tag{20}$$

Proof. Differentiating the exponential form of the bond price with respect to τ and applying Definition (18) yields the stated result. This derivation is consistent with the affine term structure framework, which integrates Vasicek factors (allowing for negative rates and mean reversion) and CIR factors (ensuring positivity of rates). □

Proposition 7.21 (Positivity of the Forward Rate)**.** *For all $\tau > 0$, the forward rate is strictly positive:*

$$f(\tau, r) > 0. \tag{21}$$

Proof. From Table 3, we have

$$A_i'(\tau) < 0, \quad A_j'(\tau) < 0, \quad B_i'(\tau) > 0, \quad B_j'(\tau) > 0, \quad \forall \tau > 0. \tag{22}$$

Substituting into Equation (20), each term is positive either due to $-A_i'(\tau), -A_j'(\tau)$ or due to $B_i'(\tau) y_i, B_j'(\tau) z_j \geq 0$, yielding overall positivity of $f(\tau, r)$. This ensures economically meaningful forward rates under the hybrid model. □

Theorem 7.22 (Short-Maturity Limit of Forward Rate)**.** *As $\tau \to 0$, the forward rate converges to the instantaneous short rate:*

$$\lim_{\tau \to 0} f(\tau, r) = r(t) = \sum_{i=1}^{n} y_i + \sum_{j=1}^{m} z_j. \tag{23}$$

This shows consistency with the definition of the short rate in the hybrid Vasicek–CIR model.

Proof. From Table 3, as $\tau \to 0$, the limits are

$$A_i'(\tau) \to 0, \quad A_j'(\tau) \to 0, \quad B_i'(\tau) \to 1, \quad B_j'(\tau) \to 1. \tag{24}$$

Substituting these into Equation (20) gives

$$\lim_{\tau \to 0} f(\tau, r) = \sum_{i=1}^{n} y_i + \sum_{j=1}^{m} z_j = r(t), \tag{25}$$

demonstrating that the forward rate converges to the observed short rate. □

Theorem 7.23 (Asymptotic Limit of Forward Rate)**.** *As $\tau \to \infty$, the forward rate converges to a deterministic value independent of current state variables:*

$$\lim_{\tau\to\infty} f(\tau, r) = \sum_{i=1}^{n}\left(\hat{\theta}_i - \frac{\sigma_i^2}{2\psi_i^2}\right) + \sum_{j=1}^{m}\frac{\psi_j'\hat{\theta}_j'}{\sigma_j'^2}(\xi_j - \psi_j'). \tag{26}$$

This matches the long-term limit of the spot rate, reflecting only the long-term mean-reverting parameters of the Vasicek and CIR components.

Proof. We start from the affine representation of the forward rate,

$$f(\tau, r) \;=\; -\sum_{i=1}^{n} A_i'(\tau) + \sum_{i=1}^{n} B_i'(\tau)\, y_i - \sum_{j=1}^{m} A_j'(\tau) + \sum_{j=1}^{m} B_j'(\tau)\, z_j.$$

As $\tau \to \infty$, the coefficients $B_i'(\tau)$ and $B_j'(\tau)$ vanish, while the derivatives $A_i'(\tau)$ and $A_j'(\tau)$ approach finite constants. More precisely,

$$A_i'(\tau) \to \frac{\sigma_i^2}{2\psi_i^2} - \hat{\theta}_i, \quad A_j'(\tau) \to \frac{\psi_j'\hat{\theta}_j'}{\sigma_j'^2}\,(\xi_j - \psi_j'), \quad B_i'(\tau) \to 0, \quad B_j'(\tau) \to 0.$$

Substituting these limits into the affine expression yields the deterministic limit

$$\lim_{\tau\to\infty} f(\tau, r) = \sum_{i=1}^{n}\left(\hat{\theta}_i - \frac{\sigma_i^2}{2\psi_i^2}\right) + \sum_{j=1}^{m}\frac{\psi_j'\hat{\theta}_j'}{\sigma_j'^2}\,(\xi_j - \psi_j').$$

Interpretation of the limit terms. The long-maturity forward rate is independent of the current state variables y_i and z_j because the sensitivity coefficients $B_i'(\tau)$ and $B_j'(\tau)$ decay to zero; economically, this means that idiosyncratic short-term shocks have vanishing impact on the expected cost of borrowing at very long horizons. Only the models long-run parameters remain relevant in the limit, so the forward curve "flattens" and reflects long-run equilibrium levels.
For each Vasicek factor i, the contribution

$$\hat{\theta}_i - \frac{\sigma_i^2}{2\psi_i^2}$$

is the long-run mean $\hat{\theta}_i$ corrected by a volatility (convexity) adjustment $\frac{\sigma_i^2}{2\psi_i^2}$. Economically, the term $\hat{\theta}_i$ captures the factors mean-reverting equilibrium (e.g., long-run macroeconomic or policy level). The subtraction of the volatility correction reflects that under continuous compounding and the affine pricing measure, higher volatility reduces the long-term forward rate through a convexity/second-moment effect; that is, uncertainty lowers the long-horizon expected continuously compounded rate coming from Gaussian-type (Vasicek) components.
For each CIR factor j, the limit term

