Journal of Mathematics and Modeling in Finance (JMMF)
Vol. 6, No. 2, Summer & Fall 2026
Research paper

Fair Profit Sharing Ratios of Islamic Investment Contracts

Abass Sagna[1]

[1] ENSIIE & labouratoire de Mathématiques et Modélisation d'Evry (LaMME), Université Évry Paris-Saclay, UMR CNRS8071, 23 Boulevard de France, 91037 Evry

abass.sagna@univ-evry.fr

Abstract:
The aim of this work is to calculate the fair profit-sharing ratios and the expected payoffs at maturity for each partner in islamic investment contrats (or instruments), based on profit and loss-sharing (PL-sharing). These investment contracts, known as *mudarabah* and *musharakah*, can be compared to *limited partnerships* and *joint ventures* (including all types of venture, such as joint-stock companies, partnerships, etc) in conventional finance. To compute these quantities, we introduce the notion of c-fair profit-sharing ratios, where $c = (c_1, \ldots, c_d) \in (\mathbb{R}^\star)^d$ and d is the number of partners. This constitutes an equilibrium approach that accounts for the contributions of the contracting parties in terms of both capital and labour. We show that the c-fair profit-sharing ratio of each partner is the sum of their contributions to capital and labour, weighted by some economic factors that we identify as *investment risk and opportunity* respectively. We deduce that, in the c-fair model, the expected investment profit is distributed among the contracts partners according to the shares $\varpi_\ell = c_\ell \,/\, (c_1 + \ldots + c_d)$, that correspond to their respective contribution weights to the venture's overall success. We extend these results to mixed contract that combine one or both previous contracts with an agency (known as *wakalah*) contract.

Keywords: Islamic finance, Mudarabah, Musharakah, Wakalah, Fair profit-sharing ratios.
MSC2010 Classification: 00A69; 62P05; 91G80.

1 Introduction

The goal of this research is to determine the c-fair profit-sharing ratios and expected profit shares of partners in Islamic investment contracts (*mudarabah* and *musharakah* contracts), possibly combined with a *wakalah* (agency) contract.

In short, A *mudarabah* contract is a form of active partnership in which the capital contributor, or limited partner, provides the capital, while the entrepreneur, or general partner, contributes expertise to help the funds grow. The entrepreneur (*mudharib*) contributes to the business venture through his knowledge and skills alone, without investing any property or making claim to any wages for conducting the business venture (see e.g. [13]). A *musharakah* contract is a partnership between

[1]Corresponding author
Received: 01/12/2025 Accepted: 26/06/2026
https://doi.org/10.22054/jmmf.2026.90083.1245

two or more partners who contribute capital to some profit seeking endeavour. The two contracts can then be compared to *limited partnerships* and *joint ventures* (including all types of ventures, such as joint-stock companies, partnerships, etc) in conventional finance. This broaden the scope of application of our work to all forms of joint venture in which the following two principles, common to islamic finance, apply:

Rule 1. Losses are absorbed by all partners in proportion to their capital contributions.

Rule 2. Profit-sharing ratios are determined by mutual agreement between partners.

This also justifies our choice to some times use the terms *PL-sharing-type limited partnership* and *PL-sharing musharakah* to refer to *musharakah* and *mudarabah* contracts, respectively, even through more specific rules of islamic finance regarding these contracts distinguish them from conventional finance (see e.g. [3], [4], [6], [7] or [9]).

Remark 1.1. It should be noted that what distinguishes a limited partnership from a joint venture is capital contribution. In the joint ventures we are considering, all partners contribute capital, whereas in the limited partnership, the limited partner provides the capital and the general partner contributes their expertise (or labor) to make it grow. Thus, *Rule* 1. above states that in the event of a loss in a *PL-sharing-type limited partnership*, such losses are borne by the limited partner, in exchange for the general partner's loss of the fruits of its labor. This corresponds to the standard framework of islamic finance.

These two financial contracts are the primary sources of the funding in an interest-free economy. However, they are rarely used by financial institutions operating within the framework of islamic finance, and are often replaced by other financial instruments (see, for example, [8], which presents an empirical study of the use of islamic finance instruments by banks in 11 countries). While these alternative instruments are more profitable and (structurally) less risky, their social impact is less significant. Another raison that explains the unpopularity of these funding instruments, and which also explains part of the associated risk, is that their economic and financial implications are not well understood. This motivates our work, which goal is to determine the profit-sharing ratios and the expected payoff at maturity for each partner of such a contract. We extend our results to mixed contracts, which are a combination of one or both of the previous funding instruments and an agency contract.

The results presented here are new and are bases on a equilibrium approach between the contreparties to the contracts under consideration. This concept of equilibrium is used in [12] for the specific case of a *PL-sharing musharakah*, within the framework of a modelling approach that differs from ours (see Remark 3.3 for a comparison of the two approaches). In fact, denote by γ_ℓ and by κ_ℓ the profit sharing and the capital contribution ratios, respectively, of partner ℓ. Let $\text{Pay}_\ell(\gamma_\ell, \kappa_\ell)$ denotes its expected payoff at maturity. We introduce the notion of *c-fair* contract, with $c = (c_1, \ldots, c_d)$, $c_\ell > 0$ for all ℓ, d standing for the number of partners, that takes into account (making the novelty of the idea with respect to [12]) the contribution of each partner to the project's overall success. The partners' contributions are reflected in a weighted allocation system that assigns a weight coefficient of $1/c_\ell$ to each partner, according to their respective contributions of capital and labour. We then determine the profit-sharing ratios by solving the equilibrium system $\text{Pay}_1(\gamma_1, \kappa_1) / c_1 = \ldots = \text{Pay}_d(\gamma_d, \kappa_d) / c_d$. The profit-sharing ratios resulting from this equilibrium system are called the *c-fair* profit-sharing ratios.

We show that the c-fair profit-sharing ratio of each partner is the sum of their contributions to capital and labour, weighted by some economic factors that we identify as *investment risk and opportunity* respectively. These financial instruments introduce thus a kind of competition between capital contribution and labor with regard to the investment risk, which increases the share of profit going to capital providers as the risk increases. We also deduce that, in the c-fair model, the expected investment profit is distributed among the contracts partners according to the shares $\varpi_\ell = c_\ell / (c_1 + \ldots + c_d)$, that correspond to their respective contribution weights to the venture's overall success. We extend these results to mixed contract that combine one or both previous contracts with an agency contract.

