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A Heston Fractional Vasicek Framework for Option Pricing

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Abstract:
We develop an option-pricing framework that couples the Heston stochastic volatility model with a fractional Vasicek short-rate process to incorporate long-memory effects in interest rates. Using a regularized semimartingale approximation of fractional Brownian motion and an affine surrogate representation, we derive a tractable pricing PDE and a semi-closed-form characteristic function suitable for Fourier-based valuation. The numerical implementation employs adaptive integration for fast and accurate pricing, and sensitivity analysis highlights the role of the memory parameter α, the smoothing term ε, and the short-rate volatility σ_r. Empirical calibration to S&P 500 option data demonstrates that the proposed model improves the fit to market prices relative to the classical Heston model, particularly for longer maturities. These results indicate that persistent interest-rate dynamics can materially influence equity option valuation and motivate further development of fractional interest-rate modelling.

Keywords: Heston stochastic volatility, Fractional Vasicek process, Option pricing, Fourier transform methods
Classification: 91G30, 91G60.

1 Introduction

Modern financial markets are increasingly shaped by the joint stochastic dynamics of asset prices, volatility, and interest rates. Among the most influential models for capturing such interactions is the Heston stochastic volatility framework [9], which models the variance of asset returns as a mean-reverting square-root process and accounts for empirically observed phenomena such as volatility clustering and the leverage effect. While the Heston model has been extended in various directions to improve empirical fit [5, 8, 12, 13, 17], many implementations retain the simplifying assumption of constant interest rates or rely on short-memory dynamics such as the Vasicek or CIR models when incorporating stochastic rates.

A growing body of empirical research indicates, however, that short-term interest rates frequently exhibit long-range dependence, meaning that their autocorrelations decay at a hyperbolic rather than exponential rate. This persistence implies that shocks to interest rates may exert influence over extended horizons, a feature that

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standard Markovian specifications fail to capture. Evidence for such long-memory behaviour has been documented across maturities, currencies, and market regimes: [11] detect multifractal persistence in interest rate futures; [2] identify fractional integration patterns in real interest rate differentials over 150 years; [1] note persistent inflation uncertainty impacting the term structure; [3] find strong persistence in European monetary policy spreads; and [15] show that real interest rates display structural breaks and centuries-long persistence. These findings underscore the relevance of incorporating long-memory features into models used for pricing interest rate–sensitive derivatives.

One natural approach to modelling such persistence is to drive the short rate with a fractional Brownian motion (fBm), leading to a fractional Vasicek process [14, 16, 19]. The fBm-driven specification enriches the autocorrelation structure of the short rate while preserving a familiar SDE framework. The degree of memory is governed by the parameter $\alpha \in (0, 0.5)$, corresponding to a Hurst index $H = \alpha + 0.5$, which offers a transparent interpretation of persistence intensity.

In this paper, we propose a hybrid model that couples the Heston stochastic volatility dynamics with an fBm-driven, fractional Vasicek-type short rate. By introducing a fractional component into the interest-rate process, the model aims to capture the persistent effects of monetary policy and macroeconomic conditions documented in empirical studies. Our framework builds on [7], which integrated a classical Vasicek rate into the Heston model, and extends it by incorporating fractional characteristics.

Importantly, the objective of this paper is *not* to derive pricing formulas for the exact fractional Vasicek model, whose non-Markovian, Volterra-type structure necessitates an infinite-dimensional state augmentation to obtain a rigorous generator. Instead, our aim is to construct a *computationally efficient, approximately affine surrogate model* obtained through a regularized (semi-martingale) representation of fBm and an affine approximation analogous to those used in hybrid Heston interest-rate models. This approximation intentionally restores tractability, enabling a semi-closed-form characteristic function and fast Fourier-based pricing. The resulting pricing formulas are therefore mathematically valid *for the approximate affine model*, whose role is to provide a practical and numerically efficient framework for incorporating long-memory effects into derivative valuation. All pricing results in this paper apply to the regularized affine surrogate model and therefore are fully compatible with the classical arbitrage-free, risk-neutral framework. Since the surrogate short-rate process is a semimartingale and Markovian, standard Feynman–Kac arguments apply without modification. Our goal is therefore not to construct a full interest-rate derivative calibration framework, but to study how persistent short-rate dynamics affect equity option valuation.

Within this approximate setting, we derive a semi-analytical expression for the characteristic function of the joint dynamics, facilitating efficient European option pricing via Fourier inversion techniques [6]. We also develop a numerical implementation based on adaptive integration and conduct a detailed sensitivity analysis to quantify the influence of the memory parameter α, the smoothing parameter ϵ, and the short-rate volatility σ_r. Finally, we illustrate the practical estimation of the memory parameter using the Whittle likelihood, a frequency-domain method well suited for long-memory Gaussian processes.

The remainder of the paper is organized as follows. Section 2 reviews the mathematical background on fractional Brownian motion and its smoothing approximation. Section 3 presents the joint Heston–fractional Vasicek model. Section 4 derives the option pricing formula and the associated characteristic function. Section 5 describes the numerical scheme, validates it via convergence tests, and reports simulation results. Section 6 concludes and outlines directions for future research.

2 Preliminaries

Fractional Brownian motion (fBm) with Hurst parameter $0 < H < 1$ is a continuous, centered Gaussian process that generalizes classical Brownian motion by capturing long–range dependence and self–similarity. Denoting the process by $B_H = \{B_H(t),\, 0 \le t \le T\}$, its covariance structure is given by

$$R_H(t,s) = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big), \qquad t,s \in [0,T].$$

For $H = 0.5$, fBm reduces to standard Brownian motion. When $H > 0.5$, the increments exhibit positive long–range dependence, a feature that makes fBm particularly relevant for modeling persistent temporal structures in financial time series, including interest rates.

A significant challenge, however, is that B_H is not a semimartingale when $H \neq 0.5$. Consequently, Itô calculus does not apply in its classical form, complicating any attempt to incorporate B_H directly into arbitrage-free asset–pricing frameworks.

To circumvent this issue, one strategy is to approximate fBm by a process that belongs to the semimartingale class while retaining its long-memory characteristics in an L^2 sense. A widely used representation expresses fBm as

$$B_H(t) = \frac{1}{\Gamma(H+\frac{1}{2})}\left(U_t + \int_0^t (t-s)^{H-\frac{1}{2}}\, \mathrm{d}W_s\right),$$

where W is a standard Brownian motion and U_t is an absolutely continuous process. The Volterra integral term captures the persistent autocorrelation structure of fBm.

To obtain a tractable semimartingale approximation, Thao [18] introduced the regularized process

$$B_{H,\varepsilon}(t) = \int_0^t (t-s+\varepsilon)^{H-\frac{1}{2}}\, \mathrm{d}W_s, \qquad \varepsilon > 0.$$

Define $\alpha = H - \frac{1}{2}$. The kernel $K_\varepsilon(t,s) := (t-s+\varepsilon)^\alpha$ is smooth in t, and the process $B_{H,\varepsilon}(t) = \int_0^t K_\varepsilon(t,s)\mathrm{d}W_s$ falls within the class of Volterra Gaussian semimartingales.

