*Journal of Mathematics and Modeling in Finance (JMMF)*
Vol. 6, No. 1, Winter & Spring 2026
Research paper

# Application of Radial Basis Functions Meshless Method for Solving an Inverse Parabolic Problem

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**Abstract:**
In this paper, we propose an approximate solution to a one-dimensional inverse parabolic problem using radial basis functions (RBFs) and the Levenberg–Marquardt (LM) regularization method. This problem involves the backward heat equation. In particular, we first transform the well-known Black–Scholes equation into the heat equation through an appropriate change of variables. The resulting heat equation is then solved using the proposed numerical method. To obtain the approximate solution for the unknown temperature values at the initial time, an optimization problem is formulated to minimize a cost functional. Since the system of equations is ill-conditioned, the LM regularization method is applied. We derive convergence rates for the LM iterates under a Holder stability estimate. Finally, a numerical example is presented to illustrate the method's accuracy and effectiveness.

*Keywords:*Inverse parabolic problem; Radial basis functions method; Levenberg-Marquardt method; Approximate solution.
*Classification:* 65M32, 65N20, 65F22.

## 1 Introduction

In recent years, financial problems, especially in option pricing, have received significant attention. Parabolic partial differential equations (PDEs), like the Black–Scholes equation, are key tools for modeling financial phenomena. They are used in derivative pricing, risk management, interest rate modeling, credit risk, and portfolio optimization. These equations help us to better understanding of financial markets and make informed decisions [1]-[16].

Many problems in financial mathematics are inverse problems. These problems are significant because they appear in many scientific and engineering applications. They often involve PDEs of parabolic, hyperbolic, or elliptic types [1]-[5].

This paper focuses on a specific inverse parabolic problem from finance: the

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Received: 28/09/2025 Accepted: 31/01/2026
**https://doi.org/10.22054/jmmf.2026.74612.1091**

one-dimensional Black–Scholes equation for option pricing. By change of variables, we transform this equation into the classical heat equation. Specifically, we use the substitutions $S = Xe^x$, $t = \tilde{T} - \frac{2\tau}{\sigma^2}$, and $V(S,t) = Xe^{\alpha x+\beta\tau}u(x,\tau)$. This converts the financial model into a standard heat equation [6]. The problem then becomes an inverse heat conduction problem: we want to find the unknown initial condition $u(x, 0)$ from the known final data $u(x, T)$. This inverse problem is challenging because the heat equation, when solved backward in time, is ill-posed. Small errors in the final data can cause large errors in the initial condition. Therefore, regularization methods are necessary to obtain stable solutions.

Many researchers have studied inverse parabolic problems using different numerical methods. For example, Chen and Liu [7] used finite element methods, Dehghan [8] applied a variational iteration method, and Hasanov [9] used weak solutions. However, these methods often require meshing and can be computationally expensive.

To address these issues, we propose a meshless method based on RBFs. RBFs method is well-suited for problems with scattered data and complex domains [10]-[17]. Our approach approximates the unknown initial condition using RBFs. We then construct a cost functional based on the approximate solution and the final data. Minimizing this functional with the LM regularization stabilizes the ill-posed problem [11]-[18].

Therefore, we consider the following backward inverse problem(BIP)

$$u_t - u_{xx} = f(x,t), \qquad (x,t) \in Q_T, \tag{1}$$
$$u(x,0) = g(x), \qquad x \in \Lambda, \tag{2}$$
$$u(0,t) = u(l,t) = 0, \qquad 0 \le t \le T, \tag{3}$$
$$u(x,T) = \psi(x), \qquad 0 \le x \le l, \tag{4}$$

in which $\Lambda = (0, l)$, $Q_T = \Lambda \times (0, T)$, $g = g(x)$ represents the unknown initial data and $\psi = \psi(x)$ is a given measured data. In this study, our main goal is as follows

(i) Obtaining an approximate solution to the BIP (1)-(4),

(ii) Convergence analysis.

For objective (1), a numerical algorithm is presented. This algorithm solves BIP (1)-(4) using the RBFs method to find the solution of u(x,t;g). This solution depends on the function g, leading to an ill-posed problem. Therefore, to solve this problem, we use the LM regularization method. By doing this, we achieve a stable solution to the BIP (1)-(4). The implementation of this numerical algorithm is presented in section 2.