To be more explicit, consider a *PL-sharing-type joint venture* contract involving d partners. Suppose the partners contributed amounts L_1, ..., and L_d, to the capital $L = L_1 + \ldots + L_d$. Let $\kappa_\ell = L_\ell / L$ be the equity stake of partners ℓ, for every $\ell \in \{1, \ldots, d\}$ and let $\gamma_1, \ldots, \gamma_d$, with $\gamma_1 + \cdots + \gamma_d = 1$, be the respective profit sharing ratios of the partners. As previously, R_T denotes the net income process of the investment at the maturity T of the contract. Then, the expected payoff of each partner ℓ at the maturity of the contract is given by

$$\text{Pay}_\ell(\gamma_\ell, \kappa_\ell) = \gamma_\ell \, \mathbb{E}(R_T - L)^+ - \kappa_\ell \, \mathbb{E}(L - R_T)^+. \tag{1}$$

The fair profit sharing ratios γ_ℓ, $\ell = 1, \ldots, d$ is determined by solving the system of equations

$$\text{Pay}_1(\gamma_1, \kappa_1) = \ldots = \text{Pay}_\ell(\gamma_\ell, \kappa_\ell) = \ldots = \text{Pay}_d(\gamma_d, \kappa_d).$$

If all the partners contributed equally to the management of the project, we will see that this makes sense and leads to the solution $\gamma_\ell = \kappa_\ell$, $\ell = 1, \ldots, d$. However, we know that this is not usually the case. Therefore, it is important to understand how to take into account the additional management or labour contributions of certain partners, who may only be compensated for these contributions through their profit-sharing ratios. To illustrate this, let us consider a *PL-sharing-type joint venture* contract between two partners whose contributions to the capital are denoted by κ_1 and κ_2. If the partners manage the project equally, contribution to management, the difference in their overall contributions to the project lies in their respective capital contributions, κ_1 and κ_2. It is the reasonable to assume that the fair profit-sharing ratios are $\gamma_1 = \kappa_1$ and $\gamma_2 = \kappa_2$. In fact, in this case, we have

$$\begin{aligned} & \mathrm{Pay}_1(\gamma_1, \kappa_1) \,/\, \kappa_1 = \mathrm{Pay}_2(\gamma_2, \kappa_2) \,/\, \kappa_2 \\ \text{or, equivalently,}\quad & \mathrm{Pay}_1(\gamma_1, \kappa_1) = c\, \mathrm{Pay}_2(\gamma_2, \kappa_2), \text{ with } c = \kappa_1/\kappa_2. \end{aligned}$$

Now suppose that the two partners (Partner 1 and Partner 2) contribute the same amount of capital ($\kappa_1 = \kappa_2$) but participate to different degrees in management. They may then agree to weight their respective contributions to the project's overall success using the coefficients c_1 and c_2 to determine the profit-sharing ratios that reflect their respective payoffs. Our aim is therefore to find γ_1 and γ_2 such that $\mathrm{Pay}_1(\gamma_1, \kappa_1) \,/\, c_1 = \mathrm{Pay}_2(\gamma_2, \kappa_2) \,/\, c_2$. We will show (see Proposition 4.1) that, for any $c_1, c_2 > 0$,

$$\gamma_1 = \frac{c_1}{c_1 + c_2}\big(1 - \rho(R_T, L)\big) + \kappa_1\, \rho(R_T, L), \tag{2}$$

$$\gamma_2 = \frac{c_2}{c_1 + c_2}\big(1 - \rho(R_T, L)\big) + \kappa_2\, \rho(R_T, L), \tag{3}$$

where $\rho(R_T, L) = \mathbb{E}(L - R_T)^+ \,/\, \mathbb{E}(R_T - L)^+$ (observe that $\mathbb{E}(R_T - L)^+ > 0$ if $R_T \not\equiv L$) is seen as the *investment risk* whereas $1 - \rho(R_T, L)$ is seen as the *investment opportunity*. It is very important to remark that the *viability* assumption (**??**) guarantees $\rho(R_T, L) \in [0, 1]$ and then, $\gamma_\ell \in [0, 1]$, for $\ell \in \{1, 2\}$. Also note that $\gamma_1 + \gamma_2 = 1$.

Furthermore, setting $\varpi_\ell = c_\ell \,/\, (c_1 + c_2)$, $\ell = 1, 2$, we show that the expected payoff of each partner $\ell \in \{1, 2\}$ is given by

$$\mathrm{Pay}_\ell(\gamma_\ell, \kappa_\ell) = \varpi_\ell\, \Delta(R_T, L), \qquad \ell = 1, 2. \tag{4}$$

In other words, the c-fair profit sharing ratios leads the two partners to share the *expected investment profit* $\Delta(R_T, L)$ according the respective weights ϖ_ℓ, $\ell = 1, 2$.

It is also worth noting that a PL-sharing *PL-sharing-type limited partnership* contract appears to be a special case of a *PL-sharing-type joint venture* contract where the manager's (or Partner 2's) share of the capital is zero. In fact, setting $\kappa_1 = 1$ and $\kappa_2 = 0$ in Equations (2) and (3) i yields the c-fair profit-sharing ratios obtained for a *PL-sharing-type limited partnership* contract:

$$\gamma_1 = \frac{c_1}{c_1 + c_2}\big(1 - \rho(R_T, L)\big) + \rho(R_T, L), \tag{5}$$

$$\gamma_2 = \frac{c_2}{c_1 + c_2}\big(1 - \rho(R_T, L)\big). \tag{6}$$

We can compare the previous result with that of [12] by setting $c_1 = c_2$ (see Remark 3.3 for more details).

In addition to applying Equations (2) and (3) to the *PL-sharing-type limited partnership* contract, it is important to understand the following remarks. Indeed, for every $\ell \in \{1,2\}$, we can decompose it as follows:

$$\gamma_\ell = \underbrace{\varpi_\ell \big(1 - \rho(R_T, L)\big)}_{\text{reward of labour}} + \underbrace{\kappa_\ell \, \rho(R_T, L)}_{\text{reward of capital}} . \tag{7}$$

The previous equation shows that the profit-sharing ratio is :

- ◇ a function of investment asset (or investment area), either through investment risk $\rho(R_T, L)$ or investment opportunity $1 - \rho(R_T, L)$;
- ◇ is proportional to the investment risk $\rho(R_T, L)$ and the invested capital;
- ◇ is proportional to the the investment opportunity $1 - \rho(R_T, L)$ and the contribution to labour (or to the management).

From Equation (7), we can deduce that a moderately risky investment (i.e. when the investment opportunity is greater than the investment risk, or equivalently, when $\rho(R_T, L) \le 1/2$) rewards labour contributions more than capital contributions. In contrast, a highly risky investment (i.e. when $\rho(R_T, L) \ge 1/2$) rewards capital contribution more. In other words, if a firm is looking to raise capital to finance its business, it is in its best interest to find a highly profitable (or low-risk) activity that will increase its share of the profits.

We also consider the case in which the two partners of a *PL-sharing-type joint venture* contract appoint a third partner (Partner 3), who is not one of the original two partners, and in which they are bound to Partner 3 by a *PL-sharing-type limited partnership* contract or a *wakalah* contract. In the former case, where management is carried out via a *PL-sharing-type limited partnership* contract, the c-fair profit-sharing ratios are similar to those obtained for a *PL-sharing-type joint venture* contract with three partners, with the manager's capital contribution set to zero. When management is arranged via a *wakalah* contract with fixed remuneration (an amount p paid k times up to maturity T), if partners 1 and 2 and manager 3 agree to rate their respective payoffs by coefficients c_1, c_2 and c_3, then, the (c_1, c_2, c_3)-fair profit sharing ratios and periodic remuneration p satisfy:

$$\theta(r,T,k)\, p \;=\; \varpi_3 \, \Delta(R_T, L) \tag{8}$$

$$\gamma_\ell \;=\; \left(\frac{\varpi_3}{2} + \varpi_\ell\right)\big(1 - \rho(R_T, L)\big) + \kappa_\ell \, \rho(R_T, L), \qquad \ell \in \{1,2\} \tag{9}$$

where the coefficient $\theta(r,T,k)$ comes from the updating value of the expected payoff of the manager. In fact, we may consider that there is a discounted rate of return r that could be earned from the investment in a good and is given by $\theta(r,T,k) = ((1+r)^{\frac{T}{k}} - 1) \,/\, ((1+r)^{T} - 1)$. When there is no discounting, $\theta(r,T,k) = k$. The weights $(\varpi_\ell)_{\ell \in \{1,2,3\}}$ still be defined by

$$\varpi_\ell = \frac{c_\ell}{c_1 + c_2 + c_3}, \qquad \text{for all } \ell \in \{1,2,3\}. \tag{10}$$

Observe that $\gamma_1 + \gamma_2 = 1$. Furthermore, the expected payoff of each partner ℓ (even Partner 3 who has a fixed k-periodic remuneration up to maturity) is given by

$$\mathrm{Pay}_\ell(\gamma_\ell, \kappa_\ell) = \varpi_\ell\, \Delta(R_T, L), \qquad \ell = 1, 2, 3. \tag{11}$$

Hence, the c-fair profit sharing ratios leads the three partners to share the *expected investment profit* $\Delta(R_T, L)$ with the respective weights ϖ_ℓ, $\ell \in \{1, 2, 3\}$.

