Journal of Mathematics and Modeling in Finance (JMMF)
Vol. 6, No. 2, Summer & Fall 2026
Research paper

Fuzzy Estimation of Value at Risk for a Portfolio with Triangular Fuzzy Returns

Batoul Salamah[1], Reza Zarei[2], Farshid Mehrdoust[3], Mohammad Ghasem Akbari[4]

[1] Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Guilan, Iran.

batoul.salamah91@gmail.com

[2] Department of Statistics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Guilan, Iran.
r.zarei@guilan.ac.ir

[3] Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Guilan, Iran.

fmehrdoust@guilan.ac.ir

[4] Department of Statistics, Faculty of Mathematical Sciences, University of Birjand, Birjand, Iran.
g_z_akbari@birjand.ac.ir

Abstract:
Value at Risk (VaR) is a key measure in financial risk management. However, traditional VaR models are often challenged by the inherent uncertainty and ambiguity in market data. This paper introduces a novel method for estimating VaR under fuzzy conditions to address this limitation. In this study, we consider a linear portfolio consisting of ten stocks whose returns are imprecise and vague. To handle this vagueness, we assume that the portfolio returns follow a normal distribution and are represented as triangular fuzzy numbers. The proposed method employs α-cut sets to compute the fuzzy VaR for the portfolio. Additionally, we use daily log returns to estimate the returns for each stock over the specified period. By applying this method, we can calculate the lower and upper bounds, as well as the core values of the α-cuts, for the fuzzy VaR metric of the portfolio. The numerical results demonstrate that fuzzy VaR yields more accurate estimates compared to traditional VaR. This study illustrates how fuzzy VaR techniques improve decision-making under ambiguity by providing a more realistic representation of financial uncertainty.

Keywords: Fuzzy value at risk; α-cuts sets; Variance-covariance method; Portfolio model; Triangular fuzzy return.
Classification: 91G70, 03E72, 62P05.

1 Introduction

Financial risk is a fundamental characteristic of contemporary markets, significantly impacting investment strategies, regulatory frameworks, and overall economic stability. The term "risk" refers to the likelihood that the actual outcomes of an investment will diverge from expectations, potentially leading to unfavorable results, such as the loss of part or all of the initial investment. In finance, a portfolio is a collection of

[2] Corresponding author
Received: 24/11/2025 Accepted: 25/04/2026
https://doi.org/10.22054/jmmf.2026.89886.1242

financial assets held to achieve diversification and balance between risk and return [24]. Portfolio risk denotes the variability of a portfolio's returns, typically evaluated through probabilistic models. In today's environment, effective management of financial risk stands as one of the primary challenges in modern finance, encompassing the processes of identifying, assessing, quantifying, and mitigating risks [8].

Value at Risk (VaR) has emerged as one of the most widely applied statistical measures for quantifying portfolio-level risk. VaR summarizes the maximum expected loss over a specified time horizon at a given confidence level. In the late 1980s, major financial institutions began employing VaR, and its use expanded rapidly after J.P. Morgan's release of the RiskMetrics™ system in 1994 [25], which established it as a market standard. Today, VaR is extensively used by financial institutions, corporations, and institutional investors, and has been endorsed by regulatory bodies like the Basel Committee for calculating capital requirements for market risk [3].

Several researchers have analyzed VaR as a benchmark in risk measurement [1, 14, 17, 32, 35, 41]. It possesses key features that contribute to its widespread adoption: (1) it is a simple, single summary measure; (2) it consolidates risks across different assets and portfolios, aiding in management; (3) it is a forward-looking measure based on current holdings; (4) it applies consistently across various asset classes, enabling direct comparison; (5) it focuses on the tail of the return distribution, measuring downside risk; and (6) its methodology can be adapted to enhance risk measurement techniques [28].

The VaR measure has also been integrated into portfolio optimization problems to control downside risk. For instance, Campbell et al. [6] created a model to maximize expected return subject to VaR limits, while others have applied VaR to stock exchange portfolios [2] and developed models incorporating VaR constraints [16, 40, 47].

However, traditional VaR has well-established limitations. First, it often assumes that past data accurately forecasts future risk, which can lead to underestimation during market shifts. Second, it frequently assumes normally distributed returns, ignoring the fat tails and skewness that characterize real financial data and increase the risk of extreme losses. Third, by focusing solely on a quantile, it ignores the magnitude of losses beyond that point [11]. Fourth, and most critically for this paper, it generates a single "crisp" number that conceals the inherent ambiguity in financial markets, creating a misleading sense of precision.

As Knight [18] distinguished between risk and uncertainty, VaR's effectiveness depends on well-defined probabilities, which limits its ability to handle situations dominated by imprecision. To address this, recent research has employed fuzzy set theory, founded by Zadeh [43], in financial risk management. Fuzzy VaR models extend the traditional concept by allowing risk factors to be represented with degrees of membership, which more realistically captures uncertainty. Fuzzy logic is well-suited for cases with incomplete or imprecise data and can incorporate expert opinions, making it valuable for assessing risks that are not fully understood by probability-based models alone [22, 33].

Research on fuzzy VaR has grown significantly, with numerous studies developing methodologies to model VaR under uncertainty [5, 19, 23, 26, 27, 29, 30, 37, 39, 42, 44–46]. For example, Guerra and Sorini [12] developed fuzzy-based models that provided more accurate capital allocation than historical simulation. Shiraz et al. [34] integrated VaR and CVaR into portfolio selection with random-fuzzy returns. Reyes et al. [31] applied fuzzy logic to credit risk, incorporating a fuzzy triangular VaR adjustment. Despite these advances, some existing studies rely heavily on specific, complex algorithms, limiting their methodological diversity and practical applicability [9, 30, 37, 46].

To overcome these limitations, this paper introduces a fuzzy VaR method using α-cut sets within the variance–covariance (linear-normal) VaR framework. In this approach, the returns of assets and the portfolio are represented as asymmetric triangular fuzzy numbers and are assumed to be dependent and normally distributed. The method constructs confidence intervals for the mean and variance of returns to derive fuzzy mean and variance numbers. We use daily log-returns for each asset and compute α-cuts for the fuzzy mean and variance. By applying these bounds to the VaR formula, we obtain a fuzzy VaR measure for the portfolio. Additionally, we apply the WU interval method [38] based on MLE estimators to compute an approximate membership function for the fuzzy VaR. The performance of the proposed method is explored using daily closing stock prices for ten companies from the Tehran Stock Exchange.

The remainder of this paper is structured as follows: Section 2 reviews fundamental concepts of fuzzy numbers and introduces the variance-covariance method for VaR in portfolios. It also covers additional topics such as maximum likelihood estimation and fuzzy estimators of fuzzy parameters. The proposed approach used to estimate the fuzzy VaR for the portfolio model is discussed in Section 3. Section 4 provides an illustrative numerical example. Finally, Section 5 presents the conclusions and suggests avenues for future research.

2 Preliminaries and Fundamental Concepts

In this section, we review the key terms and concepts that will be used throughout this paper.

2.1 Fuzzy numbers

In this subsection, we recall some definitions pertaining to fuzzy numbers [10, 43].

Assume that $\widetilde{B}$ is a fuzzy subset of $\mathbb{R}$. The definition of $\widetilde{B}$ is provided by a membership function, expressed as $\mu_{\widetilde{B}} : \mathbb{R} \to [0, 1]$.

Definition 2.1. For every $\alpha \in (0,1]$, the α-cut of a fuzzy set $\widetilde{B}$, denoted by $\widetilde{B}[\alpha]$, is defined as $\{y \in \mathbb{R} : \mu_{\widetilde{B}}(y) \geq \alpha\}$.

Definition 2.2. The lower and upper bounds of $\widetilde{B}[\alpha]$ for $\alpha \in [0,1]$ are denoted by $\widetilde{B}^L_\alpha = \inf\{y : y \in \widetilde{B}[\alpha]\}$ and $\widetilde{B}^U_\alpha = \sup\{y : y \in \widetilde{B}[\alpha]\}$, respectively.

Definition 2.3. A fuzzy set $\widetilde{B}$ of $\mathbb{R}$ is called a fuzzy number if it is normal and $\widetilde{B}[\alpha]$ is a non-empty, bounded, closed interval of $\mathbb{R}$ for every $\alpha \in [0,1]$.

Among the various types of fuzzy numbers, triangular fuzzy numbers (TFNs) are the most frequently used in examples and applications due to their simplicity. A TFN $\widetilde{B}$ is represented as $\widetilde{B} = (b_1, b_2, b_3)_T$, and its membership function is defined as follows

$$\mu_{\widetilde{B}}(y) = \begin{cases} 0, & y < b_1, \\ \dfrac{y - b_1}{b_2 - b_1}, & b_1 \leq y \leq b_2, \\ \dfrac{b_3 - y}{b_3 - b_2}, & b_2 \leq y \leq b_3, \\ 0, & y > b_3, \end{cases} \tag{1}$$

where $b_1, b_2, b_3 \in \mathbb{R}$ and $b_1 \leq b_2 \leq b_3$.

Definition 2.4. Consider $\widetilde{B} = (b_1, b_2, b_3)_T$ and $\widetilde{D} = (d_1, d_2, d_3)_T$, and let γ be a scalar. Then,

$$1) \quad \widetilde{B} + \widetilde{D} = (b_1 + d_1, b_2 + d_2, b_3 + d_3),$$

$$2) \quad \widetilde{B} - \widetilde{D} = (b_1 - d_3, b_2 - d_2, b_3 - d_1),$$

$$3) \quad \gamma \times \widetilde{B} = \begin{cases} (\gamma b_1, \gamma b_2, \gamma b_3), & \text{if } \gamma > 0, \\ (\gamma b_3, \gamma b_2, \gamma b_1), & \text{if } \gamma < 0. \end{cases}$$

2.2 Value at risk

According to John C. Hull [15], “VaR is a tool that gives a possibility to estimate the total risk of the portfolio.” A common definition of VaR is as follows: “Value at risk is the maximum amount of money that may be lost on a portfolio over a given period of time, with a given level of confidence” [4]. As an illustration, a daily VaR of 100 euros with a 95% confidence level indicates that, under typical market circumstances, the loss will not exceed 100 euros in 95 out of 100 days.

We assume a linear portfolio composed of n financial assets or stocks with current value $P(t)$ at time t. Over a time period $[0,t]$, its profit or loss is determined by

$$\Delta P(t) = P(t) - P(0) = \delta_1 R^1_t + \delta_2 R^2_t + \cdots + \delta_n R^n_t, \tag{2}$$

where $R_t^1, \ldots, R_t^n$ represent the returns of its constituents over the same period. Also, $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ is the portfolio weight vector such that $\delta_i \geq 0$ and $\delta_1 + \delta_2 + \cdots + \delta_n = 1$ for all $i = 1, \cdots, n$.