For objective (2), under standard assumptions on the solution operator and a Holder stability estimate, we present in Section 3 a convergence theorem for the LM method. Section 4 provides numerical results that validate the proposed algorithm. Finally, in Section 5, we describe some conclusions.

## 2 Implementation of Numerical Algorithm

In this section, we describe our numerical approach to solving the BIP (1)-(4). We begin by introducing the RBFs method, followed by its specific application to the forward problem(FP)(1)-(3).

### 2.1 RBFs Method

The RBFs method extends the multi-quadratic technique first introduced by Hardy in 1971 [12]. Since its inception, this versatile methodology has found applications across diverse scientific disciplines, including geophysics, computer graphics, artificial intelligence, statistical learning theory, neural networks, signal processing, systems engineering, and applied mathematics particularly for numerical solutions of PDEs. For solving PDEs specifically, Edward Kansa developed the eponymous Kansa method in 1990 [10], which we adopt in this paper. Using the RBFs approach, we approximate the solution to our problem as:

$$u(x,.) \approx \sum_{j=0}^{N} c_j(.)\varphi\left(\|x - x_j\|\right),$$

where $c_j$ represents unknown coefficients, $\varphi(\cdot)$ denotes the basis function, and $N$ indicates the number of node points. With this approximation function $u$, we will arrive at a system of linear equations such as $Ax = b$ that must be solved.

RBFs can be classified into two main categories: infinitely smooth and piecewise smooth functions. Infinitely smooth RBFs incorporate a shape parameter $\alpha$ and includes the Gaussian, multi-quadratic, inverse multi-quadratic, and inverse quadratic functions. In contrast, piecewise-smooth RBFs operate without a shape parameter and include the thin-plate spline, cubic functions, and monomial functions [10]. Their distinct advantages guide the selection of appropriate RBFs type in practical applications. RBFs effectively interpolate scattered data, making them particularly valuable in scenarios where data points are irregularly distributed. They can be implemented in higher-dimensional settings without the need for mesh generation, which simplifies the numerical implementation. In the present work, although the problem is one-dimensional, the main advantage of the RBFs method lies in its meshless formulation and flexibility in spatial discretization. Additionally, certain RBFs type particularly Inverse Quadratic, Multiquadric, and Gaussian basis functions [13]—provide exponential convergence rates, enabling rapid attainment of accurate solutions. Furthermore, the RBFs method adapts well to problems involving domains with complex geometric configurations, making it versatile across engineering, physics, and data science applications.

In our study, we address time-dependent PDEs using the RBFs method through two alternative approaches:

(i) Full discretization: applying the RBFs method to both time and space dimensions of the unknown function $u(x,t)$,

(ii) Combined discretization: employing the finite difference method for time discretization while using the RBFs method for spatial discretization.

While the first approach yields highly accurate approximations, it generates coefficient matrices with prohibitively high condition numbers. As the matrix dimension increases, solving the corresponding linear system becomes computationally challenging. Consequently, we adopt the second approach, which balances accuracy with computational efficiency.

## 2.2 RBFs Method for Solving FP (1)-(3)

Since solving an inverse problem necessitates first addressing the corresponding direct problem, we begin by solving the FP (1)-(3) using the RBFs method. We consider the following formulation:

$$\begin{aligned} u_t - u_{xx} &= f(x,t), \qquad (x,t) \in Q_T, & (5)\\ u(0,t) &= u(l,t) = 0, \quad 0 \le t \le T, & (6)\\ u(x,0) &= g(x), \qquad x \in \Lambda. & (7) \end{aligned}$$

For the temporal discretization, we partition the interval $[0,T]$ with $m \in \mathbb{N}$ times $t_k = k\tau$ for $k = 0,1,...,m$, and $\tau = \frac{T}{m}$. Implementing the forward difference scheme yields:

$$\frac{u(x,t_{k+1}) - u(x,t_k)}{\tau} - u_{xx}(x,t_k) = f(x,t_k), \quad k = 0,1,...,m-1. \tag{8}$$