A typical application of this result is a situation involving three partners: an investor (Partner 1), who provides all the capital ($\kappa_1 = 1$); a bank (Partner 2), which acts as the investor's agent through a *wakalah* contract, arranging a fixed remuneration p, paid k times up to maturity T; and a company (Partner 3), which requires funding and is bound to the bank by a *PL-sharing-type limited partnership* contract ($\kappa_3 = 0$). This case is studied in Example 4.12.

We also generalize the previous results to contracts involving multiple partners.

However, it is important to note that computing the profit-sharing ratios and related quantities in all previous results only requires the computation of the quantities $\rho(R_T, L)$ and $\Delta(R_T, L)$. These quantities can be calculated explicitly in certain models, such as when the stochastic process (R_t) is assumed to evolve according to the well-known Black-Scholes model, which assumes, among other things, that the distribution of R_T is log-normal. Nevertheless, while the values obtained from the Black-Scholes model can be considered as benchmarks, particularly in a lower volatility framework where these values will be close to those expected from more general models, it is well known that most financial datasets are not log-normally distributed. Consequently, one of the most challenging tasks is to use econometric models, such as ARIMA or (G)ARCH, or more general stochastic diffusion models, such as local volatility or stochastic volatility, to determine the most suitable model for R, given an asset dataset. This task will be addressed in a separate paper in order to reduce the scope of this one.

The paper is organised as follows. First, we introduce and discuss the viability assumption and its implications. We then study c-fair profit-sharing ratios for a *PL-sharing mudarabah* contract and a *PL-sharing musharakah* contract. In the latter case, we first consider the matter of two and three partners to help the reader better understand the quantities at stake before generalising.

2 Viable Investment

To explain what we mean by a "viable investment", consider an investor with initial capital L (which may be in the form of cash or labour). This investor invests in an asset R and expects to receive income R_T at a future time $T > 0$. There are two possible outcomes:

- ◇ The operation succeeds, so the income R_T at time T is greater than L. In this case, the *profit* is $(R_T - L)^+$,

⋄ The operation fails, so $R_T \leq L$. In this case, the *loss* is $-(L - R_T)^+$.

Let us define the expected profit of such a given investor as $\mathbb{E}(R_T - L)^+$, and its expected loss as $\mathbb{E}(L - R_T)^+$.

We also define the *expected investment profit* as $\Delta(R_T, L) = \mathbb{E}(R_T - L)^+ - \mathbb{E}(L - R_T)^+$, *the investment risk* (supposing that $R_T \not\equiv L$, in which case $\mathbb{E}(R_T - L)^+ > 0$) as

$$\rho(R_T, L) = \frac{\mathbb{E}(L - R_T)^+}{\mathbb{E}(R_T - L)^+} \tag{12}$$

and the *expected investment return* (or the *investment opportunity*) as

$$\beta(R_T, L) := \frac{\Delta(R_T, L)}{\mathbb{E}(R_T - L)^+} = 1 - \rho(R_T, L).$$

We will say that the investment in asset R with T-time maturity is viable if the investor's expected profit over the T-term is greater than their expected loss, i.e. $\Delta(R_T, L) > 0$. This is equivalent to say that

$$\beta(R_T, L) > 0 \quad \text{or} \quad \rho(R_T, L) \in [0, 1[. \tag{13}$$

It then appears that any T-term and R-asset-based investment with capital L, for which the investment risk $\rho(R_T, L) > 1$, is not viable, as its expected loss exceeds the expected profit. Therefore, a T-term and R-asset based investment (or contract) will be considered to be viable if statement (13) holds true. Consequently, $\rho(R_T.L)$ seems to be inherent in any productive activity, such as trading or manual labour. The quantity L may generally be considered the production costs (or their cash equivalent, as in manual labour, for example). Finally, as previously mentioned, $\rho(R_T.L)$ may also be viewed as the investment risk associated with an activity whose productive costs are L and which generates a random income R_T at time T.

Remark 2.1. We can rewrite $\rho(R_T, L)$ in an universal way as

$$\rho(\tilde{R}_T) = \frac{\mathbb{E}(1 - \tilde{R}_T)^+}{\mathbb{E}(\tilde{R}_T - 1)^+}, \tag{14}$$

by considering the process $\tilde{R} = R \,/\, L$.

Definition 2.2. A T-term and R-asset based investment (or contract) of capital L is said to be *viable* if

$$\Delta(R_T, L) > 0 \quad \text{or, equivalently, if} \quad \rho(R_T, L) \in [0.1[. \tag{15}$$

We have the following easy result which characterises a viable investment (or contract).

Proposition 2.3. *A T-term and R-asset based investment (or contract) of the capital L is* viable *if and only if*

$$\mathbb{E}(R_T) \geq L. \tag{16}$$

Proof. We have

$$\begin{aligned}
\Delta(R_T, L) &= \mathbb{E}(R_T - L)^+ - \mathbb{E}(L - R_T)^+ \\
&= \mathbb{E}\Big((R_T - L)\mathbb{1}_{\{R_T \le L\}} + (L - R_T)\mathbb{1}_{\{R_T > L\}}\Big) \\
&= \mathbb{E}\Big((R_T - L)\big(1 - \mathbb{1}_{\{R_T \le L\}}\big) + (L - R_T)\mathbb{1}_{\{R_T > L\}}\Big) \\
&= \mathbb{E}(R_T - L).
\end{aligned}$$

Thus the result. □

In the following sections, we will discuss the concept of "c-fair" profit-sharing ratios in the context of *PL sharing - mudarabah.* To make things easier to understand, we will first study the fair profit-sharing ratios case before introducing the c-fair one. First, we will recall the definition of a *PL-sharing mudarabah* contract.

3 PL Sharing - Mudarabah Contract

We start by describing a *mudarabah* contract and then study the fair profit sharing ratios of these contracts.

3.1 What is a *Mudarabah* Contract?

A *mudarabah* contract is a partnership agreement between two entities (which may be, for example, a bank and a company). One partner provides the capital (the owner of which is called the *rabb al-maal*), while the other (the *mudarib*) provides labour or expertise to manage the funds and generate a profit. Any profit is shared between the partners according to a predetermined key. Each partner may terminate the contract at any time, provided that a fixed maturity date has not been set or the contract has not been executed. Once the contract has been completed and the maturity date has been set, the contract will remain irreversible until the execution of the book value of the cash assets or until the maturity date. However, the profit-sharing ratios may be updated at any time before the maturity date with a mutual agreement between the partners.