Since $\partial_t K_\varepsilon(t,s) = \alpha(t-s+\varepsilon)^{\alpha-1} \in L^2$ for every fixed $\varepsilon > 0$, the stochastic Fubini theorem yields the Itô differential (see, e.g., standard results on Volterra stochastic integrals):

$$\mathrm{d}\left(\int_0^t K_\varepsilon(t,s)\,\mathrm{d}W_s\right) = \left(\int_0^t \partial_t K_\varepsilon(t,s)\,\mathrm{d}W_s\right)\mathrm{d}t + K_\varepsilon(t,t)\,\mathrm{d}W_t.$$

Substituting the explicit kernel derivatives,

$$\partial_t K_\varepsilon(t,s) = \alpha(t-s+\varepsilon)^{\alpha-1}, \qquad K_\varepsilon(t,t) = \varepsilon^\alpha,$$

we obtain the Itô differential of the regularized fBm:

$$\mathrm{d}B_{H,\varepsilon}(t) = \alpha\left(\int_0^t (t-s+\varepsilon)^{\alpha-1}\,\mathrm{d}W_s\right)\mathrm{d}t + \varepsilon^\alpha\,\mathrm{d}W_t.$$

Introducing the auxiliary Volterra process

$$\varphi_t^\varepsilon := \int_0^t (t-s+\varepsilon)^{\alpha-1}\,\mathrm{d}W_s,$$

we finally arrive at the semimartingale representation:

$$\mathrm{d}B_{H,\varepsilon}(t) = \alpha\,\varphi_t^\varepsilon\,\mathrm{d}t + \varepsilon^\alpha\,\mathrm{d}W_t.$$

In the remainder of the paper, we adopt this semimartingale approximation of fBm for modeling interest-rate dynamics with long–range dependence. This formulation enables the use of classical Itô calculus under the risk-neutral measure, facilitates numerical simulation, and avoids the theoretical difficulties associated with the non-semimartingale nature of genuine fBm.

3 Model Setup

We consider a financial market over a finite time horizon $[0,T]$, consisting of a risky asset with price process S_t and a stochastic short-term interest rate process r_t. The dynamics of the asset price are governed by a generalized Heston stochastic volatility model, while the interest rate follows a fractional Vasicek process approximated via a semi–martingale formulation.

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. The asset price process $S = \{S_t,\ 0 \leq t \leq T\}$ satisfies the following stochastic differential equation (SDE):

$$\mathrm{d}S_t = r_t S_t\,\mathrm{d}t + \sqrt{v_t}S_t\,\mathrm{d}W_t^S,$$

where r_t is the short-term stochastic interest rate, and v_t denotes the stochastic variance of the asset. The variance process v_t evolves according to the classical Heston dynamics:

$$\mathrm{d}v_t = \kappa(\theta - v_t)\,\mathrm{d}t + \sigma\sqrt{v_t}\,\mathrm{d}W_t^v, \tag{1}$$

where $\kappa > 0$ is the rate of mean reversion, $\theta > 0$ is the long-run variance level, $\sigma > 0$ is the volatility of volatility, and W_t^S, W_t^v are standard Brownian motions with correlation $\mathbb{E}[\mathrm{d}W_t^S \mathrm{d}W_t^v] = \rho_1$ for some $|\rho_1| < 1$.

The classical Vasicek model assumes a mean-reverting Ornstein–Uhlenbeck process for the short-term interest rate r_t. To account for long-range dependence observed in empirical interest rate data, we extend this model by incorporating a fractional Brownian motion (fBm) component with Hurst parameter $H > 0.5$. However, since fBm is not a semi–martingale, we work with its semi–martingale approximation $B^{H,\epsilon}(t)$ as discussed in Section 2.

Accordingly, the fractional Vasicek interest rate dynamics are modeled as:

$$\mathrm{d}r_t = a(b - r_t)\,\mathrm{d}t + \sigma_r\,\mathrm{d}B^{H,\epsilon}(t),$$

where $a > 0$ is the speed of mean reversion, b is the long-term mean of the interest rate, $\sigma_r > 0$ is the interest rate volatility, and $B^{H,\epsilon}(t)$ is defined via the regularized semi–martingale representation:

$$\mathrm{d}B^{H,\epsilon}(t) = \alpha\varphi_t^\epsilon\,\mathrm{d}t + \epsilon^\alpha\,\mathrm{d}W_t^r,$$

with $\alpha = H - \frac{1}{2}$ and $\varphi_t^\epsilon = \int_0^t (t - s + \epsilon)^{\alpha-1}\,\mathrm{d}W_s^r$.

Substituting this into the Vasicek model yields the following tractable formulation:

$$\mathrm{d}r_t = \left(a(b - r_t) + \sigma_r\alpha\varphi_t^\epsilon\right)\,\mathrm{d}t + \sigma_r\epsilon^\alpha\,\mathrm{d}W_t^r.$$

Combining the Heston asset price model with the semi–martingale approximation of the fractional Vasicek interest rate process, we obtain the full system:

$$\begin{cases} \mathrm{d}S_t &= r_t S_t\,\mathrm{d}t + \sqrt{v_t}S_t\,\mathrm{d}W_t^S, \\ \mathrm{d}v_t &= \kappa(\theta - v_t)\,\mathrm{d}t + \sigma\sqrt{v_t}\,\mathrm{d}W_t^v, \\ \mathrm{d}r_t &= \left(a(b - r_t) + \sigma_r\alpha\varphi_t^\epsilon\right)\,\mathrm{d}t + \sigma_r\epsilon^\alpha\,\mathrm{d}W_t^r, \end{cases} \tag{2}$$

where (W_t^S, W_t^v, W_t^r) is a three-dimensional Brownian motion with instantaneous correlations:

$$\mathbb{E}[\mathrm{d}W_t^S\,\mathrm{d}W_t^v] = \rho_1, \quad \mathbb{E}[\mathrm{d}W_t^S\,\mathrm{d}W_t^r] = \rho_2, \quad \mathbb{E}[\mathrm{d}W_t^v\,\mathrm{d}W_t^r] = 0,$$

where $|\rho_1| < 1$ and $|\rho_2| < 1$.

This hybrid model captures both stochastic volatility in asset returns and long-range dependence in interest rate dynamics, providing a richer framework for derivative pricing and risk assessment in financial markets.

4 Pricing Formula for European Options

European options, which may only be exercised at maturity, constitute some of the most fundamental instruments in financial markets. Their valuation is essential for hedging, risk management, and model calibration. In this section, we derive a semi-analytical pricing formula for European options under the hybrid Heston–fractional Vasicek model presented in Section 3.

We work under the risk-neutral measure $\mathbb{Q}$ and assume frictionless and arbitrage-free markets. Let $\phi(x, v, r, t)$ denote the price at time t of a European contingent claim written on the underlying asset S_t, where $x_t := \log S_t$, v_t denotes the instantaneous variance, and r_t denotes the short-term interest rate. The latter evolves according to the regularized fractional Vasicek specification discussed in Section 3.

The log-transformation $x_t = \log S_t$ is standard in stochastic volatility modelling and is particularly convenient for Fourier-based valuation techniques. Formally, $\phi(x, v, r, t)$ may be regarded as a sufficiently regular candidate solution to the backward pricing equation under the relevant risk-neutral dynamics.