The VaR metric at a confidence level of $1 - \theta$ (or a risk level of θ) is determined by the solution of the following equation:

$$P\big\{\Delta P(t) < -\mathrm{VaR}_\theta\big\} = \theta. \tag{3}$$

It is common practice to record portfolio losses as negative numbers, but equation (3) expresses VaR as a positive amount of money [26].

An alternative mathematical representation of the VaR metric is as follows:

$$P(\Delta P_t \leq F^{-1}_{\Delta P,t}(\theta)) = \theta, \tag{4}$$

$$VaR_{\Delta P,t}(\theta) = F^{-1}_{\Delta P,t}(\theta), \tag{5}$$

where ΔP_t is the portfolio's return rate and $F^{-1}_{\Delta P,t}(\theta)$ is the quantile of the return distribution corresponding to the probability θ [13].

Variance-Covariance method

The variance-covariance approach is a parametric method for determining VaR, also referred to as Linear VaR or Delta-Normal VaR. It is the most popular and straightforward approach to estimating VaR.

We presume to calculate the VaR of a portfolio without allocating the VaR to various risk factors. We also assume that all stock returns in the portfolio are dependent and identically distributed, adhering to a normal distribution where $\Delta P_t \sim \mathcal{N}(\mu_p, \sigma_p^2)$.

In this context, we take into account the following

- $\delta = [\delta_1, \delta_2, \cdots, \delta_n]^T$ is a vector of portfolio weights.
- Expected portfolio return:

  $$\mu_p = \delta^T \mu, \tag{6}$$

  where $\mu = [\mu_1, \mu_2, \cdots, \mu_n]^T$ is the vector of expected returns.
- Portfolio variance:

  $$\sigma_p^2 = \delta^\top \mathbf{\Sigma} \delta, \tag{7}$$

  where Σ represents the $n \times n$ covariance matrix of all stock returns, with elements $\sigma_{ij} = cov(X_i, X_j)$.
- z_θ denotes the quantile of the standard normal distribution corresponding to the confidence level $1 - \theta$.

The VaR analytical formula for a portfolio with dependent and identically distributed normal returns is given by:

$$\text{VaR}_\theta = -\mu_p - z_\theta\,\sigma_p. \tag{8}$$

Due to the symmetry of the normal distribution, $z(\theta) = -z(1-\theta)$. Thus, we can reformulate equation (8) as follows

$$\text{VaR}_\theta = -\mu_p + z_{1-\theta}\,\sigma_p. \tag{9}$$

To provide a more precise time horizon for our VaR estimate, we can express it as follows

$$\text{VaR}_\theta = -\mu_p T + z_{1-\theta}\,\sigma_p\sqrt{T}. \tag{10}$$

This is a straightforward formula for the $100\theta\%$ T-day VaR, expressed as a percentage of the portfolio value.

Taking the percentage VaR and multiplying it by the portfolio's current value yields the VaR in value terms

$$\text{VaR}_\theta = (-\mu_p T + z_{1-\theta}\,\sigma_p\sqrt{T}) \times P(t), \tag{11}$$

where P is the current value of the portfolio, T represents the time horizon (years if returns are annualized; days if daily returns are used), μ_p is the expected portfolio return and σ_p denotes the standard deviation of the portfolio.

2.3 Maximum likelihood estimation

Assume that the returns of stocks comprising the portfolio are $R_1, \ldots, R_n$, and let m be the number of return observations for each stock, where $j = 1, \cdots, m$. That is, for each stock, we have returns $R = (R_1, R_2, \cdots, R_m)$.

Assume that the return observations associated with each stock, $R = (R_1, R_2, \cdots, R_m)$, are independent and identically distributed from a normal distribution, i.e., $R_j \sim \mathcal{N}(\mu, \sigma^2)$.

The normal distribution $\mathcal{N}(\mu, \sigma^2)$ has probability density function

$$f(R \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(R-\mu)^2}{2\sigma^2}\right). \tag{12}$$

The likelihood function is given by:

$$f(R_1, \ldots, R_m \mid \mu, \sigma^2) = \prod_{j=1}^{m} f(R_j \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{m/2} \exp\left(-\frac{\sum_{j=1}^{m}(R_j-\mu)^2}{2\sigma^2}\right). \tag{13}$$

The normal distribution has two parameters $\eta = (\mu, \sigma)$, so we maximize the likelihood $\mathcal{L}(\mu, \sigma^2) = f(R_1, \ldots, R_m \mid \mu, \sigma^2)$ over both parameters.

The logarithm function is continuous and strictly increasing over the likelihood range, meaning that the values that maximize the likelihood will also maximize the logarithm of that function. The log-likelihood can be expressed as follows

$$\log(\mathcal{L}(\mu,\sigma^2)) = -\frac{m}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{j=1}^{m}(R_j-\mu)^2. \tag{14}$$

Next, we calculate the derivatives of this log-likelihood as follows

$$0 = \frac{\partial}{\partial\mu}\log\left(\mathcal{L}(\mu,\sigma^2)\right) = 0 - \frac{-2m(\bar{R}-\mu)}{2\sigma^2}. \tag{15}$$

Here, the sample mean is denoted by $\bar{R}$. This is solved by

$$\widehat{\mu} = \bar{R} = \frac{1}{m}\sum_{j=1}^{m} R_j. \tag{16}$$

We differentiate the log-likelihood with respect to σ in a similar manner and then set the result to zero

$$0 = \frac{\partial}{\partial\sigma}\log\left(\mathcal{L}(\mu,\sigma^2)\right) = -\frac{m}{\sigma} + \frac{1}{\sigma^3}\sum_{j=1}^{m}(R_j-\mu)^2. \tag{17}$$

This has a solution given by

$$\widehat{\sigma}^2 = \frac{1}{m}\sum_{j=1}^{m}(R_j-\mu)^2. \tag{18}$$

Recalling that $\mu = \widehat{\mu}$, we obtain

$$\widehat{\sigma}^2 = \frac{1}{m}\sum_{j=1}^{m}(R_j-\widehat{\mu})^2. \tag{19}$$

The maximum likelihood estimator for $\theta = (\mu,\sigma^2)$ is $\widehat{\theta} = (\widehat{\mu},\widehat{\sigma}^2)$.

2.4 Fuzzy estimators of fuzzy parameters

Let Θ represent the parameter space. The fuzzy parameter $\widetilde{\theta}$ is a fuzzy subset of Θ, with a membership function denoted by $\zeta_{\widetilde{\theta}}(y) : \Theta \to [0,1]$. Here, the fuzzy parameter $\widetilde{\theta}$ is considered to be a fuzzy real number. If $\widetilde{\theta}$ is a fuzzy parameter, then both $\widetilde{\theta}^L_\alpha$ and $\widetilde{\theta}^U_\alpha$ exist within the parameter space Θ for all $\alpha \in [0,1]$. Consequently, we can examine the estimates of $\widetilde{\theta}^L_\alpha$ and $\widetilde{\theta}^U_\alpha$ for all $\alpha \in [0,1]$.

Assume that the distribution of the random variable R is defined by the parameters $\theta_1, \theta_2, ..., \theta_n$, and assume that $\widetilde{R}$ is a fuzzy random variable. We assert that $\widetilde{R}$ possesses a distribution characterized by fuzzy parameters $\widetilde{\theta}_1, ..., \widetilde{\theta}_n$, induced by R, if $\widetilde{R}^L_\alpha$ and $\widetilde{R}^U_\alpha$ exhibit the same distribution as R with parameters $(\widetilde{\theta}_1)^L_\alpha, \cdots, (\widetilde{\theta}_n)^L_\alpha$ and $(\widetilde{\theta}_1)^U_\alpha, \cdots, (\widetilde{\theta}_n)^U_\alpha$, respectively [38].

Example 2.5. We assert that $\widetilde{R}$ possesses a (fuzzy) gamma distribution characterized by fuzzy parameters $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$ if $\widetilde{R}^L_\alpha$ and $\widetilde{R}^U_\alpha$ exhibit gamma distributions with parameters $(\widetilde{\theta}_1)^L_\alpha$, $(\widetilde{\theta}_2)^L_\alpha$ and $(\widetilde{\theta}_1)^U_\alpha$, $(\widetilde{\theta}_2)^U_\alpha$, respectively, for all $\alpha \in [0,1]$.

Let $\widetilde{\theta}$ denote any fuzzy parameter of the fuzzy random variable $\widetilde{R}$. We aim to estimate $\widetilde{\theta}$ in fuzzy environments by using the α-level set $\widetilde{\theta}[\alpha] = [\widetilde{\theta}^L_\alpha, \widetilde{\theta}^U_\alpha]$. Then, for any parameter value in $\widetilde{\theta}[\alpha] = [\widetilde{\theta}^L_\alpha, \widetilde{\theta}^U_\alpha]$, we have $\theta = \widetilde{\theta}^L_\beta$ or $\theta = \widetilde{\theta}^U_\beta$ for some $\beta \geqslant \alpha$. Therefore, we can associate an estimate $\widehat{\theta}$ with each such θ. If we construct an interval

$$B[\alpha] = \left[\min\left\{ \inf_{\alpha\leqslant\beta\leqslant 1} \widehat{\widetilde{\theta}^L_\beta}, \inf_{\alpha\leqslant\beta\leqslant 1} \widehat{\widetilde{\theta}^U_\beta} \right\}, \max\left\{ \sup_{\alpha\leqslant\beta\leqslant 1} \widehat{\widetilde{\theta}^L_\beta}, \sup_{\alpha\leqslant\beta\leqslant 1} \widehat{\widetilde{\theta}^U_\beta} \right\} \right], \tag{20}$$

this interval will contain all of the estimates associated with each $\widetilde{\theta}[\alpha] = [\widetilde{\theta}^L_\alpha, \widetilde{\theta}^U_\alpha]$. The fuzzy estimator of $\widetilde{\theta}$ is represented by $\widehat{\widetilde{\theta}}$, and its corresponding membership function is defined as follows:

$$\zeta_{\widehat{\widetilde{\theta}}}(y) = \sup_{0\leqslant\alpha\leqslant 1} \alpha\, \mathbf{1}_{B_\alpha}(y), \tag{21}$$

where $\mathbf{1}_{B_\alpha}(y)$ denotes the indicator function of the set B_α.

3 A Fuzzy VaR Estimation Method Using α-Cut Sets

This section introduces the α-cut sets method for estimating the fuzzy VaR metric for a portfolio model. We assume that we have a portfolio consisting of n stocks. Furthermore, we consider the stock returns as triangular fuzzy numbers, and these returns are dependent and follow a normal distribution, i.e., $R \sim \mathcal{N}(\mu, \sigma^2)$.

Based on historical stock data, we assume that the dataset includes the daily closing stock prices for each stock on working days within a specified time period. We apply the daily log return formula to the daily closing prices to estimate the returns of each stock over that period.