For spatial discretization, we divide the interval $[0,1]$ using step size $h = \frac{1}{n}$, which provides the grid points $x_i = ih, (i = 0,...,n)$. Then we select the Gaussian RBFs $\varphi(r) = \exp(-\alpha r^2)$ with shape parameter $\alpha = 1$. This generates the basis functions $\{\xi_j = \varphi(\| x - x_j \|), j = 0,1,...,n\}$. Using these basis functions, we approximate the solution as:

$$u(x,t) \approx \sum_{j=0}^{n} c_j(t)\xi_j(x).$$

Substituting this approximation into equation (8) and applying the collocation method at points $x = x_i$ for $i = 1,...,n-1$, we obtain:

$$\frac{1}{\tau}\sum_{j=0}^{n} c_j(t_{k+1})\xi_j(x_i) - \frac{1}{\tau}\sum_{j=0}^{n} c_j(t_k)\xi_j(x_i) - \sum_{j=0}^{n} c_j(t_k){\xi_j}''(x_i) = f(x_i,t_k) \tag{9}$$

The initial and boundary conditions turn to:

$$u(x_i,0) = \sum_{j=0}^{n} c_j(0)\xi_j(x_i) = g(x_i), \qquad i = 0,1,...,n,$$
$$u(0,t_k) = \sum_{j=0}^{n} c_j(t_k)\xi_j(0) = 0, \qquad k = 0,1,...,m,$$
$$u(l,t_k) = \sum_{j=0}^{n} c_j(t_k)\xi_j(l) = 0, \qquad k = 0,1,...,m.$$

Also, the matrix form of the above discretization is as follows:

$$\mathbf{A}\mathbf{c} = \mathbf{f}, \tag{10}$$

where

- $\mathbf{c} = [c(t_0), c(t_1), \ldots, c(t_m)]^T$ contains all temporal coefficient vectors
- $\mathbf{f} = [g, f(x_0,t_0), \ldots, f(x_{m-1},t_{m-1}), 0, \ldots, 0]^T$ combines initial conditions, source terms, and boundary constraints
- $\mathbf{A}$ is as

$$\begin{bmatrix} \Phi & 0 & \cdots & 0 \\ -(\frac{1}{\tau}\Phi + \Phi_{xx}) & \frac{1}{\tau}\Phi & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & -(\frac{1}{\tau}\Phi + \Phi_{xx}) & \frac{1}{\tau}\Phi \\ B_0 & 0 & \cdots & 0 \\ 0 & B_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & B_m \end{bmatrix} \tag{11}$$

in which $\Phi := \xi_{ij}$, $B_k = [\xi_{0,j}; \xi_{n,j}]$ are the boundary conditions.

## 2.3 RBFs Method for Solving the BIP (1)-(4)

Since the initial condition $g(x)$ is unknown, the system (10) cannot be solved directly. We therefore define the following cost functional using the overposed measured data (4)

$$S(g) := \sum_{i=1}^{n} \left(u(x_i^*, T; g) - \psi(x_i^*)\right)^2 \tag{12}$$

where $x_i^* = \frac{il}{n}$. Accordingly, we are led to the minimization problem $\min_{g\in\mathcal{G}} S(g)$, where $\mathcal{G}$ denotes the set of admissible solutions.

To minimize the cost function (12), as before, we approximate the unknown function $g(x)$ with Gaussian basis functions

$$g(x) \approx \sum_{i=0}^{n} g_i \varphi(\|x - x_i\|) \tag{13}$$

in which the coefficients $g_i$ are unknown parameters that must be determined computationally. Using equation (13), we solve the BIP (1)-(4) via the RBFs method. Once the coefficients $g_i$ are computed, the solution $u(x, T; g)$ can be evaluated at the final time.

Since both the BIP (1)-(4) and the parameter estimation in the minimization problem (12) are ill-posed, we employ the LM regularization method to ensure stability.

#### Computational Algorithm for Determining the Unknown Function $g(x)$

Our complete computational approach consists of the following steps.

**Step 1.** Approximate the unknown function $g(x)$ using (13).

**Step 2.** Apply the RBFs method to obtain the approximate solution $u(x, T; g)$ at the final time.

**Step 3.** Determine the $n + 1$ unknown parameters $g_i$ by minimizing the cost functional (12). Since the resulting system of algebraic equations is ill-conditioned, we apply the LM regularization method in the next step.