The pricing PDE under the *full* regularized fractional model

Applying Itô's lemma to the function $\phi(x_t, v_t, r_t, t)$ under the dynamics of the system (2) yields the formal backward equation:

$$
\begin{aligned}
0 = \frac{\partial \phi}{\partial t} + \left(r - \frac{1}{2}v\right)\frac{\partial \phi}{\partial x} + \kappa(\theta - v)\frac{\partial \phi}{\partial v} + \left(a(b - r) + \sigma_r \alpha \varphi_t^{\epsilon}\right)\frac{\partial \phi}{\partial r} \\
+ \frac{1}{2}v\frac{\partial^2 \phi}{\partial x^2} + \rho_1 \sigma v \frac{\partial^2 \phi}{\partial x \partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 \phi}{\partial v^2} + \frac{1}{2}\sigma_r^2 \epsilon^{2\alpha}\frac{\partial^2 \phi}{\partial r^2} + \rho_2 \sigma_r \epsilon^{\alpha}\sqrt{v}\,\frac{\partial^2 \phi}{\partial x \partial r} - r\phi.
\end{aligned}
$$

A key observation is that the PDE presented here does *not* represent the Kolmogorov equation associated with the full fractional Vasicek dynamics. The auxiliary process φ_t^{ε} is a Volterra-type Gaussian process and therefore inherently non-Markovian. As a result, the augmented state vector $(x_t, v_t, r_t, \varphi_t^{\varepsilon})$ possesses an infinite-dimensional memory structure. Consequently, a PDE posed solely in the variables (x, v, r, t) cannot correspond to the generator of the original fractional model, whose dynamics depend on the entire past history through φ_t^{ε}.

To retain analytic tractability, we therefore introduce an *affine surrogate model* via structurally motivated approximations, following the methodology of Grzelak and Oosterlee [7] and similar affine approximations used in hybrid Heston interest-rate models.

Affine Approximation of Nonlinear and Non-Markovian Terms

Two specific simplifications are introduced:

(i) The non-Markovian fractional term φ_t^ε appears only in the drift of r_t. Since φ_t^ε is a centered Gaussian Volterra process, we approximate its drift contribution by zero, i.e.,

$$\sigma_r \alpha \varphi_t^\varepsilon \approx 0,$$

thereby removing path-dependence in the drift of r_t. This step does *not* render the original fractional model Markovian; rather, it defines a Markovian *approximate* Vasicek-type model under which analytical pricing becomes feasible.

(ii) The mixed second-order term $\partial^2\phi/(\partial x\,\partial r)$ contains the non-affine factor $\sqrt{v}$, preventing closed-form Fourier methods. Following [7], we replace $\sqrt{v_t}$ by its deterministic approximation $\Lambda(t)$, constructed via moment-matching for the CIR distribution:

$$\sqrt{v_t} \approx \Lambda(t) := \mathbb{E}[\sqrt{v_t}].$$

This preserves the affine form of the pricing operator while maintaining a meaningful coupling between equity and interest-rate factors.

Under these approximations, the pricing equation corresponds to an *affine Markovian surrogate* with generator:

$$\begin{aligned} 0 = \frac{\partial \phi}{\partial t} + \left(r - \frac{1}{2}v\right)\frac{\partial \phi}{\partial x} + \kappa(\theta - v)\frac{\partial \phi}{\partial v} + a(b-r)\frac{\partial \phi}{\partial r} + \frac{1}{2}v\frac{\partial^2 \phi}{\partial x^2} \\ + \rho_1 \sigma v \frac{\partial^2 \phi}{\partial x \partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 \phi}{\partial v^2} + \frac{1}{2}\sigma_r^2 \epsilon^{2\alpha}\frac{\partial^2 \phi}{\partial r^2} + \rho_2 \sigma_r \epsilon^{\alpha}\,\Lambda(t)\,\frac{\partial^2 \phi}{\partial x \partial r} - r\phi. \end{aligned} \tag{3}$$

Equation (3) is therefore not the generator of the original fractional Vasicek model, but the generator of the *approximated affine hybrid model* used for pricing. This clarification resolves the issues related to mathematical consistency and the rigorous applicability of the Feynman–Kac framework.

Characteristic-Function-Based Pricing

With the affine surrogate structure restored, we may apply transform techniques in the spirit of Duffie et al. [6]. Let $x_t = \log S_t$, and define the conditional characteristic function:

$$\phi(u,\tau) = \mathbb{E}^{\mathbb{Q}}\left[e^{iux_T} \,\middle|\, x_t = x,\, v_t = v,\, r_t = r\right], \qquad \tau := T - t.$$

Motivated by the affine form of (3), we postulate the exponential–affine representation:

$$\phi(u,\tau) = \exp(A(u,\tau) + B(u,\tau)x + C(u,\tau)v + D(u,\tau)r)\,,$$

with terminal conditions $A(u,0) = 0$, $B(u,0) = iu$, $C(u,0) = 0$, $D(u,0) = 0$.

Substituting this exponential–affine representation into (3) and matching coefficients yields the usual Riccati and linear ODEs for (A, B, C, D), consistent with the affine structure. The resulting characteristic function enables efficient valuation of European options via Fourier inversion. The explicit ODE system and its closed-form solutions are presented below; their structure is unchanged from the classical Heston model, except for the additional deterministic coupling term $\Lambda(t)$ in the coefficient of $A(u, \tau)$, which captures the (approximated) interaction between the equity variance and the fractional interest-rate dynamics.

Since $B(u, \tau)$ does not depend on τ (as there is no second-order derivative in x that depends on any state variable), we immediately find:

$$\frac{\partial B}{\partial \tau} = 0 \quad \Rightarrow \quad B(u, \tau) = iu.$$

The ODE for $C(u, \tau)$ corresponds to the component associated with the stochastic volatility v_t, and takes the form of a Riccati equation:

$$\frac{\partial C}{\partial \tau} = \frac{1}{2}B(B-1) + (\rho_1 \sigma B - \kappa)C + \frac{1}{2}\sigma^2 C^2.$$

Substituting $B(u, \tau) = iu$ and defining constants

$$c_1 := -\frac{1}{2}u(u+i), \quad c_2 := \rho_1 \sigma iu - \kappa, \quad c_3 := \frac{1}{2}\sigma^2,$$

the Riccati equation becomes:

$$\frac{\partial C}{\partial \tau} = c_1 + c_2 C + c_3 C^2,$$

which admits the closed-form solution:

$$C(u, \tau) = \frac{-(c_2 + d)}{2c_3(1 - ge^{-d\tau})}\left(1 - e^{-d\tau}\right),$$

with

$$d := \sqrt{c_2^2 - 4c_1 c_3}, \quad g := \frac{c_2 + d}{c_2 - d}.$$

Next, the ODE for $D(u, \tau)$ corresponds to the interest rate factor and is linear:

$$\frac{\partial D}{\partial \tau} = B - (aD + 1).$$

Using the initial condition $D(u, 0) = 0$ and solving by the integrating factor method yields:

$$D(u, \tau) = \frac{iu - 1}{a}\left(1 - e^{-a\tau}\right).$$

The function $A(u, \tau)$ aggregates contributions from the drift and variance terms and satisfies:

$$\frac{\partial A}{\partial \tau} = abD + \kappa\theta C + \frac{1}{2}\sigma_r^2 \epsilon^{2\alpha} D^2 + \rho_2 \sigma_r \epsilon^\alpha \Lambda(\tau) BD,$$