If P_t represents the price of a stock at the end of day t, and P_{t-1} is the price at the end of the previous day, then the daily log return, denoted by R_t, is defined as $R_t = \ln\left(\dfrac{P_t}{P_{t-1}}\right)$.

Thus, for each stock, the daily log returns during the specified time period are represented as $R = (R_1, R_2, \dots, R_{m-1})$, where $j = 1, \dots, m$ and m denotes the number of price observations for each stock within that period.

Subsequently, we construct the $(1-\beta)100\%$ confidence interval for the mean and variance of the stock's fuzzy returns, as outlined below.

1. The $(1-\beta)100\%$ confidence interval for the mean return of one stock

$$CI_{\mu_R} = \left(\bar{R} - t_{\{m-1,1-\frac{\beta}{2}\}} \times \frac{s}{\sqrt{m}},\ \bar{R} + t_{\{m-1,1-\frac{\beta}{2}\}} \times \frac{s}{\sqrt{m}} \right), \tag{22}$$

where $t_{\{m-1,1-\frac{\beta}{2}\}}$ is the $(1-\frac{\beta}{2})$-th quantile of the Student's t-distribution with $(m-1)$ degrees of freedom, $\bar{R}$ is the sample mean, and s is the sample standard deviation of the stock's fuzzy returns.

Hence, we set

$$\mu^c = \bar{R}, \quad \mu^l = \bar{R} - t_{\{m-1,1-\frac{\beta}{2}\}} \times \frac{s}{\sqrt{m}}, \quad \mu^u = \bar{R} + t_{\{m-1,1-\frac{\beta}{2}\}} \times \frac{s}{\sqrt{m}},$$

which represent the central value, the lower bound, and the upper bound, respectively, for all α-cuts of the mean of the stock's fuzzy returns.

Thus, the mean of the stock's fuzzy returns can be described as the triangular fuzzy number $\widetilde{\mu}_R = (\mu^l, \mu^c, \mu^u)$.

2. The $(1-\beta)100\%$ confidence interval for the variance of returns for one stock

$$CI_{\sigma_R^2} = \left(\frac{(m-1)s^2}{\chi^2_{m-1,\frac{\beta}{2}}},\ \frac{(m-1)s^2}{\chi^2_{m-1,1-\frac{\beta}{2}}} \right), \tag{23}$$

where $\chi^2_{m-1,\frac{\beta}{2}}$ and $\chi^2_{m-1,1-\frac{\beta}{2}}$ are the $\frac{\beta}{2}$-th and $(1-\frac{\beta}{2})$-th quantiles of the chi-squared distribution with $(m-1)$ degrees of freedom, and s is the sample standard deviation of the stock's fuzzy returns.

We then define

$$\sigma^c = \sqrt{s^2}, \quad \sigma^l = \sqrt{\frac{(m-1)s^2}{\chi^2_{m-1,\frac{\beta}{2}}}}, \quad \sigma^u = \sqrt{\frac{(m-1)s^2}{\chi^2_{m-1,1-\frac{\beta}{2}}}},$$

which represent the central value, the lower bound, and the upper bound, respectively, for all α-cuts of the standard deviation of the stock's fuzzy returns.

Consequently, the standard deviation of the stock's fuzzy returns is represented as the triangular fuzzy number $\widetilde{\sigma}_R = (\sigma^l, \sigma^c, \sigma^u)$.

We represent the α-cuts of the stock's fuzzy returns $\widetilde{R} = (R^l, R^c, R^u)$ for $\alpha \in [0,1]$ as follows

$$\widetilde{R}[\alpha] = \left[R^l + \alpha(R^c - R^l),\ R^u - \alpha(R^u - R^c) \right], \tag{24}$$

where $\widetilde{R}^L_\alpha = R^l + \alpha(R^c - R^l)$ and $\widetilde{R}^U_\alpha = R^u - \alpha(R^u - R^c)$ are the lower and upper bounds, respectively, for all α-cuts of the stock's fuzzy returns. Moreover, $\widetilde{R}^L_\alpha \sim \mathcal{N}(\mu^L_\alpha, {\sigma^2}^L_\alpha)$ and $\widetilde{R}^U_\alpha \sim \mathcal{N}(\mu^U_\alpha, {\sigma^2}^U_\alpha)$.

The α-cuts of the mean and standard deviation of the fuzzy returns for one stock are given, respectively, by

$$\widetilde{\mu}_R[\alpha] = [\mu^L_\alpha, \mu^U_\alpha] = [\mu^l + \alpha(\mu^c - \mu^l),\ \mu^u - \alpha(\mu^u - \mu^c)], \tag{25}$$

and

$$\widetilde{\sigma}_R[\alpha] = [\sigma^L_\alpha, \sigma^U_\alpha] = [\sigma^l + \alpha(\sigma^c - \sigma^l),\ \sigma^u - \alpha(\sigma^u - \sigma^c)]. \tag{26}$$

To calculate the fuzzy VaR measure for one stock in the portfolio at a confidence level of $1-\theta$, using the variance-covariance method described in Subsection 2.2 and employing α-cuts, we apply the following steps:

(a) Compute the lower bound of the α-cuts associated with the fuzzy VaR metric

$$VaR^l = \widetilde{VaR}^L_\alpha = VaR_\theta(\widetilde{R}^L_\alpha) = -\mu^L_\alpha + z_{1-\theta}\sigma^L_\alpha, \tag{27}$$

where $z_{1-\theta}$ is the $(1-\theta)$-th quantile of $\mathcal{N}(0,1)$, and μ^L_α and σ^L_α denote the lower bounds of the α-cuts of the fuzzy mean and standard deviation of the stock's fuzzy returns.

(b) Compute the upper bound of the α-cuts associated with the fuzzy VaR metric:

$$VaR^u = \widetilde{VaR}^U_\alpha = VaR_\theta(\widetilde{R}^U_\alpha) = -\mu^U_\alpha + z_{1-\theta}\sigma^U_\alpha, \tag{28}$$

where μ^U_α and σ^U_α denote the upper bounds of the α-cuts of the fuzzy mean and standard deviation of the stock's fuzzy returns.

(c) Compute the core value of the α-cuts for the fuzzy VaR metric:

$$VaR^c = VaR_\theta(R^c) = -\mu^c + z_{1-\theta}\sigma^c, \tag{29}$$

where μ^c and σ^c represent the core values of the fuzzy mean and standard deviation of the stock's fuzzy returns.

Finally, we can determine the α-cut of the fuzzy VaR metric at a confidence level of $1-\theta$ as follows:

$$\widetilde{VaR}[\alpha] = [\widetilde{VaR}^L_\alpha, \widetilde{VaR}^U_\alpha]. \tag{30}$$

Thus, the fuzzy VaR for one stock or asset is expressed as the triangular fuzzy number

$$\widetilde{VaR} = (VaR^l, VaR^c, VaR^u). \tag{31}$$

Then, for any parameter value $\widetilde{VaR} \in [\widetilde{VaR}^L_\alpha, \widetilde{VaR}^U_\alpha]$, we have $\widetilde{VaR} = \widetilde{VaR}^L_\beta$ or $\widetilde{VaR} = \widetilde{VaR}^U_\beta$ for some $\beta \geq \alpha$. Thus, we can associate the parameter $\widetilde{VaR} \in$

$[\widetilde{VaR}_\alpha^L, \widetilde{VaR}_\alpha^U]$ with its estimate $\widehat{\widetilde{VaR}}$ by constructing the following interval, as proposed by Wu [38]

$$B[\alpha] = \left[\min\left\{\inf_{\alpha\leqslant\beta\leqslant1} \widehat{\widetilde{VaR}}_\beta^L, \inf_{\alpha\leqslant\beta\leqslant1} \widehat{\widetilde{VaR}}_\beta^U\right\}, \max\left\{\sup_{\alpha\leqslant\beta\leqslant1} \widehat{\widetilde{VaR}}_\beta^L, \sup_{\alpha\leqslant\beta\leqslant1} \widehat{\widetilde{VaR}}_\beta^U\right\}\right]. \tag{32}$$

This interval contains all estimates related to each $\widetilde{VaR} \in [\widetilde{VaR}_\alpha^L, \widetilde{VaR}_\alpha^U]$.

For computing the fuzzy mean number for the stock's fuzzy returns, we construct the same confidence interval as in Equation 22 using the MLE estimator for the mean stock returns. We also express the α-cuts for the fuzzy mean of stock returns $\widetilde{\mu} = (\mu^l, \mu^c, \mu^u)$ as follows

$$\widehat{\widetilde{\mu}}_R[\alpha] = [\widehat{\mu}_\alpha^L, \widehat{\mu}_\alpha^U] = [\widehat{\mu}^l + \alpha(\widehat{\mu}^c - \widehat{\mu}^l),\ \widehat{\mu}^u - \alpha(\widehat{\mu}^u - \widehat{\mu}^c)]. \tag{33}$$

Based on the MLE estimator for the mean, $\bar{R} = \frac{1}{M}\sum_{j=1}^{M} R_j$, where $M = m - 1$ (since the number of log returns for each stock is $m-1$), we compute the central value, the lower bound, and the upper bound for all α-cuts for the MLE estimator of the mean of the stock's fuzzy returns as

$$\widehat{\mu}^c = \bar{R}, \quad \widehat{\mu}^l = \bar{R} - t_{\{m-1,1-\frac{\beta}{2}\}} \times \frac{s}{\sqrt{m}}, \quad \widehat{\mu}^u = \bar{R} + t_{\{m-1,1-\frac{\beta}{2}\}} \times \frac{s}{\sqrt{m}}.$$

For computing the fuzzy variance number for the stock's fuzzy returns, we construct its confidence interval as follows:

$$CI_{\sigma_R^2} = \left(\frac{(m-1)\widehat{s}^2}{\chi^2_{m-1,\frac{\beta}{2}}}, \frac{(m-1)\widehat{s}^2}{\chi^2_{m-1,1-\frac{\beta}{2}}}\right), \tag{34}$$

where the MLE estimator of variance is $\widehat{s}^2 = \frac{1}{M}\sum_{j=1}^{M}(R_j - \bar{R})^2$.

Thus, we set

$$\widehat{\sigma}^c = \sqrt{\widehat{s}^2}, \quad \widehat{\sigma}^l = \sqrt{\frac{(m-1)\widehat{s}^2}{\chi^2_{m-1,\frac{\beta}{2}}}}, \quad \widehat{\sigma}^u = \sqrt{\frac{(m-1)\widehat{s}^2}{\chi^2_{m-1,1-\frac{\beta}{2}}}},$$

which represent the central value, the lower bound, and the upper bound, respectively, for all α-cuts of the MLE estimator for the standard deviation of the stock's fuzzy returns.