**Step 4: The LM regularization method.** First, introduce the following notations for the discrete vectors involved in the cost functional.

$$\begin{aligned} u(g) &= [u_1, u_2, \ldots, u_n]^T, \quad \text{where } u_i = u(x_i^*, T; g), \\ \psi &= [\psi_1, \psi_2, \ldots, \psi_n]^T, \quad \text{where } \psi_i = \psi(x_i^*), \\ g &= [g_0, g_1, \ldots, g_n]^T. \end{aligned}$$

Next, define the sensitivity matrix $J(g)$ as the Jacobian of the solution vector $u(g)$ with respect to the coefficients $g$

$$J(g) = \left[ \frac{\partial u^T(g)}{\partial g} \right]^T. \tag{14}$$

The LM algorithm iteratively updates the parameter vector $g$ to minimize the cost functional. The steps of the algorithm are summarized as follows [7]:

## 3 Convergence analysis

In this section, we consider the BIP (1)-(4), which involves recovering the unknown initial condition $g(x)$ from the final-time measurements $\psi(x)$. The direct problem was solved numerically using the RBFs method in subsection 2.2. At the same

**LM Regularization Algorithm**
**Input:** Initial guess $g^{(0)}$, initial regularization parameter $\mu_0 = 0.001$, tolerance threshold `tol`, and iteration counter $k = 0$.
**1.** Compute $u(g^{(0)})$ and evaluate the cost functional $S(g^{(0)})$.
**2.** Calculate the sensitivity matrix $J^{(k)}$ using (14), and form the diagonal scaling matrix $\Omega^{(k)} = \mathrm{diag}[(J^{(k)})^T J^{(k)}]$.
**3.** Solve the following linear system for $\Delta g^{(k)}$:

$$\left[(J^{(k)})^T J^{(k)} + \mu^{(k)}\Omega^{(k)}\right] \Delta g^{(k)} = (J^{(k)})^T \left[\psi - u(g^{(k)})\right].$$

**4.** Update the estimate: $g^{(k+1)} = g^{(k)} + \Delta g^{(k)}$.
**5.** If $S(g^{(k+1)}) \geq S(g^{(k)})$, increase the regularization parameter: $\mu^{(k+1)} = 10\mu^{(k)}$ and return to Step 3.
**6.** If $S(g^{(k+1)}) < S(g^{(k)})$, accept the update and reduce the regularization parameter: $\mu^{(k+1)} = 0.1\mu^{(k)}$.
**7.** If the stopping criterion $\|g^{(k+1)} - g^{(k)}\| <$ `tol` is satisfied, terminate the algorithm. Otherwise, set $k \to k+1$ and return to Step 2.

time, the LM algorithm was employed in subsection 2.3 to minimize the cost functional associated with the discrepancy between the solution to the problem and the measured data.

Now, to analyze the convergence of the proposed algorithm, our goal is to establish an upper bound for

$$\left|u(x,t,g^*) - u(x,t,g^\dagger)\right| \leq \left|u(x,t,g^*) - u(x,t,\bar{g})\right| + \left|u(x,t,\bar{g}) - u(x,t,g^\dagger)\right|.$$

We begin by analyzing the convergence of the LM method. To this end, we introduce the following assumption.

## Assumption 1 [15]

Let the operator $F : \mathcal{G} \to \mathbb{R}^n$ map the initial parameter vector $g = (g_0, g_1, \ldots, g_n)$ to the solution vector $u(g) = (u_1, u_2, \ldots, u_n)$, where $u_i = u(x_i^*, T; g)$ denotes the solution of the FP (1)-(3) at points $x_i^*$ at the final time $T$.

We assume that the operator $F$ is Frechet differentiable on the admissible set $\mathcal{G}$, and that its derivative $F'(g)$ is Lipschitz continuous with Lipschitz constant $L > 0$. Moreover, we suppose that a Holder stability estimate holds:

$$\|g - g^\dagger\|^2 \leq C_F \|F(g) - F(g^\dagger)\|^{\frac{2}{1+\varepsilon}},$$

for some constant $C_F > 0$ and $\varepsilon \in (0, 1]$, where $g^\dagger$ denotes the exact solution. Based on Assumption 1, we state the following convergence theorem for the LM method.