which integrates to:

$$A(u,\tau) = abI_1(\tau) + \kappa\theta I_2(\tau) + \frac{1}{2}\sigma_r^2\epsilon^{2\alpha}I_3(\tau) + \rho_2\sigma_r\epsilon^{\alpha}I_4(\tau),$$

where the integral terms are given explicitly as:

$$I_1(\tau) = \int_0^\tau D(s)\,\mathrm{d}s = \frac{iu-1}{a}\left(\tau - \frac{1-e^{-a\tau}}{a}\right),$$

$$I_2(\tau) = \int_0^\tau C(s)\,\mathrm{d}s = \frac{\tau}{\sigma^2}(\kappa - \sigma\rho_1 iu - d) - \frac{2}{\sigma^2}\log\left(\frac{1-ge^{-d\tau}}{1-g}\right),$$

$$I_3(\tau) = \int_0^\tau D^2(s)\,\mathrm{d}s = \frac{1}{2a^3}(i+u)^2\left(3 + e^{-2a\tau} - 4e^{-a\tau} - 2a\tau\right),$$

$$I_4(\tau) = \int_0^\tau \Lambda(\tau-s)D(s)\,\mathrm{d}s = -\frac{iu+u^2}{a}\int_0^\tau \Lambda(\tau-s)(1-e^{-as})\,\mathrm{d}s.$$

The function $\Lambda(\cdot)$ was defined earlier as the deterministic approximation to $\mathbb{E}[\sqrt{v(t)}]$ and encodes the memory effects of the variance process. The integral I_4 thus incorporates the fractional Vasicek memory structure into the pricing kernel.

The complete set of functions (A, B, C, D) fully characterizes the conditional characteristic function of the log-asset price, and allows for efficient numerical pricing of European-style derivatives via Fourier inversion techniques.

Financial Interpretation of long-memory Effects

The inclusion of a fractional Vasicek process for the short-term interest rate introduces long-range dependence, a phenomenon widely documented in empirical studies of financial time series. In contrast to classical Vasicek dynamics, which model interest rates as Markovian Ornstein–Uhlenbeck processes with exponentially decaying memory, the fractional variant exhibits power-law decay in the autocorrelation structure. This means that shocks to the interest rate tend to persist for longer periods, reflecting more realistic behavior of interest rates observed in practice, particularly in low-rate or slowly adjusting economic environments.

From a financial perspective, this long-memory has critical implications for the valuation of interest rate–sensitive securities. In standard models, the future path of interest rates is largely independent of distant past behavior. However, in the fractional Vasicek setting, past interest rate levels influence the present and future values to a much greater degree. This persistence introduces a form of path dependence into the model, where the interest rate's historical trajectory can significantly affect discounting and, thus, the present value of expected payoffs.

For European-style options, particularly those with longer maturities, the impact of this memory becomes nontrivial. A more persistent (higher Hurst parameter) interest rate process can lead to a slower mean reversion, often keeping rates lower for longer periods. Since the risk-neutral discount factor is directly tied to the short rate, this persistence translates into a smaller effective discount rate. Consequently, the expected present value of the option payoff increases, raising the overall price of the option. This is consistent with the numerical results obtained in Section 5, where higher values of the memory-related parameter α (equivalent to $H - 0.5$) result in increased option prices.

Moreover, incorporating long-memory provides more robust modeling in environments characterized by economic inertia, such as during monetary policy transitions or macroeconomic shocks. In such settings, traditional Markovian models may underestimate the true risk or overstate the speed of reversion. The fractional Vasicek model addresses this by embedding a more flexible and empirically consistent framework for interest rate dynamics, enabling better pricing, hedging, and risk management of interest rate–sensitive derivatives.

5 Numerical Implementation and Empirical Analysis

In this section, we turn our attention to the practical implications of the model developed in previous sections. Having derived the option pricing partial differential equation and its solution via the characteristic function, we now focus on how this theoretical framework can be utilized to compute option prices in practice. Specifically, we derive a semi-closed-form expression for European option prices via Fourier inversion techniques and discuss how the presence of fractional interest rate dynamics influences numerical implementation. Following this, we present a set of numerical experiments to assess the model's pricing performance, sensitivity to parameters, and its ability to capture market-implied volatility surfaces. The results highlight both the benefits and limitations of incorporating long-memory interest rate effects in the Heston-style stochastic volatility framework.

5.1 Pricing Formula via Fourier Inversion

After deriving the conditional characteristic function of the log-asset price process $x_T = \log S_T$ in exponential–affine form, we now proceed to compute European option prices via Fourier inversion techniques. This approach is well suited to models with (approximate) affine dynamics and was pioneered in the context of stochastic volatility models by Heston [9], and generalized by Carr and Madan [4] and Duffie et al. [6].

Under the risk-neutral measure $\mathbb{Q}$, the price of a European call option with strike price K, maturity T, and current time t is given by the discounted expected payoff:

$$C(S_t, K, \tau) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s\,\mathrm{d}s}\,(S_T - K)^+ \,\middle|\, S_t\right],$$

where $\tau = T - t$ is the time to maturity.

Due to the presence of stochastic interest rates r_t governed by a fractional Vasicek process, the discount factor $e^{-\int_t^T r_s\,\mathrm{d}s}$ is random and nontrivially correlated with S_T. However, since the model remains (approximately) affine, the joint characteristic function of $x_T := \log S_T$ and the integral of r_s over $[t, T]$ remains exponentially affine. Thus, following standard methodology, we define:

$$C(S_t, K, \tau) = S_t\, P_1 - K\, \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s\, \mathrm{d}s}\right]\, P_2,$$

where P_1 and P_2 are risk-neutral probabilities defined as:

$$P_1 = \mathbb{Q}\left[\log S_T > \log K \mid \mathcal{F}_t\right],$$
$$P_2 = \mathbb{Q}^T\left[\log S_T > \log K \mid \mathcal{F}_t\right],$$

where $\mathbb{Q}^T$ denotes the T-forward measure. In affine models, these probabilities can be computed via Fourier inversion of the characteristic function.

Following Carr and Madan (1999), we define a dampened option price function to ensure integrability:

$$\tilde{C}(k) := e^{\alpha k} C(S_t, K = e^k, \tau), \quad \alpha > 0,$$

and compute its Fourier transform:

$$\mathcal{F}[\tilde{C}](u) := \int_{-\infty}^{\infty} e^{iuk}\tilde{C}(k)\, \mathrm{d}k.$$

Using the characteristic function $\phi(u, \tau)$ derived in Section 4, the closed-form representation of the call price is then given by:

$$\begin{aligned} C(S_t, K, \tau) = S_t &\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-iu\log K}\,\phi_1(u)}{iu}\right]\, \mathrm{d}u\right) \\ &- K\, \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s\, \mathrm{d}s}\right]\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-iu\log K}\,\phi_2(u)}{iu}\right]\, \mathrm{d}u\right), \end{aligned} \tag{4}$$

where $\phi_1(u)$ and $\phi_2(u)$ denote the characteristic functions of x_T evaluated under the risk-neutral and forward measures, respectively.