We can express the α-cuts for the fuzzy standard deviation of stock returns $\widehat{\widetilde{\sigma}} = (\widehat{\sigma}^l, \widehat{\sigma}^c, \widehat{\sigma}^u)$ as follows

$$\widehat{\widetilde{\sigma}}_R[\alpha] = [\widehat{\sigma}_\alpha^L, \widehat{\sigma}_\alpha^U] = [\widehat{\sigma}^l + \alpha(\widehat{\sigma}^c - \widehat{\sigma}^l),\ \widehat{\sigma}^u - \alpha(\widehat{\sigma}^u - \widehat{\sigma}^c)]. \tag{35}$$

Therefore, the MLE estimator for the lower bound of α-cuts for the fuzzy VaR measure for one stock is

$$\widehat{\widetilde{VaR}}^{L}_{\alpha} = -\widehat{\mu}^{L}_{\alpha} + z_{1-\theta}\widehat{\sigma}^{L}_{\alpha}, \tag{36}$$

where $\widetilde{R}^{L}_{\alpha} \sim \mathcal{N}(\widehat{\mu}^{L}_{\alpha}, \widehat{\sigma}^{2L}_{\alpha})$ and $\widetilde{R}^{L}_{\alpha} = (\widetilde{R}^{L}_{\alpha,1}, \widetilde{R}^{L}_{\alpha,2}, \ldots, \widetilde{R}^{L}_{\alpha,m-1})$.

The MLE estimator for the upper bound of α-cuts for the fuzzy VaR measure for one stock is:

$$\widehat{\widetilde{VaR}}^{U}_{\alpha} = -\widehat{\mu}^{U}_{\alpha} + z_{1-\theta}\widehat{\sigma}^{U}_{\alpha}, \tag{37}$$

where $\widetilde{R}^{U}_{\alpha} \sim \mathcal{N}(\widehat{\mu}^{U}_{\alpha}, \widehat{\sigma}^{2U}_{\alpha})$ and $\widetilde{R}^{U}_{\alpha} = (\widetilde{R}^{U}_{\alpha,1}, \widetilde{R}^{U}_{\alpha,2}, \ldots, \widetilde{R}^{U}_{\alpha,m-1})$.

We can express the membership function for $\widehat{\widetilde{VaR}}$ as follows:

$$\zeta_{\widehat{\widetilde{VaR}}}(y) = \sup_{0\leq\alpha\leq 1} \alpha\, \mathbf{1}_{B_{\alpha}}(y). \tag{38}$$

However, our objective is to estimate the fuzzy VaR measure for a portfolio containing n stocks. Thus, we consider a linear portfolio characterized by a weight vector $(\delta_1, \ldots, \delta_n)$, where $\delta_1 + \delta_2 + \cdots + \delta_n = 1$, and $\delta_i \geq 0$ for all $i = 1, \ldots, n$.

The imprecise returns of the n components in the portfolio are represented as asymmetric triangular fuzzy numbers. Specifically, the fuzzy returns of stock i are defined as $\widetilde{R}_i = (R^l_i, R^c_i, R^u_i)$, for $i = 1, \ldots, n$, where R^c_i represents the core value, and R^l_i and R^u_i denote the left and right spreads, respectively. We propose that the fuzzy return of the portfolio can be expressed as

$$\Delta\widetilde{P} = \delta_1\widetilde{R}_1 + \delta_2\widetilde{R}_2 + \cdots + \delta_n\widetilde{R}_n = \sum_{i=1}^{n}\delta_i\widetilde{R}_i. \tag{39}$$

For $\alpha \in [0,1]$, the α-cuts of the fuzzy VaR at a confidence level of $1-\theta$ for the portfolio are defined as

$$\widetilde{VaR}_{\theta}[\alpha] = [VaR_{\theta}(\Delta\widetilde{P}^{L}_{\alpha}), VaR_{\theta}(\Delta\widetilde{P}^{U}_{\alpha})], \tag{40}$$

where the lower and upper bounds of the portfolio's fuzzy return are given by

$$\Delta\widetilde{P}^{L}_{\alpha} = \delta_1\widetilde{R}^{L}_{\alpha,1} + \delta_2\widetilde{R}^{L}_{\alpha,2} + \cdots + \delta_n\widetilde{R}^{L}_{\alpha,n} = \sum_{i=1}^{n}\delta_i\widetilde{R}^{L}_{\alpha,i}, \tag{41}$$

$$\Delta\widetilde{P}^{U}_{\alpha} = \delta_1\widetilde{R}^{U}_{\alpha,1} + \delta_2\widetilde{R}^{U}_{\alpha,2} + \cdots + \delta_n\widetilde{R}^{U}_{\alpha,n} = \sum_{i=1}^{n}\delta_i\widetilde{R}^{U}_{\alpha,i}. \tag{42}$$

We have $\widetilde{R}^{L}_{\alpha} = (\widetilde{R}^{L}_{\alpha,1}, \widetilde{R}^{L}_{\alpha,2}, \ldots, \widetilde{R}^{L}_{\alpha,n})$ and $\widetilde{R}^{U}_{\alpha} = (\widetilde{R}^{U}_{\alpha,1}, \widetilde{R}^{U}_{\alpha,2}, \ldots, \widetilde{R}^{U}_{\alpha,n})$, with $\widetilde{R}^{L}_{\alpha} \sim \mathcal{N}(\boldsymbol{\mu}^{L}_{\alpha}, \boldsymbol{\Sigma}^{L}_{\alpha})$ and $\widetilde{R}^{U}_{\alpha} \sim \mathcal{N}(\boldsymbol{\mu}^{U}_{\alpha}, \boldsymbol{\Sigma}^{U}_{\alpha})$.

3.1 Fuzzy covariance matrix computation with crisp correlation

In this subsection, we compute the fuzzy covariance matrix required to determine the lower bound, upper bound, and central value for computing fuzzy VaR. The procedure involves the following steps

Step 1: Computing log returns

For each stock $i = 1, \ldots, n$, the daily log return on day t is calculated as

$$R_{i,t} = \ln\left(\frac{P_{i,t}}{P_{i,t-1}}\right), \qquad t = 1, \ldots, T. \tag{43}$$

Step 2: Computing the crisp correlation matrix

Using the observed return series $\{R_{i,t}\}$, we estimate the correlation coefficients as follows

$$\widehat{\rho}_{ik} = \frac{\sum_{j=1}^{M}(R_{i,j} - \bar{R}_i)(R_{k,j} - \bar{R}_k)}{\sqrt{\sum_{j=1}^{M}(R_{i,j} - \bar{R}_i)^2}\sqrt{\sum_{j=1}^{M}(R_{k,j} - \bar{R}_k)^2}}, \quad \bar{R}_i = \frac{1}{M}\sum_{j=1}^{M} R_{i,j}, \tag{44}$$

where $j = 1, \ldots, m$ and $M = m - 1$, since the number of computed log returns for each stock is M, and m denotes the number of price observations for each stock. The matrix $\widehat{\mathbf{R}} = (\widehat{\rho}_{ik})_{1 \leq i,k \leq n}$ represents the estimated correlation matrix.

Step 3: Computing the α-cuts of fuzzy mean and standard deviations

For each stock i, we represent the mean and standard deviation by triangular fuzzy numbers as follows

$$\widetilde{\mu}_i = (\mu_i^l, \mu_i^c, \mu_i^u), \qquad \widetilde{\sigma}_i = (\sigma_i^l, \sigma_i^c, \sigma_i^u). \tag{45}$$

Hence, their associated α-cuts are given by

$$\widetilde{\mu}_i[\alpha] = [\mu_{\alpha,i}^L, \mu_{\alpha,i}^U] = [\mu_i^l + \alpha(\mu_i^c - \mu_i^l),\ \mu_i^u - \alpha(\mu_i^u - \mu_i^c)], \tag{46}$$

and

$$\widetilde{\sigma}_i[\alpha] = [\sigma_{\alpha,i}^L, \sigma_{\alpha,i}^U] = [\sigma_i^l + \alpha(\sigma_i^c - \sigma_i^l),\ \sigma_i^u - \alpha(\sigma_i^u - \sigma_i^c)]. \tag{47}$$

Step 4: Computing the fuzzy covariance matrices for fuzzy VaR

Using the relationship $\sigma_{ik} = \rho_{ik}\sigma_i\sigma_k$ and setting $\rho_{ik} = \widehat{\rho}_{ik}$, where $\widehat{\rho}_{ik}$ is the empirical correlation coefficient between assets i and k estimated from historical log returns, the lower, upper, and core covariance values are defined as follows

(1) Core (central) covariance

$$\sigma_{ik}^c = \widehat{\rho}_{ik}\, \sigma_i^c \sigma_k^c. \tag{48}$$

(2) Lower and upper covariance bounds at level α-cuts

The α-cut covariance bounds are chosen as

$$\sigma^L_{\alpha,ik} = \begin{cases} \hat{\rho}_{ik}\, \sigma^L_{\alpha,i}\sigma^L_{\alpha,k}, & \text{if } \hat{\rho}_{ik} \geq 0, \\ \hat{\rho}_{ik}\, \sigma^U_{\alpha,i}\sigma^U_{\alpha,k}, & \text{if } \hat{\rho}_{ik} < 0, \end{cases} \tag{49}$$

and

$$\sigma^U_{\alpha,ik} = \begin{cases} \hat{\rho}_{ik}\, \sigma^U_{\alpha,i}\sigma^U_{\alpha,k}, & \text{if } \hat{\rho}_{ik} \geq 0, \\ \hat{\rho}_{ik}\, \sigma^L_{\alpha,i}\sigma^L_{\alpha,k}, & \text{if } \hat{\rho}_{ik} < 0. \end{cases} \tag{50}$$

For $i = k$, the diagonal elements reduce to

$$\sigma^L_{\alpha,ii} = \left(\sigma^L_{\alpha,i}\right)^2, \qquad \sigma^U_{\alpha,ii} = \left(\sigma^U_{\alpha,i}\right)^2. \tag{51}$$

This construction ensures that the covariance bounds are consistent with the sign of dependence between assets and preserves the variance-covariance structure required for parametric VaR estimation under fuzzy uncertainty.

We then define the lower, upper, and core covariance matrices associated with the VaR measure as

$$\mathbf{\Sigma}^L_\alpha = (\sigma^L_{\alpha,ik})_{1\leq i,k\leq n}, \quad \mathbf{\Sigma}^U_\alpha = (\sigma^U_{\alpha,ik})_{1\leq i,k\leq n}, \quad \mathbf{\Sigma}^c = (\sigma^c_{ik})_{1\leq i,k\leq n}. \tag{52}$$

Consequently, we can calculate the portfolio mean and standard deviation bounds as follows.