It is important to note that this is a participative contract, since the investor does not contribute labour or expertise, but rather provides capital in exchange for the labour or expertise of the worker. As a basic principle, the capital owner is only compensated on the profit of the investment, in the same way as the manager. Like the manager (who loses the reward for their labour in case of loss), the capital owner has to support any losses, except in the event of mismanagement (a risk that the owner of the funding may cover by requiring guarantees from the worker).

Finally, the *mudarabah* may be absolute or restrictive. It is restrictive when the funding owner imposes restrictions on the scope of activity or on any other aspect he consider appropriate. Otherwise, the *mudarabah* is said to be absolute.

In the next section, our aim is to determine fair profit-sharing ratios for the partners of a mudarabah contract, in the sense of equalising their payoffs at the maturity of the contract.

3.2 Fair Profit Sharing Ratio

Consider a *PL-sharing mudarabah* contract between a capital owner (who may be a bank) and a worker (who may be a firm, for example). We denote the profit-sharing ratios between the partners (the funding owner and the worker) by (γ_1, γ_2), with $\gamma_1 + \gamma_2 = 1$. The aim is to determine a fair profit-sharing arrangement between the funding owner and the worker. We denote the net income process of the investment at the maturity T of the contract by R_T. Therefore, at the maturity of the contract,

- ◇ the payoff of the funding owner is

$$\gamma_1 \,(R_T - L)^+ - (L - R_T)^+, \tag{17}$$

- ◇ and the payoff of the worker is

$$\gamma_2 \,(R_T - L)^+. \tag{18}$$

Definition 3.1. A T-term viable *PL sharing - mudarabah* contract based on asset R is said to be *fair* if the funding owner's expected maturity payoff is equal to the worker's expected payoff. This means that

$$\gamma_1 \,\mathbb{E}[(R_T - L)^+] - \mathbb{E}[(L - R_T)^+] = \gamma_2 \,\mathbb{E}[(R_T - L)^+]. \tag{19}$$

Remark that we may also consider the actualised values of the expected payoffs in Definition 3.1 by considering that there is a discounted rate r of return that could be earned from the investment on some good of the expected payoffs. However, this is not necessary since Equation (19) will still hold true by simplifying the discounted factor on both sides of the equality. In the sequel we will consider actualised values of the payoff when it is necessary.

The following proposition follows directly from Definition 3.1.

Proposition 3.2. *In a fair* PL sharing - mudarabah *contract, the funding owner and the worker ratios are given by*

$$\begin{aligned}
\gamma_1 &= \frac{1}{2}\big(1 + \rho(R_T, L)\big) && (20)\\
&= \frac{1}{2}\big(1 - \rho(R_T, L)\big) + \rho(R_T, L), && (21)\\
\gamma_2 &= \frac{1}{2}\big(1 - \rho(R_T, L)\big), && (22)
\end{aligned}$$

and their common expected payoffs at maturity is given by

$$\frac{1}{2}\Delta(R_T, L).$$

This means that they share equally the expected investment profit
$\Delta(R_T, L) = \mathbb{E}(R_T) - L > 0$.

The proof is in fact easy and is given after the following remark, which discusses on the comparison with the results of [12].

Remark 3.3. Define the success probability of the project to be $\beta = \mathbb{P}(R_T > L)$. Observing that
$\mathbb{E}((R_T - L)^+) = \beta\, \mathbb{E}(R_T - L|\{R_T > L\})$, $\mathbb{E}((L - R_T)^+) = (1-\beta)\, \mathbb{E}(L - R_T|\{R_T \leq L\})$ and setting

$$R^-(L) := \mathbb{E}(R_T|\{L \geq R_T\}) \quad \text{and} \quad R^+(L) := \mathbb{E}(R_T|\{R_T > L\}), \tag{23}$$

we obtain

$$\gamma_1 = \frac{1}{2}\Big(1 + \frac{(1-\beta)}{\beta}\, \frac{L - R^-(L)}{R^+(L) - L}\Big). \tag{24}$$

This must be compared with the result of Proposition 3.2.4. in [12] which states that if an investor invests an amount L in a company that may generate revenue $R^+(L) > L$ with probability β, or revenue $R^-(L) < L$ with a probability $1 - \beta$, then the *fair* profit-sharing ratio is

$$\gamma_1 = \frac{R^+(L) - R^-(L)}{2(R^+(L) - L)} + \frac{R^-(L) - L}{2\beta(R^+(L) - L)} = \frac{1}{2}\Big(1 + \frac{1-\beta}{\beta}\, \frac{L - R^-(L)}{R^+(L) - L}\Big).$$

This shows that our model matches with that of [12] if the generated revenue $R^-(L)$ and $R^+(L)$ considered in [12] are defined as in (23).

Proof. The proof follows directly from Equation (19). On the other hand, replacing in Equation (19) the ratios γ_1 and γ_2 by their values from (20), we see that in an T-term viable investment of amount L on the asset R, the common expected payoff at maturity of both the owner of the capital and the worker is

$$\frac{1}{2}\Delta(R_T, L).$$

This ends the proof. □

From Equation (20), we can see that, in a fair profit-sharing *mudarabah* contract, the investor's profit ratio γ_1 is at least 50%, increased by a quantity that is a non-decreasing function of the investment risk $\rho(R_T, L)$. This implies that, in such a contract, the manager's ratio increases with the expected profit (or decreases with the expected loss) of the investment. Furthermore, the greater the difference between the expected profit and loss, the greater the worker's ratio will be. If the project is so risky that the expected profit of the investment is almost equal to the expected loss, then γ_1 will be almost equal to 100%. This means that the manager must agree to work without pay (for example, during the short term of a high-risky *mudarabah* contract).

Now, consider the situation in which we wish to determine the profit-sharing ratios γ_1 and γ_2 such that the expected payoff for the capital owner at maturity is c times (with $c > 0$) the payoff for the worker. Denote by

$$\begin{aligned} &\mathrm{Pay}_1(\gamma_1) := \gamma_1\, \mathbb{E}[(R_T - L)^+] - \mathbb{E}[(L - R_T)^+ \\ \text{and} \quad &\mathrm{Pay}_2(\gamma_2) := \gamma_2\, \mathbb{E}[(R_T - L)^+], \end{aligned}$$

the respective expected payoffs of the capital owner and the labourer (or the manager). We suppose that

$$\mathrm{Pay}_1(\gamma_1) = c\, \mathrm{Pay}_2(\gamma_2).$$

Setting $c = c_1/c_2$, this boils down to

$$\mathrm{Pay}_1(\gamma_1)\ /\ c_1 = \mathrm{Pay}_2(\gamma_2)\ /\ c_2.$$

This motivates the introduction of a (c_1, c_2)-fair *PL sharing - mudarabah* contract, where c_ℓ is a real number for $\ell = 1, 2$. In the notation of the (c_1, c_2)-fair contract, the weighting coefficientc_ℓ is associated with the payoff Pay_ℓ. This weighting accounts for each partner's contribution to the project's overall success. As will be seen in Equation (28), these weighting coefficients determine the weight assigned to each partner in the sharing of the expected common investment profit $\Delta(R_T, L)$.