In our setting, due to the time-dependent discounting induced by r_t, the exact computation of $\mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_s\, \mathrm{d}s}]$ may not admit a closed-form expression. However, under the affine approximation and the exponential–affine form of the solution, this expectation can be computed using the function $A(u, \tau)$ derived earlier, evaluated at $u = 0$:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s\, \mathrm{d}s}\right] \approx e^{A(0,\tau)+C(0,\tau)v_t+D(0,\tau)r_t}.$$

In practical implementation, we perform numerical integration in Equation (4) using quadrature methods such as the trapezoidal or Gauss–Laguerre rule, after discretizing the integral over a suitably truncated domain $[0, U_{\max}]$ with step size Δu. The choice of the damping parameter α, integration bounds, and discretization scheme affects numerical stability and convergence and will be discussed in the next subsection.

5.2 Numerical Implementation Details

This subsection outlines the computational strategy employed to implement the option pricing methodology developed in the preceding sections. The analytical tractability of the model hinges on the availability of a closed-form expression for the characteristic function of the log-asset price under risk-neutral dynamics, which enables efficient pricing via Fourier inversion.

The pricing algorithm is implemented in MATLAB, utilizing numerical quadrature to evaluate the relevant integrals. Specifically, we employ adaptive integration routines to ensure stability and accuracy when computing oscillatory Fourier integrals. The formulation accommodates both the stochastic volatility of the Heston model and the long-memory effects introduced by the fractional Vasicek interest rate process.

All numerical experiments are based on a consistent baseline parameter set, calibrated according to representative values from the literature on stochastic interest rate and volatility models. The model inputs include the parameters governing the variance process $(\kappa, \theta, \sigma, \rho_1)$, the interest rate process (a, b, σ_r, ρ_2), and the fractional parameters (α, ϵ).

To assess the sensitivity of the option price to individual parameters, we perform a series of one-at-a-time experiments in which a single parameter is varied while all others are held constant. These include variations in the Hurst-related parameter α, the smoothing parameter ϵ, the initial short rate r_0, and the interest rate volatility σ_r. The goal of this sensitivity analysis is to isolate and understand the marginal effects of each modeling component on European call option prices.

All numerical results are presented graphically and discussed in the following subsection.

Under the baseline parameter values specified in Section 5.2, the European call option price is computed to be approximately `9.1565`. This result is obtained using a Fourier inversion technique with adaptive integration, and the computation time is reasonably fast (under 0.01 seconds). This suggests that our implementation is both efficient and suitable for practical pricing applications.

5.3 Convergence Analysis of Fourier Integral

The Fourier inversion method used in pricing relies on numerical integration over a truncated domain. To ensure accuracy and efficiency, we investigate how the integration upper bound U affects the computed option price.

Figure 1 shows the convergence behavior of the call option price as U increases from 20 to 200. As expected, the price gradually stabilizes, with diminishing improvements for larger values of U. This supports our choice of $U = 200$ in all subsequent experiments, as it strikes a good balance between accuracy and computational speed. This analysis confirms that the Fourier inversion method is numerically stable and convergent under the fractional Heston–Vasicek model, thereby validating its use in the subsequent pricing and sensitivity experiments.

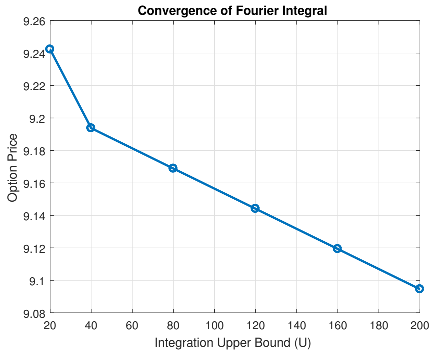

Figure 1: Convergence of the option price with respect to the upper integration bound U in the Fourier inversion method.

5.4 Estimation of the Memory Parameter α

The fractional parameter α governs the degree of long-range dependence in the short rate dynamics of the fractional Vasicek model. Specifically, it determines the Hurst index $H = \alpha + 0.5$, which measures the degree of memory and persistence in the underlying process. For $\alpha > 0$ (i.e., $H > 0.5$), the process exhibits positive long-range dependence, meaning that shocks to the short rate decay slowly over time. Accurately estimating α is therefore crucial for calibration, pricing, and risk management applications.

Despite its importance, the estimation of α is often neglected in theoretical modeling. To address this gap, we conduct a numerical experiment to evaluate the feasibility of recovering α from simulated data. In our experiment, we simulate sample paths of the short rate under the fractional Vasicek model over a five-year horizon, using $N = 10{,}000$ time steps and the smoothing approximation method described in Section 3. This approach approximates the fractional kernel via a finite-memory expansion, enabling tractable numerical simulations of the non-Markovian short rate process.

For each simulated path, we estimate the Hurst index H using the *Whittle estimator*—a widely used frequency-domain method that approximates the Gaussian maximum likelihood estimator [20]. The Whittle method is particularly well suited for stationary Gaussian processes and long-memory time series, offering desirable statistical properties such as asymptotic unbiasedness and consistency under standard regularity conditions. The Whittle likelihood function is given by:

$$\mathcal{L}_W(\theta) = \int_{-\pi}^{\pi} \left[\log f_\theta(\lambda) + \frac{I(\lambda)}{f_\theta(\lambda)} \right] \mathrm{d}\lambda, \tag{5}$$

where $f_\theta(\lambda)$ denotes the parametric spectral density of the process, indexed by the parameter θ (here related to H), and $I(\lambda)$ is the periodogram of the observed data.

The estimator minimizes $\mathcal{L}_W(\theta)$ with respect to θ, producing an efficient estimate of the memory parameter.

The following pseudocode outlines the steps used in our MATLAB implementation to estimate α from a simulated path of the fractional Vasicek short rate process:

Algorithm 4 Estimation of α via the Whittle Estimator

1: Simulate a path $\{r_t\}_{t=0}^{T}$ of the fractional Vasicek model using a smoothing approximation.
2: Remove deterministic trend: $\tilde{r}_t = r_t - \mathbb{E}[r_t]$.
3: Estimate the Hurst index H using a spectral method (e.g., MATLAB's `wfbmesti`).
4: Compute the memory parameter as $\alpha = H - 0.5$.
5: Return $\hat{\alpha}$ as the estimated fractional parameter.

In our context, we apply the Whittle estimator to the simulated short rate path and compute α via the identity $\alpha = H - 0.5$. Table 1 reports the results from three representative simulation trials, all based on a true value of $\alpha = 0.20$.

Table 1: Whittle-based estimation of the Hurst index and the memory parameter α.

Trial	Estimated H	Estimated $\alpha = H - 0.5$
1	0.6831	0.1831
2	0.6869	0.1869
3	0.6998	0.1998

The results demonstrate that the Whittle estimator is capable of producing highly accurate estimates of the Hurst index even in relatively short sample sizes, provided that the sampling frequency is sufficiently high. The recovered values of α lie within a small neighborhood of the true parameter, with absolute errors below 0.017 in all three cases. This supports the practical viability of using spectral-domain techniques for estimating long-memory effects in interest rate modeling.