Let $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n)^\top$ be the portfolio weight vector. For each $\alpha \in [0, 1]$, we have

$$\mu^L_{\alpha,\Delta\widetilde{P}} = \sum_{i=1}^n \delta_i \mu^L_{\alpha,i}, \quad \mu^U_{\alpha,\Delta\widetilde{P}} = \sum_{i=1}^n \delta_i \mu^U_{\alpha,i}, \quad \mu_{\Delta P} = \sum_{i=1}^n \delta_i \mu^c_i, \tag{53}$$

and

$$\sigma^L_{\alpha,\Delta\widetilde{P}} = \sqrt{\boldsymbol{\delta}^\top \mathbf{\Sigma}^L_\alpha \boldsymbol{\delta}}, \quad \sigma^U_{\alpha,\Delta\widetilde{P}} = \sqrt{\boldsymbol{\delta}^\top \mathbf{\Sigma}^U_\alpha \boldsymbol{\delta}}, \quad \sigma_{\Delta P} = \sqrt{\boldsymbol{\delta}^\top \mathbf{\Sigma}^c \boldsymbol{\delta}}. \tag{54}$$

Finally, we calculate the portfolio's lower, upper, and central VaR, respectively, as:

$$VaR^l = \widetilde{VaR}^L_{\Delta\widetilde{P}} = VaR_\theta(\Delta\widetilde{P}^L_\alpha) = -\mu^L_{\alpha,\Delta P} + z_{1-\theta}\sigma^L_{\alpha,\Delta P}, \tag{55}$$

$$VaR^u = \widetilde{VaR}^U_{\Delta\widetilde{P}} = VaR_\theta(\Delta\widetilde{P}^U_\alpha) = -\mu^U_{\alpha,\Delta P} + z_{1-\theta}\sigma^U_{\alpha,\Delta P}, \tag{56}$$

$$VaR^c = VaR_{\Delta\widetilde{P}} = VaR_\theta(\Delta P) = -\mu_{\Delta P} + z_{1-\theta}\sigma_{\Delta P}. \tag{57}$$

where

- $\mu_{\alpha,i}^{L}$ and $\mu_{\alpha,i}^{U}$ are the lower and upper bounds of the α-cut of the fuzzy mean number of asset i, and μ_i^c is the central value.
- $\sigma_{\alpha,i}^{L}$ and $\sigma_{\alpha,i}^{U}$ are the lower and upper bounds of the α-cut of the fuzzy standard deviation of asset i, and σ_i^c is the central value.
- $\widehat{\rho}_{ik}$ is the crisp correlation estimated from historical log returns.
- $\sigma_{\alpha,ik}^{L}$, $\sigma_{\alpha,ik}^{U}$, and σ_{ik}^{c} denote the lower covariance bound, upper covariance bound, and central covariance value, respectively.
- $\boldsymbol{\Sigma}_{\alpha}^{L}$, $\boldsymbol{\Sigma}_{\alpha}^{U}$, and $\boldsymbol{\Sigma}^{c}$ denote the lower, upper, and central covariance matrices at the α-cut level.
- $\mu_{\alpha,\Delta P}^{L}$, $\mu_{\alpha,\Delta P}^{U}$, and $\mu_{\Delta P}$ denote the corresponding lower, upper, and central portfolio means.
- $\sigma_{\alpha,\Delta P}^{L}$, $\sigma_{\alpha,\Delta P}^{U}$, and $\sigma_{\Delta P}$ denote the corresponding lower, upper, and central portfolio standard deviations.

Based on the preceding results, we can conclude that the fuzzy VaR metric for a portfolio containing n stocks can be described as the triangular fuzzy number:

$$\widetilde{VaR}_{\Delta\widetilde{P}} = (VaR^l, VaR^c, VaR^u). \tag{58}$$

Furthermore, we use the interval method of Wu [38] to calculate the MLE estimator of the fuzzy VaR metric for the portfolio, which is detailed below

$$B[\alpha] = \left[\min\left\{\inf_{\alpha\leq\beta\leq 1} \widehat{\widetilde{VaR}}^{L}_{\beta,\Delta\widetilde{P}}, \inf_{\alpha\leq\beta\leq 1} \widehat{\widetilde{VaR}}^{U}_{\beta,\Delta\widetilde{P}}\right\}, \max\left\{\sup_{\alpha\leq\beta\leq 1} \widehat{\widetilde{VaR}}^{L}_{\beta,\Delta\widetilde{P}}, \sup_{\alpha\leq\beta\leq 1} \widehat{\widetilde{VaR}}^{U}_{\beta,\Delta\widetilde{P}}\right\}\right]. \tag{59}$$

Similarly, we calculate the α-cuts for the fuzzy mean and standard deviation of the stock's fuzzy returns within the portfolio based on the MLE estimators of mean and variance discussed previously

$$\widehat{\widetilde{\mu}}_i[\alpha] = [\widehat{\mu}_{\alpha,i}^{L}, \widehat{\mu}_{\alpha,i}^{U}] = [\widehat{\mu}_i^l + \alpha(\widehat{\mu}_i^c - \widehat{\mu}_i^l),\ \widehat{\mu}_i^u - \alpha(\widehat{\mu}_i^u - \widehat{\mu}_i^c)], \tag{60}$$

and

$$\widehat{\widetilde{\sigma}}_i[\alpha] = [\widehat{\sigma}_{\alpha,i}^{L}, \widehat{\sigma}_{\alpha,i}^{U}] = [\widehat{\sigma}_i^l + \alpha(\widehat{\sigma}_i^c - \widehat{\sigma}_i^l),\ \widehat{\sigma}_i^u - \alpha(\widehat{\sigma}_i^u - \widehat{\sigma}_i^c)]. \tag{61}$$

Here, $\widehat{\widetilde{\mu}}_i = (\widehat{\mu}_i^l, \widehat{\mu}_i^c, \widehat{\mu}_i^u)$ and $\widehat{\widetilde{\sigma}}_i = (\widehat{\sigma}_i^l, \widehat{\sigma}_i^c, \widehat{\sigma}_i^u)$ denote the MLE estimators of the fuzzy mean and standard deviation of each stock.

The MLE estimators for fuzzy mean and standard deviation are defined as

$$\begin{aligned}
\widehat{\mu}^c &= \bar{R} = \frac{1}{M}\sum_{j=1}^{M} R_j, \quad \text{where } M = m-1,\\
\widehat{\mu}^l &= \bar{R} - t_{\{m-1,1-\frac{\beta}{2}\}} \times \frac{s}{\sqrt{m}},\\
\widehat{\mu}^u &= \bar{R} + t_{\{m-1,1-\frac{\beta}{2}\}} \times \frac{s}{\sqrt{m}},\\
\widehat{\sigma}^c &= \sqrt{\widehat{s}^2}, \quad \text{where } \widehat{s}^2 = \frac{1}{M}\sum_{j=1}^{M}(R_j - \bar{R})^2,\\
\widehat{\sigma}^l &= \sqrt{\frac{(m-1)\widehat{s}^2}{\chi^2_{m-1,\frac{\beta}{2}}}},\\
\widehat{\sigma}^u &= \sqrt{\frac{(m-1)\widehat{s}^2}{\chi^2_{m-1,1-\frac{\beta}{2}}}}.
\end{aligned}$$

Furthermore, we use the Wu interval to construct the MLE-based fuzzy estimator of the portfolio's VaR. For each $\alpha \in [0,1]$, the α-cut bounds of the portfolio VaR are computed by

$$\widehat{\widetilde{VaR}}^L_{\alpha,\Delta\widetilde{P}} = -\widehat{\mu}^L_{\alpha,\Delta P} + z_{1-\theta}\widehat{\sigma}^L_{\alpha,\Delta P}, \tag{62}$$

$$\widehat{\widetilde{VaR}}^U_{\alpha,\Delta\widetilde{P}} = -\widehat{\mu}^U_{\alpha,\Delta P} + z_{1-\theta}\widehat{\sigma}^U_{\alpha,\Delta P}. \tag{63}$$

where $z_{1-\theta}$ is the $(1-\theta)$-th quantile of the standard normal distribution, $\widehat{\mu}^L_{\alpha,\Delta P} = \sum_{i=1}^{n}\delta_i\widehat{\mu}^L_{\alpha,i}$ and $\widehat{\mu}^U_{\alpha,\Delta P} = \sum_{i=1}^{n}\delta_i\widehat{\mu}^U_{\alpha,i}$ denote the lower and upper bounds of the α-cut of the portfolio mean. In addition, $\widehat{\sigma}^L_{\alpha,\Delta\widetilde{P}} = \sqrt{\boldsymbol{\delta}^\top\widehat{\boldsymbol{\Sigma}}^L_\alpha\boldsymbol{\delta}}$ and $\widehat{\sigma}^U_{\alpha,\Delta\widetilde{P}} = \sqrt{\boldsymbol{\delta}^\top\widehat{\boldsymbol{\Sigma}}^U_\alpha\boldsymbol{\delta}}$ denote the lower and upper bounds of the α-cut of the portfolio standard deviation.

Under the crisp correlation coefficient $\widehat{\rho}_{ik}$, the lower and upper covariance bounds between stocks, and the corresponding covariance matrix bounds, are given by

$$\widehat{\sigma}^L_{\alpha,ik} = \begin{cases} \hat{\rho}_{ik}\,\widehat{\sigma}^L_{\alpha,i}\widehat{\sigma}^L_{\alpha,k}, & \text{if } \hat{\rho}_{ik} \geq 0,\\ \hat{\rho}_{ik}\,\widehat{\sigma}^U_{\alpha,i}\widehat{\sigma}^U_{\alpha,k}, & \text{if } \hat{\rho}_{ik} < 0, \end{cases} \tag{64}$$

and

$$\widehat{\sigma}^U_{\alpha,ik} = \begin{cases} \hat{\rho}_{ik}\,\widehat{\sigma}^U_{\alpha,i}\widehat{\sigma}^U_{\alpha,k}, & \text{if } \hat{\rho}_{ik} \geq 0,\\ \hat{\rho}_{ik}\,\widehat{\sigma}^L_{\alpha,i}\widehat{\sigma}^L_{\alpha,k}, & \text{if } \hat{\rho}_{ik} < 0. \end{cases} \tag{65}$$

For $i = k$, the diagonal elements reduce to:

$$\widehat{\sigma}^L_{\alpha,ii} = \left(\widehat{\sigma}^L_{\alpha,i}\right)^2, \qquad \widehat{\sigma}^U_{\alpha,ii} = \left(\widehat{\sigma}^U_{\alpha,i}\right)^2. \tag{66}$$

We also have

$$\widehat{\boldsymbol{\Sigma}}^L_\alpha = (\widehat{\sigma}^L_{\alpha,ik}), \quad \widehat{\boldsymbol{\Sigma}}^U_\alpha = (\widehat{\sigma}^U_{\alpha,ik}). \tag{67}$$

The membership function for $\widehat{\widetilde{VaR}}_{\Delta \widetilde{P}}$ is given by

$$\zeta_{\widehat{\widetilde{VaR}}_{\Delta \widetilde{P}}}(y) = \sup_{0 \le \alpha \le 1} \alpha \, \mathbf{1}_{B_\alpha}(y). \tag{68}$$

4 Numerical Example

In this section, we will explain the fuzzy VaR for the portfolio model that we defined in earlier sections.