Definition 3.4. A T-term viable *PL sharing - mudarabah* contract based on asset R is said to be (c_1, c_2)-*fair* if

$$\mathrm{Pay}_1(\gamma_1)\ /\ c_1 = \mathrm{Pay}_2(\gamma_2)\ /\ c_2. \tag{25}$$

Proposition 3.5. *In a (c_1, c_2)-fair* PL sharing - mudarabah *contract, the funding owner and the worker profit ratios are given by*

$$\begin{aligned} \gamma_1 &= \frac{1}{c_1 + c_2}\big(c_1 + \rho(R_T, L)\, c_2\big) \\ &= \varpi_1\,(1 - \rho(R_T, L)) + \rho(R_T, L), \qquad (26) \\ \gamma_2 &= \varpi_2\,(1 - \rho(R_T, L)), \qquad (27) \end{aligned}$$

whit $\varpi_1 = c_1\,/(c_1 + c_2)$ and $\varpi_2 = c_2\,/(c_1 + c_2)$. Moreover, their respective expected payoff are given by

$$\frac{c_1}{c_1 + c_2}\,\Delta(R_T, L) \qquad \textit{and} \qquad \frac{c_2}{c_1 + c_2}\,\Delta(R_T, L). \tag{28}$$

This means that they will share the expected investment profit *$\Delta(R_T, L)$ with the respective weights ϖ_1 and ϖ_2.*

Comment on Proposition 3.5. We see from Proposition 3.5 that the weight ϖ_1 is the managers's share of the expected investment profit $\Delta(R_T, L)$. It is also the manager's gross compensation for the work when the investment is totally risk-free (i.e. when $\rho(R_T, L) = 0$). Consequence, the weighting coefficients, c_1 and c_2, of the overall contributions of the funding owner and the manager to the project's success can be chosen to meet a desired key of sharing the expected investment profit $\Delta(R_T, L)$. For example, the manager's share of the expected profit could be set to be half that of the funding owner: $\varpi_1 = 2\,\varpi_2$, or, equivalently, $c_1 = 2\,c_2$.

Proof. Keeping in mind that $\gamma_1 + \gamma_2 = 1$, we get (26)-(27) by solving Equation (25). The expected payoff $\mathrm{Pay}_1(\gamma_1)$ and $\mathrm{Pay}_2(\gamma_2)$ of the capital owner and the manager are obtained by replacing γ_1 and γ_2 by their values in (26)-(27). □

Example 3.6. In this example we fix the value of $\rho(R_T, L) \in \{\frac{1}{4}, \frac{1}{2}\}$ and aim to find the (c_1, c_2)-fair profit sharing ratios in a *PL sharing - mudarabah* contract between a bank and a firm.

⋄ Suppose we want to choose profit sharing ratios such that the expected payoff at maturity is $\frac{3}{2}$ times greater for the bank than for the company. In this case, we can set $c_1 = 3$ and $c_2 = 2$. Furthermore, the (c_1, c_2)-fair profit sharing ratios are given as follows:

$$\begin{aligned} &\text{when} \quad \rho(R_T, L) = \frac{1}{4}, \quad \text{then,} \quad \gamma_1 = 70\% \quad \text{and} \quad \gamma_2 = 30\%, \\ \text{and} \quad &\text{when} \quad \rho(R_T, L) = \frac{1}{2}, \quad \text{then,} \quad \gamma_1 = 80\% \quad \text{and} \quad \gamma_2 = 20\%. \end{aligned}$$

Given the previous selection of ratios γ_1 and γ_2, we anticipate that the profit $\Delta(R_T, L)$ will be shared according to the respective weighted shares $(\varpi_1, \varpi_2) = (3/5, 2/5)$.

⋄ Now, for a fairprofit sharing (when $(c_1, c_2) = (1, 1)$), we get:

$$\begin{aligned} &\text{when} \quad \rho(R_T, L) = \frac{1}{4}, \quad \text{then,} \quad \gamma_1 = 62.5\% \quad \text{and} \quad \gamma_2 = 37.5\%, \\ \text{and} \quad &\text{when} \quad \rho(R_T, L) = \frac{1}{2}, \quad \text{then,} \quad \gamma_1 = 75\% \quad \text{and} \quad \gamma_2 = 25\%. \end{aligned}$$

This leads to a equal sharing of the anticipated expected profit $\Delta(R_T, L)$.

⋄ Finally, we can choose to set the profit-sharing ratios so that the bank can expect to be paid off at maturity $2/3$ times less than the company expects. In this case, we can set $c_1 = 2$ and $c_2 = 3$, and, we have:

$$\begin{aligned} &\text{when} \quad \rho(R_T, L) = \frac{1}{4}, \quad \text{then,} \quad \gamma_1 = 55\% \quad \text{and} \quad \gamma_2 = 45\%, \\ \text{and} \quad &\text{when} \quad \rho(R_T, L) = \frac{1}{2}, \quad \text{then,} \quad \gamma_1 = 70\% \quad \text{and} \quad \gamma_2 = 30\%. \end{aligned}$$

Selecting these profit-sharing ratios (γ_1 for the bank and γ_2 for the company) results in the anticipated profit $\Delta(R_T, L)$ being shared according to the respective weighted shares $(\varpi_1, \varpi_2) = (2/5, 3/5)$.

As discussed previously, this example shows that the profit-sharing ratio depends on exposure to risk $\rho(R_T, L)$. Remember that the investor is more exposed to risk than the funding manager because they will have to cover any losses. This exposure to investment risk is reflected in the investor's profit ratio: the riskier the investment, the higher the profit ratio.

4 PL Sharing - Musharakah Contract

First, we define a *musharakah* contract, then we deal with mixed contracts that combine a *musharakah* contract with *mudarabah* and *wakalah* contracts.

4.1 Definition

In a *musharakah* contract, two or more partners combine their assets, labour or liabilities in order to generate profits. The share of each partner's capital must be determined in advance. The profit-sharing ratios are agreed by the partners and can be amended before the contract matures. However, all partners absorb losses in proportion to their capital contribution. A *musharakah* is said to be *diminishing* if one partner promises to gradually purchase the other partner's equity share until the title is fully transferred.

In this section, we will determine fair profit-sharing ratios between partners in a *PL-sharing musharakah* contract.

4.2 Fair Profit Sharing Ratio of Mixed Contracts

Suppose there are two partners who have contributed amounts L_1 and L_2 to the capital L, such that their respective contributions are given by $\kappa_1 = L_1/L$ and $\kappa_2 = L_2/L$. Let $\mathrm{Pay}_1(\gamma_1, \kappa_1)$ and $\mathrm{Pay}_2(\gamma_2, \kappa_2)$ denote the expected payoff of the partners. Then,

$$\begin{aligned} &\mathrm{Pay}_1(\gamma_1, \kappa_1) = \gamma_1\, \mathbb{E}(R_T - L)^+ - \kappa_1\, \mathbb{E}(L - R_T)^+ \\ \text{and}\quad &\mathrm{Pay}_2(\gamma_2, \kappa_2) = \gamma_2\, \mathbb{E}(R_T - L)^+ - \kappa_2\, \mathbb{E}(L - R_T)^+, \end{aligned}$$

where γ_1 and γ_2 stand respectively for the profit ratios of partner 1 and partner 2.

We want to determine the c-fair profit-sharing ratios, γ_1 and γ_2, and possibly the project manager's remuneration share, depending on the type of contract that binds him to the entrepreneurs.

First, we consider the situation in which the partners of the *PL-sharing musharakah* contract decide to manage the project themselves.