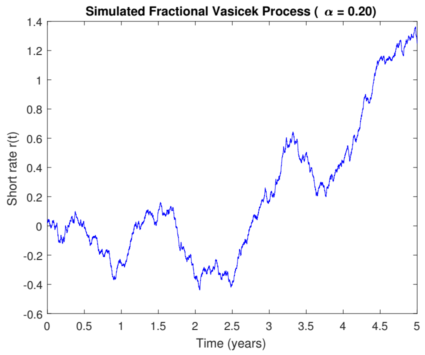

Figure 2: Sample path of the simulated fractional Vasicek process with $\alpha = 0.20$, used for estimation of the Hurst index via the Whittle method.

Figure 2 shows a sample path of the simulated short rate process used for the estimation, highlighting its persistent and smooth behavior characteristic of long-memory dynamics.

5.5 Interpretation of Numerical Results

We begin our numerical analysis by examining the behavior of European call option prices across varying strikes and maturities. Figure 3 displays the resulting call price surface under the fractional Heston–Vasicek model, computed via the Fourier inversion method described in Section 5.1.

As expected, the option price increases monotonically with the time to maturity T and decreases with the strike price K. This is consistent with financial intuition: longer maturities increase the probability of the option ending in-the-money, while higher strikes reduce the likelihood of a profitable exercise.

The surface also displays curvature effects attributable to both stochastic volatility and long-memory in interest rates. In particular, the elevated values for deep in-the-money options (low K) and long maturities reflect the compounding impact of persistent interest rate fluctuations over time. These effects would be underrepresented in a Black–Scholes setting or even in standard Heston models without a fractional rate component.

The smooth gradient along the maturity axis further suggests numerical stability and coherence of the Fourier-based pricing algorithm, even in the presence of fractional noise.

Figure 4 presents the implied volatility smile generated by the fractional Heston–Vasicek model for a fixed maturity. The model successfully reproduces the well-known smile effect observed in equity markets, with higher implied volatilities for deep in-the-money and out-of-the-money options, and a trough near the at-the-money strike.

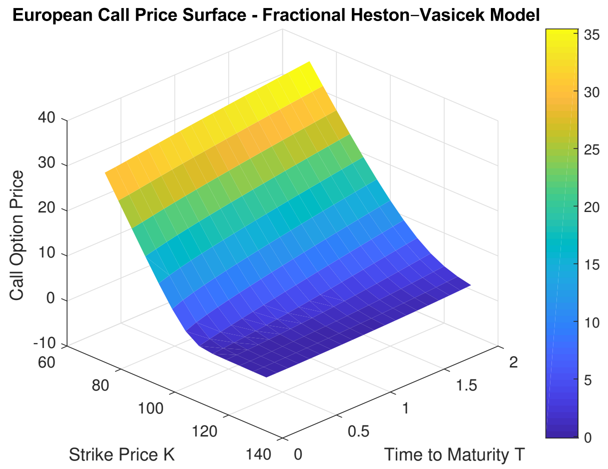

Figure 3: European call option price surface under the fractional Heston–Vasicek model. The prices are shown as a function of strike K and maturity T.

The implied volatilities are computed by inverting the Black–Scholes formula for a range of strike prices using the model-produced option prices. The ATM strike, indicated by a red marker, aligns closely with market-consistent levels, suggesting that the model is well-calibrated at the central region of the smile. The upward sloping wings of the smile are a result of the volatility-of-volatility component from the Heston model, whereas the curvature is further affected by the long-memory effects in the interest rate process.

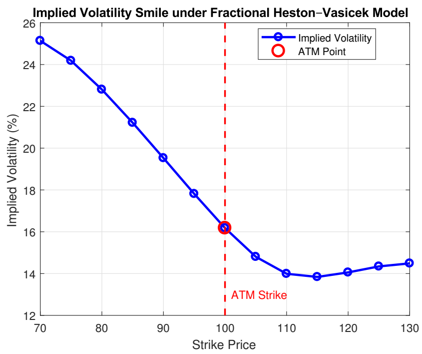

Figure 4: Implied volatility smile for maturity $T = 1$ under the fractional Heston–Vasicek model.

In addition to the smile for a fixed maturity, we compute the full implied volatility surface across a range of strikes and maturities. The result is shown in Figure 5.

The surface confirms the model's ability to jointly reproduce strike- and maturity-dependent features of implied volatilities.

As maturity increases, implied volatilities tend to decline for at-the-money and slightly out-of-the-money options, reflecting the mean-reverting nature of both the variance and interest rate processes. The fanned shape at shorter maturities is indicative of stronger effects from the fractional interest rate component, which diminishes as maturity grows. This long-memory effect is thus more pronounced for short-dated instruments, aligning with empirical findings in fixed income markets.

Overall, the model captures the essential skew and term structure observed in market data, suggesting its practical utility for derivative pricing and risk management tasks.

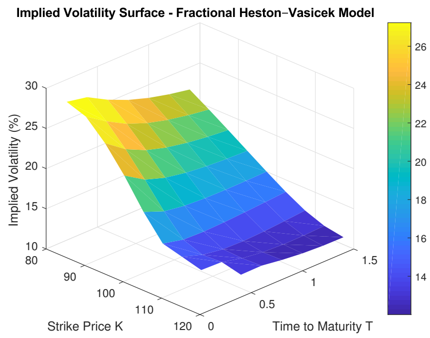

Figure 5: Implied volatility surface under the fractional Heston–Vasicek model.

5.6 Sensitivity Analysis with Respect to Model Parameters

To gain deeper insights into the influence of various model parameters on option prices, we conduct a one-at-a-time sensitivity analysis. Specifically, we vary key parameters of both the fractional Vasicek short rate process and the Heston variance dynamics while holding all other parameters fixed at their baseline values (as specified in Section 5.2).

This approach allows us to isolate the marginal impact of each parameter on the European call option price, offering valuable intuition on model behavior and guiding future calibration efforts. In each experiment, the option price is recomputed across a grid of values for the parameter of interest, and the results are visualized to highlight trends and nonlinearities in price response.

We begin with the fractional parameter α, which controls the strength of memory in the short rate process.

Effect of the Hurst Parameter α. Figure 6 illustrates the sensitivity of the European call option price to the fractional memory parameter α, which relates to the Hurst index as $H = \alpha + 0.5$. As α increases from 0.05 to 0.45 (corresponding to H ranging from 0.55 to 0.95), the option price exhibits a steadily increasing pattern.

This behavior reflects the long-memory nature of the fractional Vasicek process governing the short rate. Larger α values imply greater persistence in the interest rate path, which in turn increases the expected present value of the option's payoff due to more pronounced autocorrelation in the short rate path. The nonlinearity observed in the curve suggests diminishing marginal effects of memory persistence beyond $H \approx 0.85$, indicating a saturation point in pricing sensitivity.

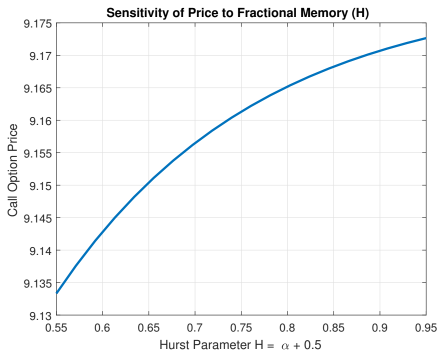

Figure 6: Sensitivity of the European call option price to the Hurst parameter $H = \alpha + 0.5$.