**Theorem 3.1** ([15]). *Suppose the assumptions above hold and the initial guess $g^{(0)}$ satisfies.*

$$\frac{1}{2}\|g^{(0)} - g^\dagger\|^2 \leq \rho,$$

*for some sufficiently small radius $\rho > 0$ depending on problem constants. Define the parameter*

$$c := \frac{q(1-q)}{2Lb^2 C_F^{\frac{4}{1+\varepsilon}}},$$

*where $q \in (0,1)$ is a chosen tuning parameter, and $b$ is related to bounds on the operator derivative. Then the iterates $\{g^{(k)}\}$ generated by the LM algorithm satisfy:*

- *If $\varepsilon = 1$,*
$$\frac{1}{2}\|g^{(k)} - g^\dagger\|^2 \leq \rho(1-c)^k.$$
- *If $\varepsilon \in (0,1)$,*
$$\frac{1}{2}\|g^{(k)} - g^\dagger\|^2 \leq \left(ck\frac{1-\varepsilon}{1+\varepsilon} + \rho^{-\frac{1-\varepsilon}{1+\varepsilon}}\right)^{-\frac{1+\varepsilon}{1-\varepsilon}}.$$

*Proof.* For a detailed proof, we refer the reader to [15]. □

We note that, from the Holder stability estimate,

$$\|F(g) - F(g^\dagger)\| \leq C_F^{-\frac{1+\varepsilon}{2}} \|g - g^\dagger\|^{1+\varepsilon}.$$

Therefore, by applying Theorem 3.1, we obtain

$$\left\|u(g^{(k)}) - u(g^\dagger)\right\| \leq B^{LM},$$

where

$$B^{LM} = \begin{cases} 2C_F^{-1}\rho(1-c)^k, & \varepsilon = 1, \\ C_F^{-\frac{1+\varepsilon}{2}}\left(ck\frac{1-\varepsilon}{1+\varepsilon} + \rho^{-\frac{1-\varepsilon}{1+\varepsilon}}\right)^{-\frac{(1+\varepsilon)^2}{2(1-\varepsilon)}}, & \varepsilon \in (0,1). \end{cases}$$

Due to this, if $u(\cdot, t) \in H^1(\Lambda)$, then by the Sobolev embedding $H^1(\Lambda) \hookrightarrow C(\Lambda)$, we have

$$|u(x,t,g^{(k)}) - u(x,t,g^\dagger)| \leq C_S \left\|u(g^{(k)}) - u(g^\dagger)\right\|_{H^1(\Lambda)} \leq C_S B^{LM}, \quad \forall\, x \in (0,\Lambda),\ t \in (0,T),$$

where $C_S > 0$ is a constant depending only on the domain $\Lambda = (0,l)$. If the solution is computed on a discrete positions $\{x_i\}_{i=1}^n$, then

$$\max_i |u(x_i,t,g^{(k)}) - u(x_i,t,g^\dagger)| \leq \|u(g^{(k)}) - u(g^\dagger)\|_\infty \leq \|u(g^{(k)}) - u(g^\dagger)\|_2 \leq B^{LM}.$$

Therefore, the LM error can be bounded as

$$|u(x,t,g^{(k)}) - u(x,t,g^\dagger)| \leq B^{LM}.$$

On the other hand, for the convergence of the RBFs method, according to [13], when implementing the discretization scheme from subsection 2.2 for the forward problem (1)-(3) with given initial data $g(x)$, we achieve an optimal convergence rate:

$$B^{RBF} \sim \mathcal{O}\Big(e^{-c/h^2}\Big),$$

where $h$ is the spatial grid size and $c > 0$ is a constant.

By combining these two error contributions, the total pointwise error of the numerical solution can be expressed as

$$|u(x,t,g^{(k)}) - u(x,t,g^{\dagger})| \leq B^{LM} + B^{RBF}.$$

Thus, for each point $(x,t)$, the LM error decreases with the number of iterations $k$, and the RBFs discretization error decreases as the spatial grid is refined ($h \to 0$).

# 4 Numerical results

In this section, we present a numerical example to evaluate the correctness and accuracy of the proposed algorithm. The example is an inverse parabolic problem arising from the Black-Scholes equation, which is widely used in financial mathematics.

**Example 4.1.** Consider the linear inverse parabolic problem of the form

$$u_t - u_{xx} = f(x,t), \quad (x,t) \in Q_T, \tag{15}$$

$$u(x,0) = g(x), \quad x \in \Lambda, \tag{16}$$

$$u(0,t) = u(l,t) = 0, \quad 0 \leq t \leq T, \tag{17}$$

$$u(x,T) = x^5(1-x), \quad 0 \leq x \leq l, \tag{18}$$

in which the exact solution is $u(x,t) = tx^5(1-x)$.