PL-Sharing Musharakah Contract Between Two Partners

If the partners are managing the project together and making an equal contribution to management, the difference in their overall contributions lies in their respective capital contributions, κ_1 and κ_2. In this case, it would be reasonable to use the following fair profit-sharing ratios: $\gamma_1 = \kappa_1$ and $\gamma_2 = \kappa_2$. So, we have:

$$\begin{aligned} &\mathrm{Pay}_1(\gamma_1, \kappa_1) \ / \ \kappa_1 = \mathrm{Pay}_2(\gamma_2, \kappa_2) \ / \ \kappa_2 \\ \text{or, equivalently,} \quad &\mathrm{Pay}_1(\gamma_1, \kappa_1) = c\,\mathrm{Pay}_2(\gamma_2, \kappa_2), \ \text{with } c = \kappa_1/\kappa_2. \end{aligned}$$

Now suppose that the two partners in the *PL-sharing musharakah* contract agree to appoint one of them (Partner 2, for example) as manager of the project. In this case, it is not permitted to specify a fixed salary for the manager, who is also a partner in the contract. However, their additional contribution to the management of the project can be rewarded by increasing their profit-sharing ratio. As before, we must determine the (c_1, c_2)-fair profit-sharing ratios (γ_1 and γ_2) by selecting the weighting constants c_1 and c_2 for their payoffs in a way that offsets Partner 2's additional responsibilities. This means:

$$\begin{aligned} &\mathrm{Pay}_1(\gamma_1, \kappa_1) \ / \ c_1 = \mathrm{Pay}_2(\gamma_2, \kappa_2) \ / \ c_2 \\ \text{or, equivalently,} \quad &\mathrm{Pay}_1(\gamma_1, \kappa_1) = c\,\mathrm{Pay}_2(\gamma_2, \kappa_2), \ \text{with } c = c_2/c_1. \end{aligned}$$

To determine γ_1 and γ_2, we have to solve the system of equations:

$$\begin{cases} \mathrm{Pay}_1(\gamma_1, \kappa_1) \ / \ c_1 = \mathrm{Pay}_2(\gamma_2, \kappa_2) \ / \ c_2 \\ \gamma_1 + \gamma_2 = 1, \ \kappa_1 + \kappa_2 = 1. \end{cases} \tag{29}$$

Its solution reads

$$\gamma_\ell = \frac{c_\ell}{c_1 + c_2} - \left(\frac{c_\ell}{c_1 + c_2} - \kappa_\ell \right) \rho(R_T, L), \qquad \ell \in \{1, 2\}. \tag{30}$$

Next, we consider the situation in which the two partners of the *PL-sharing musharakah* contract appoint an external manager to oversee the project. There are several ways to compensate him, but here we consider two types of contract that could bind the manager to the entrepreneurs: a *wakalah* contract, and a *mudarabah* contract.

When the Entreprise is Managed Through a *Wakalah* Contract

Here, we suppose that the two partners of the *PL-sharing musharakah* contract are bound to the manager by a *wakalah* contract that provides for fixed remuneration (an amount p, paid out k times up to maturity T), which will be included in their expenses. In this case, if there is a discounted rate of return r that could be earned from investing in a asset over a T-period, the actualised expected payoffs of the partners in the *PL-sharing musharakah* contract, $\mathrm{Pay}_1(\gamma_1, \kappa_1, p)$ and $\mathrm{Pay}_2(\gamma_2, \kappa_2, p)$, satisfy the following, respectively:

$$\begin{aligned} \mathrm{Pay}_1(\gamma_1, \kappa_1, p) \ &= \ (1+r)^{-T} \left(\gamma_1 \, \mathbb{E}(R_T - L)^+ - \kappa_1 \, \mathbb{E}(L - R_T)^+ \right) - \frac{\mathrm{Pay}_3(p, k)}{2} \\ \mathrm{Pay}_2(\gamma_2, \kappa_2, p) \ &= \ (1+r)^{-T} \left(\gamma_2 \, \mathbb{E}(R_T - L)^+ - \kappa_2 \, \mathbb{E}(L - R_T)^+ \right) - \frac{\mathrm{Pay}_3(p, k)}{2}. \end{aligned}$$

In the previous equations, $\mathrm{Pay}_3(p,k)$ is the (actualised expected) payoff of Partner 3 (which is supposed to be the manager). The quantity $\mathrm{Pay}_3(p,k)$ is the present value of the sum of the future cash flow p up to maturity T of the contract. So, we have

$$\mathrm{Pay}_3(p,k) = \sum_{i=1}^{k} p\,(1+r)^{-\frac{T}{k}i} = p\left(1+r\right)^{-T}\,\frac{(1+r)^T-1}{(1+r)^{\frac{T}{k}}-1}\,.$$

As in previous cases, we aim to determine the (c_1,c_2,c_3)-fair profit-sharing ratios in a viable *PL-sharing musharakah* contract. In other words, we want to calculate the periodic remuneration p up to maturity and the profit-sharing ratios γ_1 and γ_2 satisfying :

$$\begin{cases} \mathrm{Pay}_1(\gamma_1,\kappa_1,p)\ /\ c_1 = \mathrm{Pay}_2(\gamma_2,\kappa_2,p)\ /\ c_2 = \mathrm{Pay}_3(p,k)\ /\ c_3 \\ \gamma_1+\gamma_2 = 1. \end{cases} \tag{31}$$

If we want to determine first p, it is suitable to rewrite the system (31) as a system of three equations with three unknown:

$$\begin{cases} \mathrm{Pay}_1(\gamma_1,\kappa_1,p)\ /\ c_1\,c_2 = \mathrm{Pay}_3(p,k)\ /\ c_2\,c_3 \\ \mathrm{Pay}_2(\gamma_2,\kappa_2,p)\ /\ c_1\,c_2 = \mathrm{Pay}_3(p,k)\ /\ c_1\,c_3 \\ \gamma_1+\gamma_2 = 1. \end{cases} \tag{32}$$

Solving the system (32) leads to (setting $\theta(r,T,k) = ((1+r)^{\frac{T}{k}}-1)\,/\,((1+r)^T-1)$)

$$\theta(r,T,k)\,p = \varpi_3\,\Delta(R_T,L), \quad \text{with} \quad \varpi_3 = \frac{c_3}{c_1+c_2+c_3}. \tag{33}$$

Then, replacing p by its value in the two first equations gives, for $\ell\in\{1,2\}$,

$$\gamma_\ell = \left(\frac{\varpi_3}{2}+\varpi_\ell\right)\left(1-\rho(R_T,L)\right) + \kappa_\ell\,\rho(R_T,L), \tag{34}$$

with

$$\varpi_\ell = \frac{c_\ell}{c_1c_2+c_1c_3+c_2c_3}. \tag{35}$$

When the Enterprise is Managed Through a *Mudarabah* Contract

Here, we assume that the two partners of the *PL-sharing musharakah* contract are bound by a PL-profit-sharing mudarabah contract with the manager (partner 3). In this case, the expected payoffs $\mathrm{Pay}_1(\gamma_1,\kappa_1)$, $\mathrm{Pay}_2(\gamma_2,\kappa_2)$ and $\mathrm{Pay}_3(\gamma_3)$ of the partners are given respectively by:

$$\begin{aligned} \mathrm{Pay}_1(\gamma_1,\kappa_1) &= \gamma_1\,\mathbb{E}(R_T-L)^+ - \kappa_1\,\mathbb{E}(L-R_T)^+ \\ \mathrm{Pay}_2(\gamma_2,\kappa_2) &= \gamma_2\,\mathbb{E}(R_T-L)^+ - \kappa_2\,\mathbb{E}(L-R_T)^+ \\ \mathrm{Pay}_3(\gamma_3) &= \gamma_3\,\mathbb{E}(R_T-L)^+. \end{aligned}$$