Effect of the Smoothing Parameter ϵ. The parameter ϵ plays a technical but important role in the numerical approximation of fBm used to model the long-memory short rate dynamics. Smaller values of ϵ correspond to a finer approximation of fBm, but may introduce numerical instability, while larger values act as a regularization, smoothing out the singularity in the fractional kernel.

Figure 7 shows the sensitivity of the European call option price to changes in ϵ, ranging from 10^{-3} to 10^{-1}. The results exhibit a mild but consistent decrease in the option price as ϵ increases. This reflects the fact that higher smoothing dampens the effective memory component in the short rate process, thereby reducing the cumulative discounting effect associated with persistent low rates.

Overall, while the option price is relatively stable across reasonable values of ϵ, its sensitivity suggests that careful tuning of this parameter may be needed when calibrating the model to real market data.

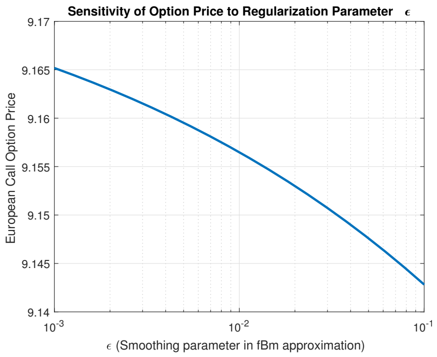

Figure 7: Sensitivity of the European call option price to the smoothing parameter ϵ.

Effect of Interest Rate Volatility σ_r. The parameter σ_r governs the instantaneous volatility of the short rate process and encapsulates how uncertain the future evolution of interest rates is. As σ_r increases, the stochastic variation in the short rate intensifies, leading to more dispersion in the stochastic discount factor.

Figure 8 presents the sensitivity of the European call option price with respect to σ_r, examined over a broader and more economically relevant range $[0, 0.5]$. The relationship is clearly nonlinear and strictly decreasing: higher interest rate volatility leads to a systematic decline in the option price. This decline is particularly steep in the lower range of σ_r, and gradually levels off for large σ_r, suggesting a diminishing marginal impact.

This behavior aligns with financial intuition: greater interest rate uncertainty tends to reduce the expected present value of future payoffs, particularly when the option payoff is heavily discounted. Hence, σ_r exerts a pronounced effect on pricing, especially for long-dated or interest rate-sensitive instruments such as callable bonds, convertible securities, and long-maturity equity options.

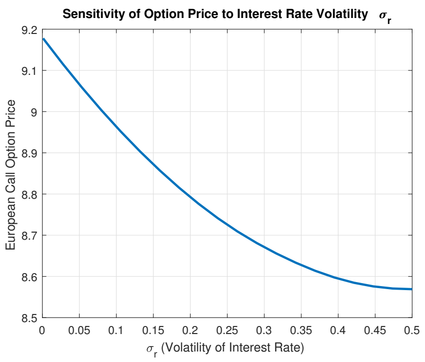

Figure 8: Sensitivity of the European call option price to the interest rate volatility parameter σ_r, examined over a broader range.

Effect of Mean Reversion Speed a and Long-Term Level b. We now turn to the classical parameters of the Vasicek short rate process: the speed of mean reversion a and the long-term mean level b. These parameters jointly govern how quickly the short rate reverts to its long-run average and what that average is. Figure 9 presents the sensitivity of the European call option price to variations in a and b. In the left panel, we observe that increasing the mean reversion speed a leads to a modest increase in the option price. This is because faster reversion reduces the uncertainty in the interest rate path, effectively narrowing the distribution of the discount factor and stabilizing its expectation. The right panel shows a much stronger sensitivity to the long-term mean level b. As b increases, the option price rises significantly. Since higher values of b raise the expected short rate over the life of the option, this leads to higher discounting of the strike component $Ke^{-\int_t^T r_s ds}$, reducing its impact and thereby increasing the net option price. The steepness of this response confirms that b is a critical driver of pricing behavior in models with stochastic interest rates.

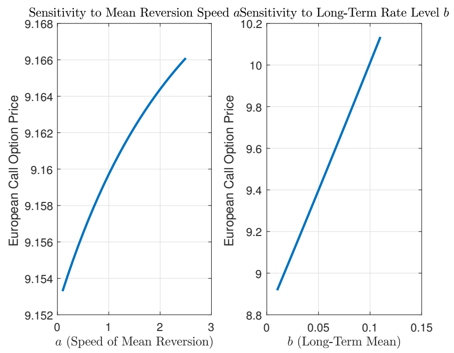

Figure 9: Sensitivity of the European call option price to the Vasicek parameters: mean reversion speed a (left) and long-term mean b (right).

Sensitivity to Initial Short Rate r_0. We also examine the impact of the initial short rate r_0 on option pricing outcomes. Figure 10 illustrates how the European call option price responds to variations in r_0, holding all other parameters fixed.

The relationship is strongly positive and approximately linear: as r_0 increases, the option price rises noticeably. This reflects the direct role that r_0 plays in the exponential discounting term $e^{-\int_t^T r_s \, \mathrm{d}s}$. Higher values of r_0 shift the entire short rate path upward, leading to increased discounting of the strike and hence a higher present value of the call payoff. This finding emphasizes the importance of accurately estimating current interest rate conditions when pricing under models with stochastic rates.

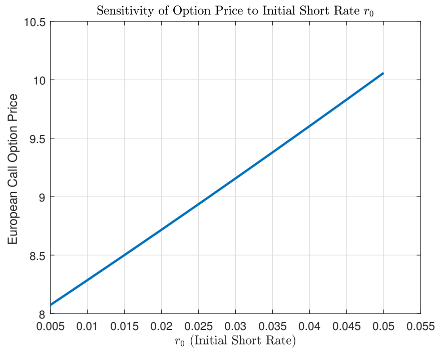

Figure 10: Sensitivity of European call option price to the initial short rate r_0.

5.7 Calibration to Market Data

To assess the empirical performance of the proposed Heston–fractional Vasicek (HFV) framework, we calibrate the model to real call option prices on the S&P 500 Index at the close of trading on 7 August 2019. On this date, the index closed at \$287.97, and the prevailing risk-free rate was approximately $r = 0.02$. We consider two maturities–$T = 100$ days and $T = 237$ days–chosen to highlight both short- and medium-term behaviour of the model and to evaluate the impact of long-memory interest-rate dynamics, which are expected to be more pronounced for longer maturities.

Calibration Procedure. The model parameters are estimated by minimizing the squared pricing error between market and model call prices. Given a set of observed market prices $\{C^{\text{mkt}}(K_i, T)\}_{i=1}^{N}$ for strikes $\{K_i\}$, the calibration problem is

$$\min_{\theta} \frac{1}{N} \sum_{i=1}^{N} \left(C^{\text{HFV}}(K_i, T; \theta) - C^{\text{mkt}}(K_i, T) \right)^2,$$

where θ denotes the vector of model parameters. The pricing function $C^{\text{HFV}}(\cdot)$ is computed via Monte Carlo simulation under the affine-approximated Heston–fractional Vasicek dynamics.