In this problem, both $u(x,t)$ and the initial condition $g(x) = u(x,0)$ are unknown. We set $\alpha = 1$, $T = 1$, and $l = 1$, and apply the proposed method for two different spatial step sizes: $h = 0.25$, and $h = 0.17$. The numerical results are presented in Table 1 for $m = n = 4$, and Table 2 for $m = n = 6$.

To solve this problem, according to subsection 2.3, we assume the radial basis function is Gaussian, i.e.,

$$\varphi(r) = e^{-\alpha r^2} = e^{-\alpha\|x-x_j\|^2}, \qquad \text{with } \alpha = 1.$$

And approximate the solution of the problem as

$$u(x,t) \approx \sum_{j=0}^{6} c_j(t)\, \varphi(\|x - x_j\|).$$

We then implement the discretization scheme (9) for the problem defined by equations (15)–(17) and obtain the coefficients $c_{i,j}$ for $h = 0.17$ as follows:

$$c_{0,0} \to 7.09339 \times 10^{-11} + 1.d_0(0) - 7.69825 \times 10^{-11} d_1(0) - \cdots - 7.08851 \times 10^{-11} d_4(0),$$
$$c_{0,1} \to -0.469962 - 26.9086 d_0(0) - 28.7013 d_1(0) - \cdots - 22.5163 d_4(0),$$
$$\vdots$$
$$c_{3,4} \to -14.6405 - 0.178732 d_0(0) - 0.218237 d_1(0) - \cdots - 0.22516 d_4(0),$$
$$c_{4,4} \to -21.8659 - 0.0354034 d_0(0) - 0.0429502 d_1(0) - \cdots - 0.0401768 d_4(0),$$

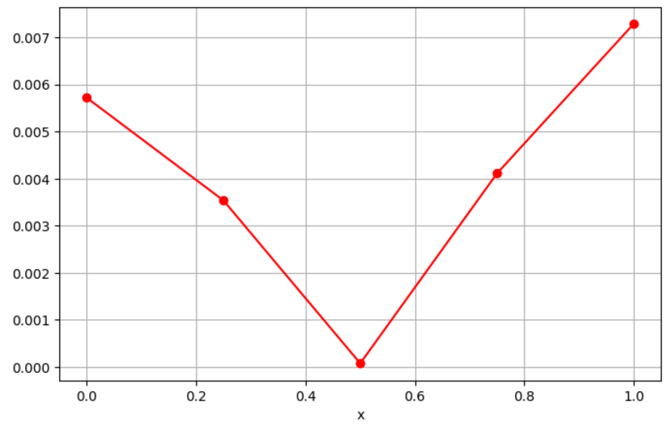


**Figure 1:** Absolute error plot of $|g(x_i) - \bar{g}(x_i)|$ for $m = n = 4$.

After applying the LM regularization method to estimate the unknown coefficients $d_i$, we observe that it converges from the 11th iteration onward.

The absolute errors between the exact solution $g(x)$ and the approximate solution $\bar{g}(x)$ at the points for $h = 0.25$ with $m = n = 4$ and $h = 0.17$ with $m = n = 6$ are presented in Table 1 respectively. Furthermore, the plots of the absolute errors obtained by the proposed method are shown in Figures 1, and 2. The approximate values of the solution $u(x, t)$ are given in Table 2, and the corresponding surface plot is shown in Figure 3. In this example, we verify the correctness and accuracy of the proposed method.

**Table 1:** The absolute error of $|g(x_i) - \bar{g}(x_i)|$, for $m = n = 4$, and $m = n = 6$.

| $x_i$ | 0 | 0.25 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|---|
| $\lvert g(x_i) - \bar{g}(x_i)\rvert$ | 0.0516536 | 0.126086 | 0.0676948 | 0.0977676 | 0.267226 |

| $x_i$ | 0 | 0.16667 | 0.33333 | 0.5 | 0.66667 | 0.83333 | 1 |
|---|---|---|---|---|---|---|---|
| $\lvert g(x_i) - \bar{g}(x_i)\rvert$ | 0.00430058 | 0.00130213 | 0.00336878 | 0.0174847 | 0.461465 | 0.068641 | 0.000147593 |