Finding the (c_1, c_2, c_3)-fair profit sharing ratio amounts to solve the system

$$\begin{cases} \mathrm{Pay}_1(\gamma_1, \kappa_1) \;/\; c_1 = \mathrm{Pay}_2(\gamma_2, \kappa_2) \;/\; c_2 = \mathrm{Pay}_1(\gamma_3, \kappa_3) \;/\; c_3 \\ \quad \gamma_1 + \gamma_2 + \gamma_3 = 1. \end{cases} \tag{36}$$

To compute γ_1, we consider the following equivalent system:

$$\begin{cases} \mathrm{Pay}_1(\gamma_1, \kappa_1) \;/\; c_1\, c_2\, c_3 = \mathrm{Pay}_1(\gamma_1, \kappa_1) \;/\; c_1\, c_2\, c_3 \\ \mathrm{Pay}_1(\gamma_1, \kappa_1) \;/\; c_1^2\, c_3 = \mathrm{Pay}_2(\gamma_2, \kappa_2) \;/\; c_1\, c_2\, c_3 \\ \mathrm{Pay}_1(\gamma_1, \kappa_1) \;/\; c_1^2\, c_2 = \mathrm{Pay}_3(\gamma_3) \;/ c_1\, c_2\, c_3 \\ \gamma_1 + \gamma_2 + \gamma_3 = 1. \end{cases}$$

Now, adding the left hand side and then the right hand side of the three first equations of the previous system gives

$$\Big(\frac{1}{c_2 c_3} + \frac{1}{c_1 c_3} + \frac{1}{c_1 c_2}\Big)\,\mathrm{Pay}_1(\gamma_1, \kappa_1) = \frac{1}{c_2\, c_3}\Big(\mathbb{E}(R_T - L)^+ - \mathbb{E}(L - R_T)^+\Big).$$

It follows that

$$\gamma_1 = \varpi_1 - (\varpi_1 - \kappa_1)\,\rho(R_T, L), \qquad \text{with } \; \varpi_1 = \frac{c_1}{c_1 + c_2 + c_3}. \tag{37}$$

The ratio γ_2 is obtained by solving the equivalent system

$$\begin{cases} \mathrm{Pay}_2(\gamma_2, \kappa_2) \;/\; c_2^2\, c_3 = \mathrm{Pay}_1(\gamma_1, \kappa_1) \;/\; c_1\, c_2\, c_3 \\ \mathrm{Pay}_2(\gamma_1, \kappa_2) \;/\; c_1\, c_2\, c_3 = \mathrm{Pay}_2(\gamma_2, \kappa_2) \;/\; c_1\, c_2\, c_3 \\ \mathrm{Pay}_2(\gamma_2, \kappa_2) \;/\; c_2^2\, c_2 = \mathrm{Pay}_3(\gamma_3) \;/\; c_1\, c_2\, c_3 \\ \gamma_1 + \gamma_2 + \gamma_3 = 1, \end{cases}$$

which solution is given by

$$\gamma_2 = \frac{c_2}{c_1 + c_2 + c_3} - \left(\frac{c_2}{c_1 + c_2 + c_3} - \kappa_2\right)\rho(R_T, L). \tag{38}$$

Using the condition $\gamma_1 + \gamma_2 + \gamma_3 = 1$ yields

$$\gamma_3 = \frac{c_3}{c_1 + c_2 + c_3}\Big(1 - \rho(R_T, L)\Big). \tag{39}$$

The following proposition summarises all the results of the previous discussions.

Proposition 4.1. *Consider a viable T-term and R-asset based* PL-sharing musharakah *contract between two partners. Then:*

(i) If they manage themselves the project and agree to rate their respective payoff by the coefficients $1/c_1$ and $1/c_2$ according to their contribution to the funding

and the management of the project, then, the (c_1, c_2)*-fair profit sharing ratios are given by*

$$\begin{aligned}\gamma_\ell &= \varpi_\ell\,(1-\rho(R_T, L)) + \kappa_\ell\,\rho(R_T, L), &(40)\\ &= \varpi_\ell\,\beta(R_T, L) + \kappa_\ell\,\rho(R_T, L), \qquad \ell = 1, 2, &(41)\end{aligned}$$

where, for every $\ell \in \{1, 2\}$, $\varpi_\ell = c_{\bar{\ell}} \,/\, (c_1 + c_2)$. *Furthermore, the expected payoff of each partner* ℓ *is given by*

$$Pay_\ell(\gamma_\ell, \kappa_\ell) = \varpi_\ell\,\Delta(R_T, L), \qquad \ell = 1, 2. \tag{42}$$

(ii) Supose they appoint a external manager and enter into a wakalah *contract with them, whereby the manager receives fixed remuneration (an amount* p *paid out* k *times up to maturity* T*). If Partner 1, Partner 2 and the manager agree to rate their respective payoffs using the coefficients* $1/c_1$, $1/c_2$ *and* $1/c_3$*, then the fair profit-sharing ratios and periodic remuneration* p *will satisfy the following:*

$$\begin{aligned}\theta(r, T, k)\,p &= \varpi_3\,\Delta(R_T, L) &(43)\\ \gamma_\ell &= \left(\frac{\varpi_3}{2} + \varpi_\ell\right)\left(1-\rho(R_T, L)\right) + \kappa_\ell\,\rho(R_T, L),\ \ell \in \{1, 2\}. &(44)\end{aligned}$$

The updating coefficient $\theta(r, T, k) = ((1+r)^{\frac{T}{k}} - 1) \,/\, ((1+r)^{T} - 1)$ *and the weights* $(\varpi_\ell)_{\ell\in\{1,2,3\}}$ *are given by* $\varpi_\ell = c_\ell \,/\, (c_1 + c_2 + c_3)$*, for all* $\ell \in \{1, 2, 3\}$*. Moreover, the expected payoff of each partner* ℓ *(including the manager) is given by*

$$Pay_\ell(\gamma_\ell, \kappa_\ell) = \varpi_\ell\,\Delta(R_T, L), \qquad \ell \in \{1, 2, 3\}. \tag{45}$$

(iii) If they appoint a manager (Partner 3) outside the company and enter into a PL-sharing mudarabah *contract with him, the* (c_1, c_2, c_3)*-fair profit sharing ratios are given by the following formula (keeping in mind that* $\kappa_3 = 0$*)*

$$\gamma_\ell = \varpi_\ell\big(1-\rho(R_T, L)\big) + \kappa_\ell\,\rho(R_T, L), \quad \textit{for all } \ell \in \{1, 2, 3\}, \tag{46}$$

with $\varpi_\ell = c_\ell \,/\, (c_1 + c_2 + c_3)$.

Still, the expected payoff of each partner ℓ *is given by*

$$Pay_\ell(\gamma_\ell, \kappa_\ell) = \varpi_\ell\,\Delta(R_T, L), \qquad \ell \in \{1, 2, 3\}. \tag{47}$$

Remark 4.2. It is important to note that setting $\kappa_1 = 1$ and $\kappa_2 = 0$ in Equation (41) yields the (c_1, c_2)-fair profit sharing ratios obtained in Equation (26) and (27) for a *PL-sharing mudarabah* contract between two partners. Consequently, a *PL-sharing mudarabah* contract between two partners is a *PL-sharing musharakah* contract in which one partner provides all the capital (so that $\kappa_1 = 1$) and the other provides labour ($\kappa_2 = 0$). Further discussions on the results of Proposition 4.1 can be found in the subsequent discussion of Proposition 4.4.