In this study, we use data from January 2025 to December 2025.This data involves the daily closing stock prices for ten companies chosen from the top 50 Tehran Stock Exchange companies. We apply the log returns formula, as detailed in Section 3, to the daily closing prices. We apply this method to estimate the returns for all stocks during the specified period. Furthermore, we assume that the returns are identically distributed but not independent. Also, we consider that the returns are asymmetric triangular fuzzy numbers. Next, we will construct a $(1 - \beta)100\%$ confidence interval for both the mean and standard deviation of the fuzzy returns associated with the stock. The mean and standard deviation of the log daily returns form the core values of the mean and standard deviation fuzzy numbers, with the lower and upper bounds of these confidence intervals indicating their spreads. By using the α-cuts sets for every fuzzy number, we compute the lower and upper bounds of the fuzzy VaR measure for each stock for $\alpha \in [0, 1]$. The calculation of VaR is performed using the variance-covariance method, as outlined in subsection 2.2 of section 2. Finally, we compute the fuzzy VaR metric for our portfolio, which consists of 10 stocks. First, we take the lower bounds of α-cuts for the mean and standard deviation for all stocks' fuzzy returns, and we compute their VaR measure. Hence, we obtain the lower VaR for the portfolio. We use the same method to calculate the upper and central values of α-cuts for the fuzzy VaR metric for our portfolio. As a result, we obtain the fuzzy VaR metric for the portfolio, which is an asymmetric triangular fuzzy number. Then, using the WU interval discussed in Section 3, we compute the approximate membership function for the resulting Fuzzy VaR metric for the corresponding portfolio.

Furthermore, all the stocks are assumed to have the same weights $(\delta_i = 1/10, i = 1, \cdots, n)$ in the portfolio.

Table 1 presents an overview of ten stocks that construct the portfolio in this example.

Table 1: Selected stocks for the reviewed portfolio

n	Portfolio
Stock1	Atmosphere
Stock2	Azmayesh
Stock3	S_I. N. C. Ind.
Stock4	Firooza Co.
Stock5	Ghadir Inv.
stock6	GoldIran
Stock7	Khalij Fars
Stock8	Inf. Services
Stock9	S_Hepco
Stock10	Iran Tractor

In addition, Figure 1 shows the changes that occurred in the returns of the portfolio in the assumed period of time.

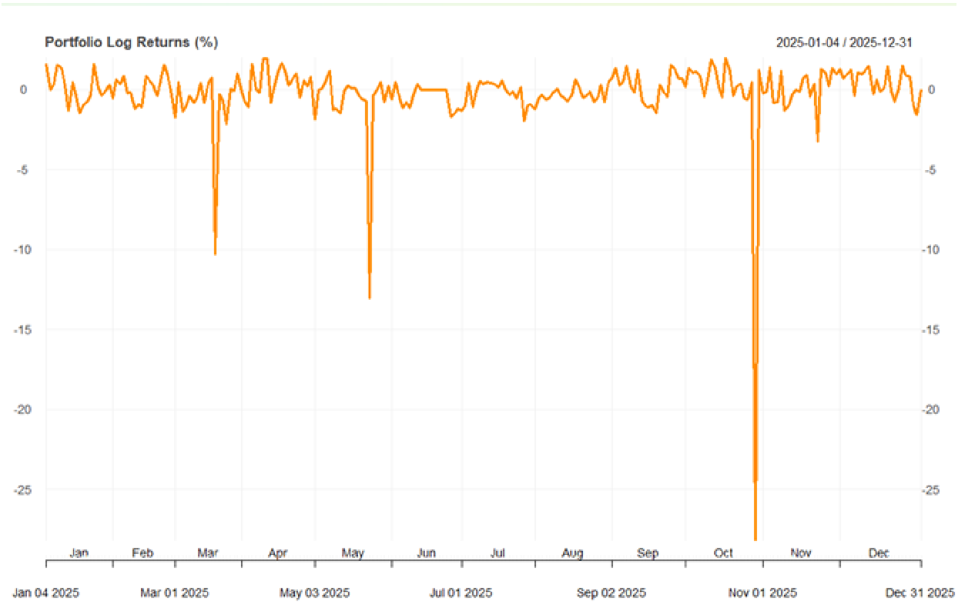

Figure 1: The portfolio's returns in percent value.

Table 2 shows the VaR for the corresponding portfolio assuming a normal distribution when the returns are precise values.

Based on the data presented in Table 2, we can conclude that the portfolio with normally distributed crisp returns exhibits the highest VaR at a 99% confidence level. Consequently, the 1% one-day VaR represents 5.5112% of the portfolio value.

Table 2: VaR for the relevant portfolio assuming crisp returns

Confidence level	95%	97.5%	99%
VaR	3.9530%	4.6735%	5.5112%

The changes in the VaR measure and the portfolio returns from January 2025 to December 2025, at various confidence levels, are illustrated in Figure 2.

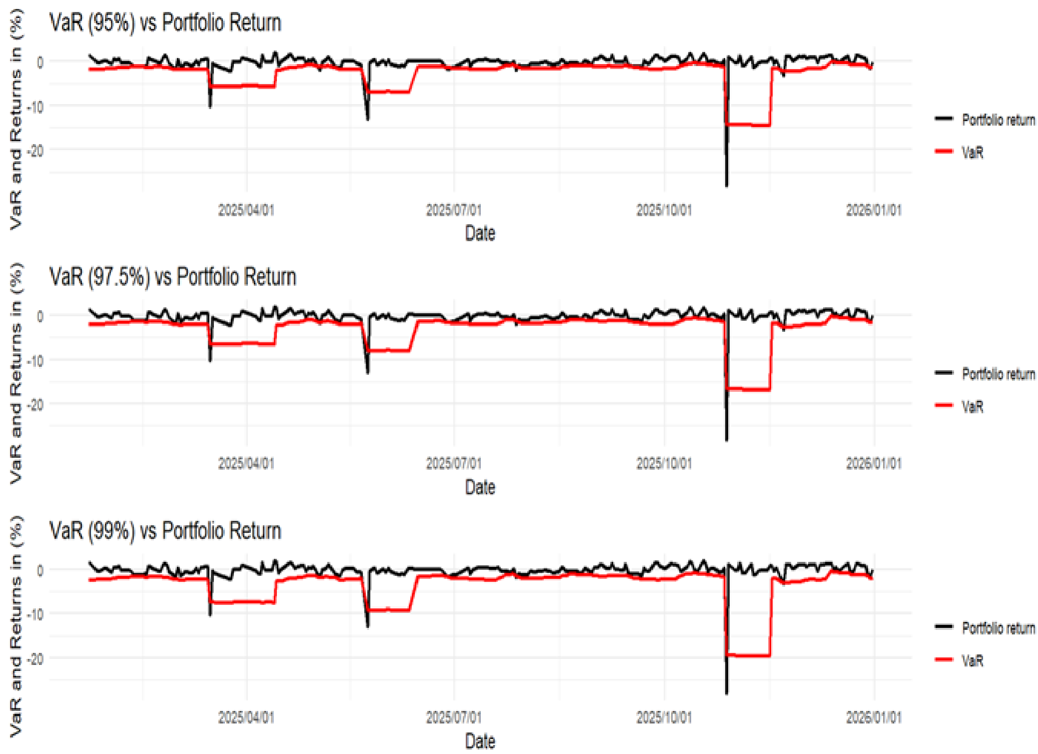

Figure 2: Value at risk and returns of portfolio with different confidence levels in percent value

Table 3 displays the upper and lower bounds, as well as the central values of the α-cuts for the fuzzy VaR metric of the portfolio at various confidence levels (95%, 97.5%, and 99%). The estimates in Table 3 are based on the assumptions that the portfolio's fuzzy returns and the stock's fuzzy returns follow a normal distribution, are dependent, and use the variance-covariance method to compute the VaR metric. In Table 3, we present the lower, central, and upper values of the fuzzy VaR measure, denoted as $lVaR$, $mVaR$, and $uVaR$, respectively.

Also, it should be noted from the resulting estimates values in Table 3 that the $\alpha = 0$ level corresponds to the full support of the fuzzy parameters, yielding the

Table 3: Fuzzy Value-at-Risk for portfolio at different confidence levels and α-cuts

α	Confidence Level								
	95%			**97.5%**			**99%**		
	lVaR	**mVaR**	**uVaR**	**lVaR**	**mVaR**	**uVaR**	**lVaR**	**mVaR**	**uVaR**
0	2.9608%	3.9530%	4.9957%	3.6101%	4.6735%	5.7971%	4.3651%	5.5112%	6.7288%
0.25	3.2101%	3.9530%	4.7359%	3.8774%	4.6735%	5.5172%	4.6533%	5.5112%	6.4256%
0.5	3.4585%	3.9530%	4.4755%	4.1437%	4.6735%	5.2367%	4.9404%	5.5112%	6.1216%
0.75	3.7061%	3.9530%	4.2146%	4.4090%	4.6735%	4.9554%	5.2263%	5.5112%	5.8169%
1	3.9530%	3.9530%	3.9530%	4.6735%	4.6735%	4.6735%	5.5112%	5.5112%	5.5112%

widest interval for the fuzzy VaR and representing the highest degree of uncertainty in the return distribution. As α increases, the corresponding α-cut intervals shrink, reflecting higher confidence levels and reduced uncertainty. Specifically, at $\alpha = 0$ with a confidence level of 99%, the lower and upper bounds of fuzzy VaR are equal to 4.3651% and 6.7288%, respectively.

Figure 3 illustrates the fuzzy VaR measure expressed in percent values for the portfolio. This measure is represented as an asymmetric triangular fuzzy number at confidence levels of (95%, 97.5%, and 99%).

Conversely, the results suggest that the smallest amounts are observed at $\alpha = 0$ with a confidence level of 95%. Here, the lowest amount recorded is 2.9608%, while the highest amount is 4.9957%.
However, at $\alpha = 1$, we observe that the lower and upper bounds, along with the central values, are identical and reflect the same value of the VaR metric.

Furthermore, Figure 4 illustrates the changes in the lower, upper, and central values of the fuzzy VaR measure for the portfolio over the period from January 2025 to December 2025, at confidence levels (95%, 97.5%, and 99%), respectively.

Table 4 displays the minimum and maximum bounds of the estimated fuzzy VaR of α-cuts based on the WU interval [38] for the assumed portfolio at various confidence levels (95%, 97.5%, and 99%).

Furthermore, in Table 4, we indicate the minimum and maximum numbers as $\min(\widetilde{\widetilde{VaR}})$ and $\max(\widetilde{\widetilde{VaR}})$, respectively.