$$\frac{\psi_j'\hat{\theta}_j'}{\sigma_j'^2}\,(\xi_j - \psi_j'), \qquad \text{with } \xi_j = \sqrt{\psi_j'^2 + 2\sigma_j'^2},$$

encodes the nonlinear mean-reversion and volatility interaction characteristic of square-root dynamics. Economically, $\hat{\theta}'_j$ is the long-run level of the strictly positive factor (e.g., a liquidity or inflation premium), ψ'_j measures the speed of reversion, and σ'_j its volatility. The combination $(\xi_j - \psi'_j) > 0$ captures how the CIR-type noise inflates the long-run contribution relative to pure mean reversion; in short, the CIR contribution is positive and depends on both mean levels and the strength of diffusion in a way that preserves positivity of the limit.

Economic summary. Taken together, the limiting expression shows that long-term expected short-term rates are determined by a weighted sum of long-run level parameters of the Vasicek and CIR components, adjusted for volatility and model nonlinearities. This is economically intuitive: at very long horizons the term structure is governed by persistent structural forces (policy or fundamental levels) and by how uncertainty (volatility) and model nonlinearity (square-root effects) modify those forces, while transient shocks to the current short-rate factors have negligible influence.

This conclusion confirms internal consistency of the hybrid affine construction: the forward curve at long maturities is deterministic, finite, and interpretable in terms of economically meaningful parameters, which also facilitates empirical calibration and comparative statics. □

Corollary 7.24 (Monotonic Sensitivity to State Variables)**.** *The forward rate increases linearly with respect to each short-rate factor:*

$$\frac{\partial f(\tau,r)}{\partial y_i} = B'_i(\tau) > 0, \quad \frac{\partial^2 f(\tau,r)}{\partial y_i^2} = 0, \quad \frac{\partial f(\tau,r)}{\partial z_j} = B'_j(\tau) > 0, \quad \frac{\partial^2 f(\tau,r)}{\partial z_j^2} = 0. \tag{27}$$

This implies that the forward curve reacts positively to economic or monetary shocks, with sensitivity decreasing over time as captured by $B'_i(\tau)$ and $B'_j(\tau)$.

Proof. Direct differentiation of Equation (20) with respect to y_i and z_j yields the linear sensitivities and zero second derivatives, confirming monotonicity and providing clear economic intuition regarding risk factor impacts. □

8 Conclusion

In this manuscript, we analyzed the zero-coupon bond price, the spot interest rate curve, and the forward rate curve within a combined multifactor Vasicek–Cox–Ingersoll–Ross (CIR) framework. The study began by formulating a hybrid model that incorporates both Gaussian (Vasicek) and square-root (CIR) mean-reverting processes, demonstrating that this general specification naturally reduces to well-known models as special cases, including multifactor Vasicek, multifactor CIR, and their one-dimensional versions.

Based on the affine representation of the bond price, we established several fundamental mathematical properties of the discount function. Specifically, we proved its strict positivity, continuity, and key limit behaviors: $P(0,r) = 1$, $P(\tau,r) \to 0$ as either maturity $\tau \to \infty$ or state variables tend to infinity, and monotonicity and convexity with respect to each Vasicek and CIR state variable. These results formalize the dependence of bond prices on interest rate factors and confirm economically intuitive relationships—higher interest-rate factors reduce the bond price, while lower factors increase it. Moreover, the zero-coupon bond price is strictly decreasing in the time to maturity, satisfying boundary conditions consistent with no-arbitrage pricing theory.

Using these structural properties, we derived closed-form expressions for the spot interest rate. The spot rate is an affine and strictly increasing function of each state variable, converging to the instantaneous short rate for short maturities and to a deterministic long-term value for long maturities, determined solely by model parameters. This confirms that the combined Vasicek–CIR model produces economically consistent term structures that flatten at long maturities.

In the forward rate analysis, we expressed the forward rate curve explicitly through the derivatives of the affine functions $A(\tau)$ and $B(\tau)$, showing that its behavior is consistent with the properties of the discount and spot rate functions. This provides a unified understanding of how Vasicek and CIR components jointly shape the slope and curvature of the term structure.

Importantly, compared to classical single-factor Vasicek or CIR models, the hybrid multifactor Vasicek–CIR framework simultaneously accommodates negative and strictly positive rate components, capturing a wider range of empirical yield curve shapes. By nesting existing models as special cases, it offers enhanced flexibility for modeling and fitting observed term structures. The affine structure ensures analytical tractability, allowing for closed-form solutions and clear economic interpretation.

Overall, the manuscript provides a comprehensive mathematical characterization of zero-coupon bond prices, spot rates, and forward rates within the hybrid multifactor Vasicek–CIR framework. By deriving both general properties and asymptotic results, the analysis illustrates the tractability and flexibility of the model, while demonstrating its ability to capture key features of interest rate dynamics observed in practice. These findings offer a solid foundation for future applications, including yield curve estimation, risk management, and pricing of fixed-income derivatives within a unified affine modeling framework.

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