**Table 2:** Approximate solution $u(x, t)$ values for $m = n = 6$.

| $x \backslash t$ | 0.00 | 0.17 | 0.33 | 0.50 | 0.67 | 0.83 | 1.00 |
|---|---|---|---|---|---|---|---|
| 0.00 | 0.000036 | 0.000006 | 0.000012 | 0.000018 | 0.000024 | 0.000030 | 0.000036 |
| 0.17 | 0.000014 | 0.000002 | 0.000005 | 0.000007 | 0.000009 | 0.000012 | 0.000014 |
| 0.33 | −0.000011 | −0.000002 | −0.000004 | −0.000005 | −0.000007 | −0.000009 | −0.000011 |
| 0.50 | −0.000035 | −0.000006 | −0.000012 | −0.000018 | −0.000023 | −0.000029 | −0.000035 |
| 0.67 | −0.000056 | −0.000009 | −0.000019 | −0.000028 | −0.000037 | −0.000046 | −0.000056 |
| 0.83 | −0.000069 | −0.000012 | −0.000023 | −0.000035 | −0.000046 | −0.000058 | −0.000069 |
| 1.00 | −0.000075 | −0.000013 | −0.000025 | −0.000038 | −0.000050 | −0.000063 | −0.000075 |

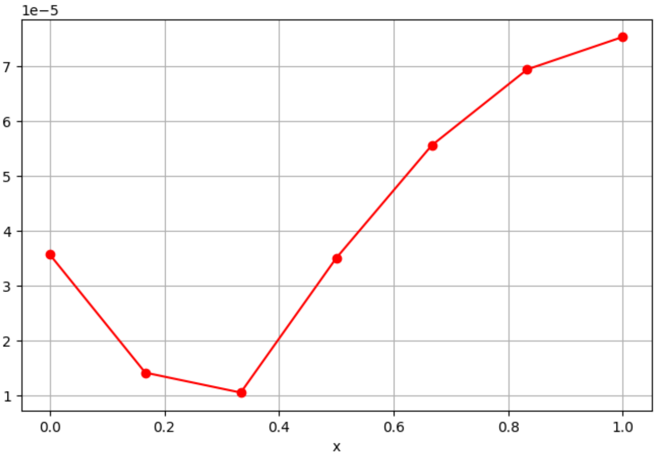


**Figure 2:** Absolute error plot of $|g(x_i) - \bar{g}(x_i)|$ for $m = n = 6$.

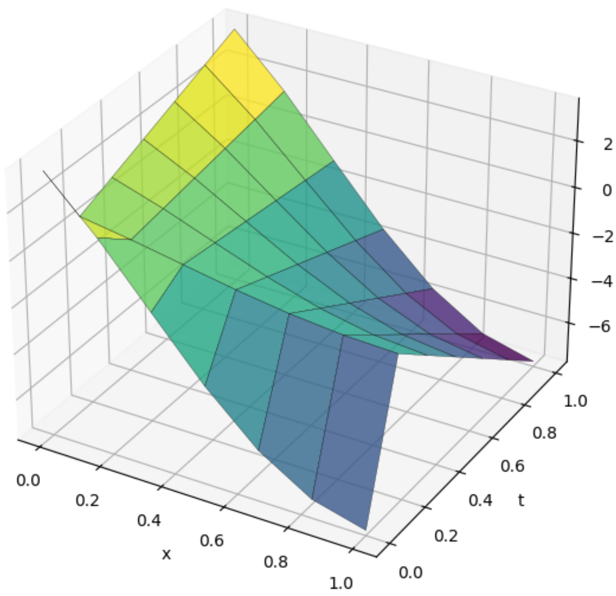


**Figure 3:** 3D surface plot of the approximate solution $u(x, t)$.

# 5 Conclusion

In this paper, we have proposed a numerical method for solving an inverse parabolic problem arising from the Black-Scholes equation, which is fundamental in financial mathematics. By employing Gaussian RBFs for spatial discretization and implementing LM regularization for coefficient estimation, we have demonstrated the effectiveness of the proposed algorithm in approximating the solution. Through numerical experiments, we have verified the method's ability to efficiently solve the inverse problem with suitable accuracy, making it a promising tool for similar problems in mathematical finance and related fields.

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*How to Cite:* Jamalpour Malekabadi, Moslem[1], Zakeri, Ali[2], Salehi Shayegan, Amir Hossein[3], *Application of Radial Basis Functions Meshless Method for Solving an Inverse Parabolic Problem*, Journal of Mathematics and Modeling in Finance (JMMF), Vol. 6, No. 1, Pages:251–263, (2026).