From the results in Table 4 at $\alpha = 1$, we observe that the minimum and maximum bounds are the same and indicate the same amount of the estimated fuzzy VaR across all confidence levels in this table. Where the estimated values of fuzzy VaR at 99% confidence level and at $\alpha = 1$ are equal to 5.5000%.

Conversely, the resulting estimates of fuzzy VaR in Table 4 refer to different

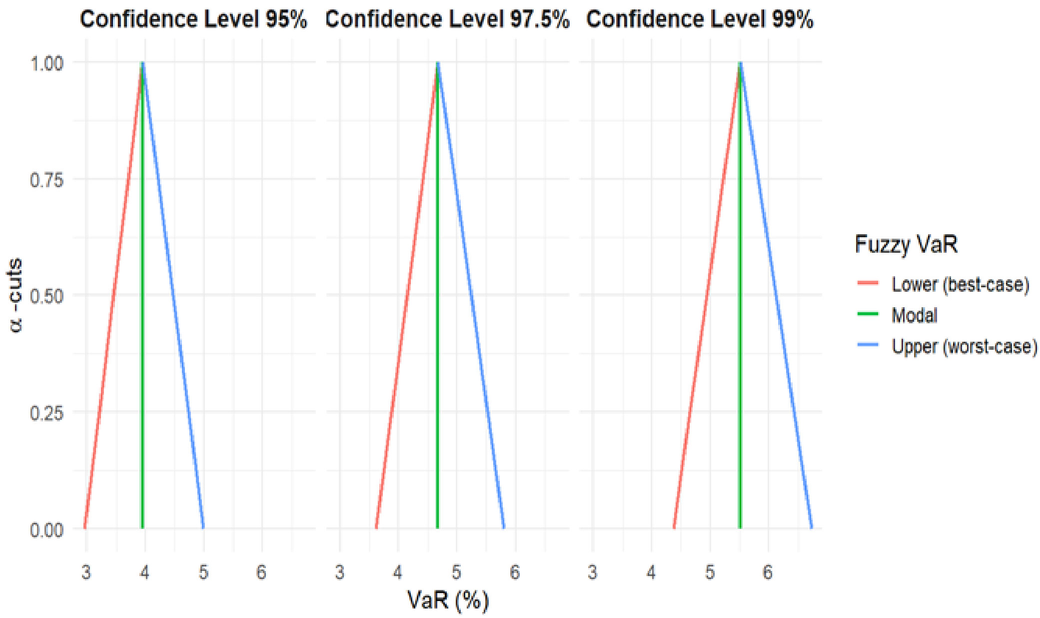

Figure 3: Fuzzy VaR for portfolio with different confidence levels and *alpha*-cuts

Figure 4: The changes in the lower, upper, and central values of the portfolio's fuzzy VaR over time

Table 4: The portfolio's fuzzy VaR by the WU method at various confidence levels and α-cuts.

α	Confidence Level					
	95%		97.5%		99%	
	$\min(\widehat{\widetilde{\mathbf{VaR}}})$	$\max(\widehat{\widetilde{\mathbf{VaR}}})$	$\min(\widehat{\widetilde{\mathbf{VaR}}})$	$\max(\widehat{\widetilde{\mathbf{VaR}}})$	$\min(\widehat{\widetilde{\mathbf{VaR}}})$	$\max(\widehat{\widetilde{\mathbf{VaR}}})$
0	2.9550%	4.9856%	3.6029%	5.7853%	4.3563%	6.7151%
0.25	3.2038%	4.7263%	3.8697%	5.5060%	4.6440%	6.4125%
0.50	3.4516%	3.4516%	4.1354%	5.2261%	4.9304%	6.1092%
0.75	3.6987%	4.2061%	4.4002%	4.9454%	5.2157%	5.8050%
1	3.9451%	3.9451%	4.6641%	4.6641%	5.5000%	5.5000%

values for the minimum and maximum bounds at $\alpha = 0$ and various confidence levels. For example, the estimated minimum and maximum values at $\alpha = 0$ and at 99% confidence level equal 4.3563% and 6.7151%, respectively. Consequently, the investor can assess the risk of her portfolio by accepting a certain level of uncertainty and making decisions based on it.

In Figures 5, 6 and 7 we show the approximate membership functions for the MLE estimator of the fuzzy VaR for a portfolio based on Equation (68) at a 95%, 97.5% and 99% confidence levels and at different α-cuts.

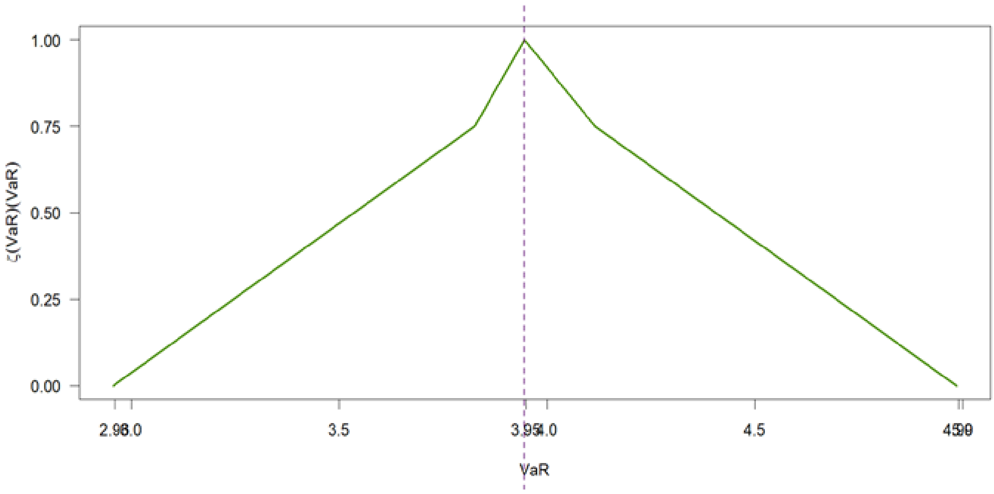

Figure 5: The approximate membership function for the estimated fuzzy VaR of the portfolio at a 95% confidence level

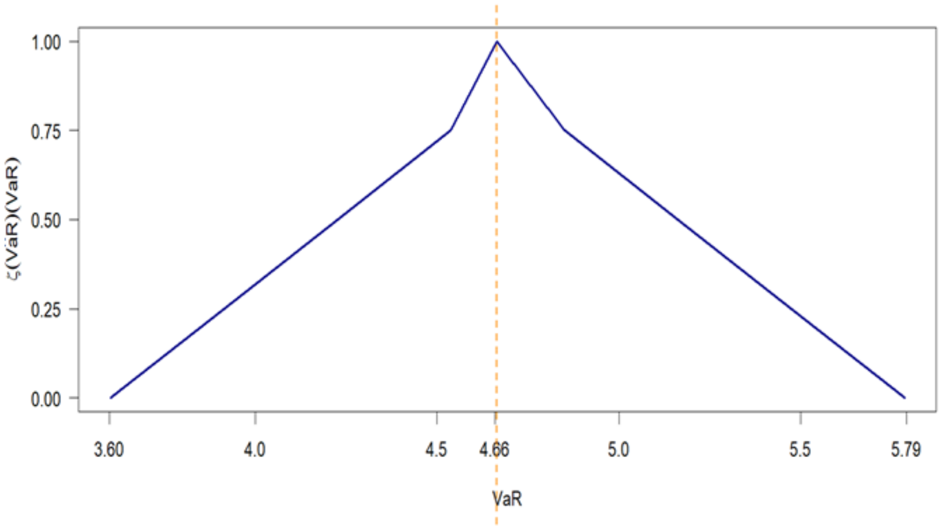

Figure 6: The approximate membership function for the estimated fuzzy VaR of the portfolio at a 97.5% confidence level

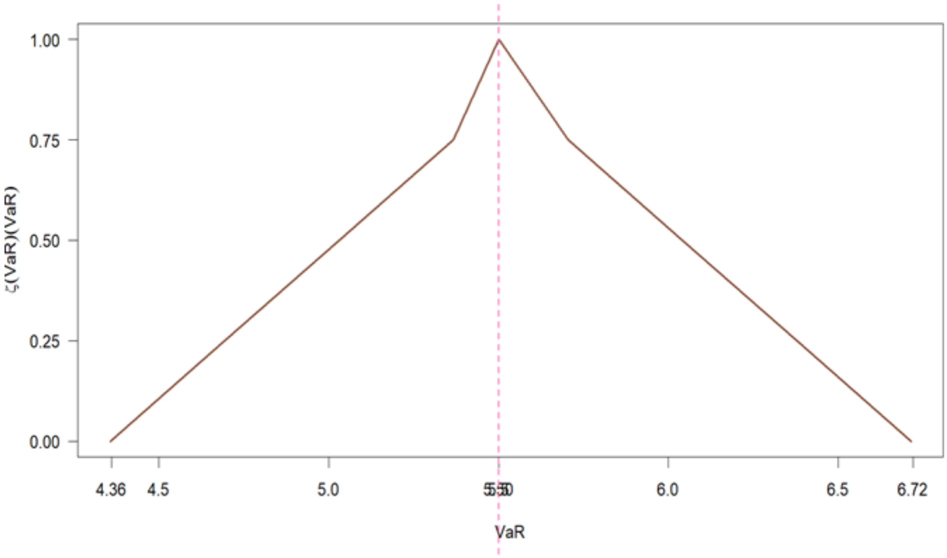

Figure 7: The approximate membership function for the estimated fuzzy VaR of the portfolio at a 99% confidence level

5 Summary and Conclusion

In this paper, we introduced a new method for computing Value at Risk for a portfolio model in a fuzzy environment. We assumed that the returns of the portfolio are imprecise and follow a normal distribution. To handle the imprecision and vagueness inherent in returns, we represented them as asymmetric triangular fuzzy numbers. In this study, we employed the α-cuts method to estimate the fuzzy VaR metric for the portfolio model.

For our empirical analysis, we used daily closing prices of ten stocks from companies listed on the Tehran Stock Exchange, covering the period from January 2025 to December 2025. We applied the log daily return formula to estimate returns for all stocks. Based on the α-cuts of the mean and standard deviation of the fuzzy returns for all stocks, we determined both the lower and upper bounds, along with the central value, for the fuzzy VaR of the portfolio. The variance–covariance method was utilized for computing fuzzy VaR. Following the Wu method [38], which is employed to compute the membership function for estimates of fuzzy parameters, we estimated the membership function for the derived fuzzy VaR.

We evaluated the effectiveness of our method using an illustrative numerical example. The numerical results indicate that fuzzy VaR serves as a flexible and comprehensive risk metric that captures uncertainty more effectively than traditional VaR. These outcomes provide decision-makers with richer insights compared to conventional single-point VaR measures.