Parameterization. The calibrated parameter vector is

$$\theta = (v_0, \bar{v}, \kappa, \eta, \rho_1,\ a, b, \sigma_r, \rho_2, \alpha, \varepsilon),$$

corresponding respectively to the initial variance, long-run variance, variance mean-reversion speed, volatility of volatility, equity–volatility correlation, interest-rate mean reversion and mean level, short-rate volatility, equity–rate correlation, fractional memory parameter, and the smoothing parameter of the regularized fBm.

Stochastic Optimization via Adam. To solve the above nonlinear optimization problem, we employ a stochastic gradient descent scheme with Adam updates. The algorithm proceeds as follows:

- Market strikes and prices are randomly shuffled, and small strike batches are used to stochastically estimate the loss and gradients.
- Gradients are computed by symmetric finite differences:
$$\frac{\partial L}{\partial \theta_j} \approx \frac{L(\theta_j + \varepsilon_{\text{fd}}) - L(\theta_j - \varepsilon_{\text{fd}})}{2\varepsilon_{\text{fd}}},$$
with $\varepsilon_{\text{fd}} = 10^{-4}$.
- Adam first- and second-moment recursions are updated with parameters $(\beta_1, \beta_2) = (0.9, 0.999)$ and learning rate $\alpha_{\text{lr}} = 0.02$.
- Gradient steps are clipped to control instability and projected onto economically reasonable parameter bounds.
- An ℓ_2-regularization penalty $\lambda_{\text{reg}}\|\theta\|^2$ with $\lambda_{\text{reg}} = 10^{-3}$ is included to enhance stability.

Training is performed for 35 epochs, with each epoch processing mini-batches of six strikes and averaging Monte Carlo losses over multiple random seeds to reduce variance. This procedure provides a robust and noise-tolerant calibration, especially in the presence of Monte Carlo pricing noise.

Monte Carlo Pricing. Model prices within the optimization loop are computed using a discretized simulation of the Heston–fractional Vasicek system under the affine approximation. For each maturity T, paths for (S_t, v_t, r_t) are simulated on a uniform grid with $n_{\text{steps}} = 200$ time steps. Correlated Gaussian innovations are generated via Cholesky factorization of the correlation matrix of (W_t^S, W_t^v, W_t^r).

The short-rate dynamics employ the regularized fBm approximation, producing an effective interest-rate volatility $\sigma_r \varepsilon^{\alpha}$. The discounted payoff

$$e^{-\int_0^T r_t dt} (S_T - K)^+$$

is estimated across $n_{\text{paths}} = 6000$–12000 Monte Carlo samples (depending on context), and averaged across batches to yield the call price.

Final Parameter Selection. After the final epoch, the optimized parameter vector θ^* is used to compute model prices across all strikes with a higher Monte Carlo budget to generate the plotted calibration curves. These results form the basis of the empirical comparison between the Heston–fractional Vasicek model and the classical Heston model reported in Figures 11–12.

For the 100-day maturity (Figure 11), both the classical Heston model and the Heston–fractional Vasicek (HFV) model capture the general shape of the market price curve. However, the HFV model provides a noticeably tighter fit across strikes. The improvement is most evident in the intermediate-strike region, where the Heston model systematically underprices options, while the HFV curve closely overlaps the observed market prices.

For the 237-day maturity (Figure 12), the difference between the two models becomes significantly more pronounced. The Heston model exhibits a persistent misfit, particularly for deep out-of-the-money strikes, where it underprices market options and produces an overly flat price profile. In contrast, the HFV model reproduces the market curvature more accurately and remains close to observed prices across the full range of strikes. This enhanced performance aligns with the increasing impact of long-memory interest-rate dynamics over longer horizons. Overall, the results demonstrate that incorporating fractional short-rate behaviour improves calibration quality, especially for medium-term maturities.

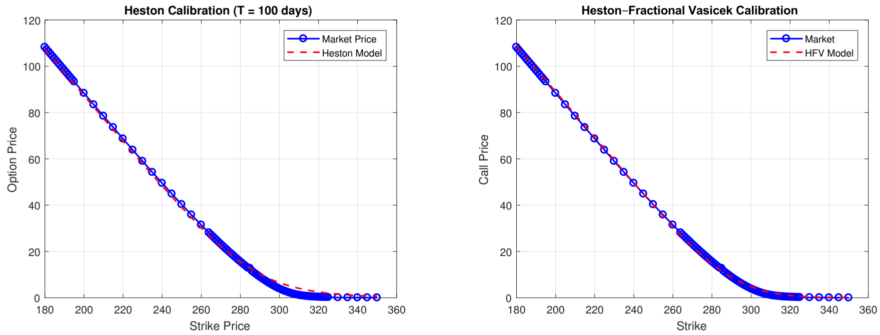

Figure 11: The comparison of the call option price estimated by the Heston (left) and the Heston Fractional Vasicek (right) models (red dashed) and the market call price (blue circles) for the time to maturity T of 100 days.

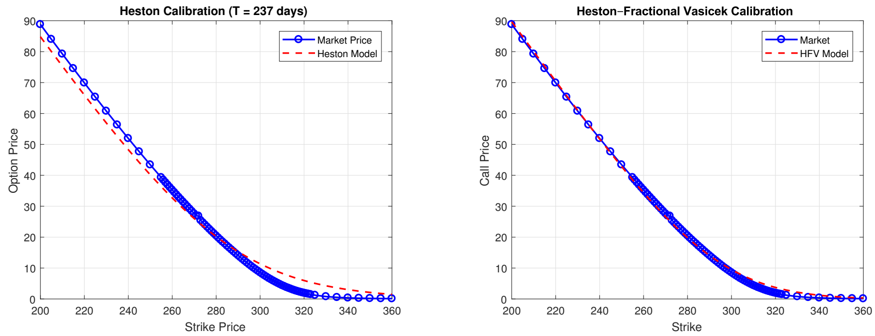

Figure 12: The comparison of the call option price estimated by the Heston (left) and the Heston Fractional Vasicek (right) models (red dashed) and the market call price (blue circles) for the time to maturity T of 237 days.

6 Conclusion

In this paper we introduced a tractable framework for option pricing that couples the Heston stochastic volatility model with an interest-rate process driven by a regularized fractional Brownian motion. By employing a semimartingale approximation of fBm together with an affine approximation of the volatility–rate interaction, we constructed an analytically convenient surrogate model that retains key long-memory features while restoring an exponential–affine structure suitable for Fourier-based pricing.

Our numerical analysis demonstrates that fractional interest-rate dynamics can materially influence equity option values through their effect on the stochastic discount factor. The sensitivity experiments highlight the role of the memory parameter α and the short-rate volatility σ_r in shaping implied-volatility smiles and term structures. Moreover, the empirical calibration results show that the proposed Heston–fractional Vasicek model achieves a closer fit to market option prices than the classical Heston model, particularly for longer maturities where persistent rate dynamics become more influential.

It is important to emphasize that the pricing formulas derived here correspond to the *approximate affine surrogate model*, not to the exact fractional Vasicek process whose generator would require an infinite-dimensional Volterra state representation. Constructing such a full Markovian lift remains an open challenge and a promising direction for future work.

Finally, while the present study focuses on equity-style options, extending the framework to fixed-income derivatives such as caps, floors, and swaptions remains an important avenue for future research. This would require developing full term-structure dynamics and calibration methodologies under the regularized fractional Vasicek specification, which we leave for future investigation.

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