It should be noted that the proposed approach relies on the normality assumption within the variance–covariance VaR framework. In a fuzzy environment, deviations from normality may become more pronounced at low α levels, corresponding to higher uncertainty. Sensitivity analysis with respect to distributional assumptions and extensions to non-normal or heavy-tailed return models constitute important directions for future research.

The proposed approach is currently limited to the variance–covariance framework and the assumption of normally distributed returns. Future research may extend this methodology by incorporating alternative VaR estimation techniques such as historical simulation or Monte Carlo methods, as well as other downside risk measures, including fuzzy Conditional Value at Risk (CVaR) or Expected Shortfall (ES). Additionally, extending the model to account for non-normal and heavy-tailed return distributions, time-varying volatility, and dynamic dependence structures would further enhance its practical applicability. Another promising direction is the application of the proposed fuzzy VaR framework to larger and more diversified portfolios across different markets and asset classes, as well as its integration into portfolio optimization and real-time risk management systems.

Data Availability

The data used in this study consist of daily closing prices of stocks traded on the Tehran Stock Exchange. These data are publicly available and can be obtained from the official Tehran Stock Exchange website at https://tsetmc.com/.

Acknowledgement

The authors would like to thank the editors and anonymous reviewers for their valuable comments and suggestions, which helped improve the quality of this paper.

Bibliography

[1] P. Abad, S. Benito, and C. López, *A comprehensive review of Value at Risk methodologies*, The Spanish Review of Financial Economics, 12(1) (2014), 15-32.

[2] E. Abasi, B. Teimurpur, and M. Barjesteh Maleki, *Optimum portfolio selection using value at risk in Tehran Stock Exchange*, Journal of Economic Research (Tahghighat-E-Eghtesadi), 44(2), (2010).

[3] Basle Committee on Banking Supervision (Basle Committee), *Amendment to the Capital Accord to Incorporate Market Risks*, Base: Basle Committee on Banking Supervision (January), 1996.

[4] P. Best, *Implementing value at risk*, John Wiley and Sons,1998.

[5] U. Cherubini, and G. Della Lunga, *Fuzzy value-at-risk: accounting for market liquidity*, Economic Notes, 30(2), (2001), 293-312.

[6] R. Campbell, R. Huisman, and K. Koedijk, *Optimal portfolio selection in a Value-at-Risk framework*, Journal of Banking and Finance, 25(9), (2001), 1789-1804.

[7] T. Coleman, *A practical guide to risk management*, CFA Institute Research Foundation, (2011), M2011-2.

[8] G. Dionne, *Risk management: History, definition, and critique*, Risk management and insurance review, 16(2), (2013), 147-166.

[9] X. Deng, and W. Li, *A novel probabilistic hesitant fuzzy portfolio selection model with value-at-risk and safety level of score*, Engineering Computations, 38 (5), (2021) 2137-2162.

[10] D. Dubois, and H. Prade, *Operations on fuzzy numbers*, International Journal of systems science, 9(6), (1978), 613-626.

[11] S. Emmer, M. Kratz, and D. Tasche, *What is the best risk measure in practice? A comparison of standard measures*, SSRN Electronic Journal, Jan, (2013), 1312.1645.

[12] M. L. Guerra,and L. Sorini, *Value at risk based on fuzzy numbers*, Axioms, 9(3), (2020), 98.

[13] O.L. Gebizlioglu, B. Şenoğlu, and Y. M. Kantar, *Comparison of certain value-at-risk estimation methods for the two-parameter Weibull loss distribution*, Journal of Computational and Applied Mathematics, 235(11), (2011), 3304-3314.

[14] J. Gaytán Cortés, *Value at Risk (VaR)*, Mercados Y Negocios, (2022), 95-106.

[15] J. C. Hull, *"Options, futures, and other derivatives"*, AMBER – ABBS Management Business and Entrepreneurship Review, 7(1), (2016), 70.

[16] F. Hooshmand, Z. Anoushirvani, and S.A. MirHassani, *Model and efficient algorithm for the portfolio selection problem with real-world constraints under value-at-risk measure*, International Transactions in Operational Research, 30(5), (2023), 2665-2690.

[17] P. Jorion, *Value at risk: the new benchmark for managing financial risk*, New York: McGraw-Hill, 2000.

[18] F. H. Knight, *Risk, uncertainty and profit*, Houghton Mifflin, 1921.

[19] M. B. Kar, S. Kar, S. Guo, X. Li, and S. Majumder, *A new bi-objective fuzzy portfolio selection model and its solution through evolutionary algorithms*, Soft Computing, 23(12), (2018), 4367–4381.

[20] J. LONGERSTAEY, AND M. SPENCER, *RiskMetrics—Technical document*, New York: RiskMetrics Group, J.P. Morgan, 1996.

[21] W. LIU, A. SEMEYUTIN, C. K. M. LAU, AND G. GOZGOR, *Forecasting value-at-risk of cryptocurrencies with riskmetrics type models*, Research in International Business and Finance, 54, (2020), 101259.

[22] J. LUKIĆ, M. MISITA, D.D. MILANOVIĆ, A. BOROTA-TIŠMA, AND A. JANKOVIĆ, *Determining the risk level in client analysis by applying fuzzy logic in insurance sector*, Mathematics, 10(18), (2022), 3268.

[23] Y. LIU, H. AHMADZADE, AND M. FARAHIKIA, *Portfolio selection of uncertain random returns based on value at risk*, Soft Computing-A Fusion of Foundations, Methodologies and Applications, 25(8), (2021).

[24] H. MARKOWITZ, *Portfolio Selection*, The Journal of Finance, 7(1), (1952), 77–91.

[25] J. P. MORGAN, (1994). *RiskMetrics Technical Document*, New York: Morgan Guaranty Trust Company, (October), 1994.

[26] A. M. MOUSSA, J. S. KAMDEM, AND M. TERRAZA, *Fuzzy value-at-risk and expected shortfall for portfolios with heavy-tailed returns*, Economic Modelling, 39, (2014), 247-256.

[27] L. MACIEL, R. BALLINI, R., AND F. GOMIDE, *An evolving possibilistic fuzzy modeling approach for value-at-risk estimation*, Applied Soft Computing, 60, (2017), 820-830.

[28] N. PEARSON, *Risk Budgeting: Portfolio Problem Solving with Value-at-Risk*, John Wiley and Sons, Hoboken, 2002.

[29] M. POURRAFIEE, A. NAFEI, S. BANIHASHEMI, AND S. P. AZIZI, *Comparing entropies in portfolio diversification with fuzzy value at risk and higher-order momen*, Fuzzy information and engineering, 12(1), (2020), 123-138.

[30] A. PEÑA, I. BONET, C. LOCHMULLER, H. A. PATIÑO, F. CHICLANA, AND M. GÓNGORA, *A fuzzy credibility model to estimate the Operational Value at Risk using internal and external data of risk events*, Knowledge-Based Systems, 159, (2018), 98-109.

[31] J. E. M. REYES, J. J. C. PÉREZ, AND S. C. AKÉ, *Credit risk management analysis: An application of fuzzy theory to forecast the probability of default in a financial institution*, Contaduría Y Administración, 69(1), (2023), 430.

[32] R. SHAYYA, M. T. SORROSAL-FORRADELLAS, AND A. TERCEÃ±O, *Value-at-risk models: a systematic review of the literature*, Journal of Risk, (2023).

[33] K. SHANG, AND Z. HOSSEN, *Applying fuzzy logic to risk assessment and decision-making*, Casualty Actuarial Society, Canadian Institute of Actuaries, Society of Actuaries, (2013), 1-59.

[34] R. K. SHIRAZ, M. TAVANA, AND H. FUKUYAMA, *A random-fuzzy portfolio selection DEA model using value-at-risk and conditional value-at-risk*, Soft Computing, 24(22), (2020), 17167-17186.

[35] G. SATHVIKA, MR.N. SURESH, AND D. THAVVA, *A Study On Value At Risk (VAR) Models In Measuring Market Risk*, International Journal of Research Publication and Reviews, 6, (2025), 5976-5980.

[36] S. TAGHIKHANI, F. BAROUGHI, AND B. ALIZADEH, *Mean–variance value at risk criterion for solving a p-median location problem on networks with type-2 intuitionistic fuzzy weights*, Journal of Computational and Applied Mathematics, 437, (2024), 115481.

[37] B. WANG, S. WANG, AND J. WATADA, *Fuzzy portfolio selection based on value-at-risk*, In 2009 IEEE International Conference on Systems, Man and Cybernetics, IEEE, (2009, October), (pp. 1840-1845).

[38] H. C. WU, *The fuzzy estimators of fuzzy parameters based on fuzzy random variables*, European Journal of Operational Research, 146(1), (2003), 101-114.

[39] B. WANG, Y. LI, S. WANG, AND J. WATADA, *A multi-objective portfolio selection model with fuzzy value-at-risk ratio*, IEEE Transactions on Fuzzy Systems, 26(6), (2018), 3673-3687.

[40] D. WANG, Y. CHEN, H. WANG, AND M. HUANG, *Formulation of the non-parametric value at risk portfolio selection problem considering symmetry*, Symmetry, 12(10), (2020), 1639.

[41] Y. WANG, X. QI, S. QIN, *Calculation of VaR — Based on the Account Manager's Perspective*, Modern Economics and Management Forum, 5, (2024), 442.

[42] Y. YOSHIDA, *An estimation model of value-at-risk portfolio under uncertainty*, Fuzzy Sets and Systems, 160(22), (2009), 3250-3262.

[43] L. A. ZADEH, *Fuzzy sets*, Information and control, 8(3), (1965),338-353.

[44] Z. ZMEŠKAL, *Value at risk methodology of international index portfolio under soft conditions (fuzzy-stochastic approach)*, International Review of Financial Analysis, 14(2),(2005), 263-275.

[45] H. ZHANG, J. WATADA, AND B. WANG, *Sensitivity-based fuzzy multi-objective portfolio model with Value-at-Risk*, IEEJ Transactions on Electrical and Electronic Engineering, 14(11), (2019), 1639-1651.

[46] Y. ZHANG, W. LIU, AND X. YANG, *An automatic trading system for fuzzy portfolio optimization problem with sell orders*, Expert Systems with Applications, 187, (2022), 115822.

[47] P. ZHAO, AND Q. XIAO, *Portfolio selection problem with Value-at-Risk constraints under non-extensive statistical mechanics*. Journal of computational and applied mathematics, 298, (2016), 64-71.

How to Cite: Batoul Salamah[1], Reza Zarei[2], Farshid Mehrdoust[3], Mohammad Ghasem Akbari[4], *Fuzzy Estimation of Value at Risk for a Portfolio with Triangular Fuzzy Returns*, Journal of Mathematics and Modeling in Finance (JMMF), Vol. 6, No. 2, Pages:99–125, (2026).