Our current objective is to extend Proposition 4.1 to mixed contracts involving multiple partners.

4.3 Fair Profit Sharing Ratio: the General Framework

We consider a *PL-sharing musharakah* contrat between d partners: partner 1, Partner 2, ..., and partner d. When $d = 2$, the partnership may be for example between a bank and a company, two companies, two banks, etc. The partners contributed respectively amounts L_1, L_2, ..., and L_d, to the capital. Let $L = L_1 + \cdots + L_d$ be the initial capital and define $\kappa_\ell = L_\ell/L$ as the equity stake of partners ℓ, for every $\ell \in \{1, \ldots, d\}$. Finally, let us denote the profit sharing ratios of the partners by $(\gamma_1, \ldots, \gamma_d)$, with $\gamma_1 + \cdots + \gamma_d = 1$. The aim is to determine the fair profit sharing between the partners.

As before, R_T denotes the net income process of the investment at maturity T. At the maturity of the contract, the expected payoff of partner ℓ is given by

$$\mathrm{Pay}_\ell(\gamma_\ell, \kappa_\ell) = \gamma_\ell \, \mathbb{E}(R_T - L)^+ - \kappa_\ell \, \mathbb{E}(L - R_T)^+. \tag{48}$$

We next define a $(c_1, \ldots, c_d)$-fair *PL-sharing musharakah* contract.

Definition 4.3. A viable T-term and R-asset based *PL-sharing musharakah* contract involving d partners is said to be $(c_1, \ldots, c_d)$-*fair* if

$$\mathrm{Pay}_1(\gamma_1, \kappa_1) \ / \ c_1 \ = \ \cdots \ = \ \mathrm{Pay}_d(\gamma_d, \kappa_d) \ / c_d. \tag{49}$$

The following result gives the profit sharing ratio of each partner of a *PL-sharing musharakah* contract. We still define the weights $(\varpi_\ell)_{1 \le \ell \le d}$ by

$$\varpi_\ell = c_\ell \ / \sum_{i=1}^{d} c_i, \text{ for every } \ \ell \in \{1, \ldots, d\}. \tag{50}$$

Proposition 4.4. *Consider a viable T-term and R-asset based* PL-sharing musharakah *contract between d partners whom manage the project themselves and agree to rate their respective payoff by the coefficients $1/c_\ell$, $\ell \in \{1, \ldots, d\}$. Then, the $(c_1, \ldots, c_d)$-fair profit sharing ratios are given by*

$$\begin{aligned} \gamma_\ell &= \varpi_\ell \, \beta(R_T, L) + \kappa_\ell \, \rho(R_T, L) && (51) \\ &= \varpi_\ell \big(1 - \rho(R_T, L)\big) + \kappa_\ell \, \rho(R_T, L), \quad \textit{for any} \ \ \ell \in \{1, \ldots, d\}. && (52) \end{aligned}$$

Furthermore, the expected payoff of each partner ℓ is given by

$$Pay_\ell(\gamma_\ell, \kappa_\ell) = \varpi_\ell \, \Delta(R_T, L), \qquad \ell = 1, \ldots, d. \tag{53}$$

This means that the partners will share the expected investment profit $\Delta(R_T, L)$ *according to the respective weights ϖ_ℓ.*

An interesting case arises when $d - 1$ partners in a *PL-sharing musharakah* contract appoint a manager (partner d), who is not one of the partners, and enter into a *PL-sharing mudarabah* contract with him. This appears to be a special instance of Proposition 4.4 which we present as a corollary.

Corollary 4.5. *Consider a viable T-term and R-asset based* PL-sharing musharakah *contract involving $d-1$ partners. Suppose the partners appoint a manager (partner d) from outside their group and enter into a* PL-sharing mudarabah *contract. If they agree to weight their respective payoff by the coefficients $1/c_\ell$, $\ell \in \{1,\dots,d\}$, according to their contribution to the funding and the management of the project, the $(c_1,\dots,c_d)$-fair profit sharing ratios are given by*

$$\begin{aligned} \gamma_\ell &= \varpi_\ell \left(1-\rho(R_T,L)\right) + \kappa_\ell\, \rho(R_T,L), \quad \ell = 1,\dots,d-1, && (54)\\ \gamma_d &= \varpi_d \left(1-\rho(R_T,L)\right). && (55) \end{aligned}$$

Furthermore, the expected payoff of each partner ℓ is given by

$$Pay_\ell(\gamma_\ell,\kappa_\ell) = \varpi_\ell\, \Delta(R_T,L), \qquad \ell = 1,\dots,d. \tag{56}$$

Before giving the proof of Proposition 4.4, let us make the following remarks.

Remark 4.6. Owing to Equation (52) (or to equations (54) and (55)), we can see that the $(c_1,\dots,c_d)$-fair profit-sharing ratio γ_ℓ of partner ℓ can be decomposed as

$$\gamma_\ell = \underbrace{\varpi_\ell \left(1-\rho(R_T,L)\right)}_{\text{reward fof labour}} + \underbrace{\kappa_\ell\, \rho(R_T,L)}_{\text{reward of capital}}.$$

Therefore, tthe $(c_1,\dots,c_d)$-fair profit-sharing ratio appears to be a reward for the contribution to capital ($\kappa_\ell\, \rho(R_T,L)$) and a reward for the contribution to the management (or labour) of the project ($\varpi_\ell\,(1-\rho(R_T,L))$). We can also deduce that a moderately risky investment (i.e. an investment opportunity greater than the investment risk, or equivalently, $\rho(R_T,L) \le 1/2$) rewards project management contributions more than capital participation, whereas a high-risk investment (i.e. $\rho(R_T,L) \ge 1/2$) rewards capital contributions more. In other words, if we (a firm, for example) are looking to raise capital to finance our business, it is in our best interests to find a highly profitable (or low-risk) activity that will increase our share of the profits.

Corollary 4.7. *We still deduce from Equation (52) that for any $\ell,\ell' \in \{1,\dots,d\}$,*

$$\gamma_\ell - \gamma_{\ell'} = (\varpi_\ell - \varpi_{\ell'})\left(1-\rho(R_T,L)\right) + (\kappa_\ell - \kappa_{\ell'})\,\rho(R_T,L). \tag{57}$$

Then, the difference in profit-sharing ratios between partners ℓ and ℓ' ($\gamma_\ell - \gamma_{\ell'}$) is balanced by the difference in their contributions to capital ($\kappa_\ell - \kappa_{\ell'}$) and the difference in their weighted contributions to management ($\varpi_\ell - \varpi_{\ell'}$). Consequently, we deduce from Equation (57) that:

(i) If $\kappa_\ell \ge \kappa_{\ell'}$ and $\varpi_\ell \ge \varpi_{\ell'}$ (i.e., the partner ℓ contributes more than partner ℓ' in terms of both capital and labour) then, $\gamma_\ell \ge \gamma_{\ell'}$.

(ii) If $\kappa_\ell < \kappa_{\ell'}$ and $\varpi_\ell > \varpi_{\ell'}$, that is, partner ℓ contributes less capital than partner ℓ' but, more labour to the project's overall success, then,

$$\gamma_\ell \geq \gamma_{\ell'} \qquad \iff \qquad \rho(R_T, L) \leq \frac{\varpi_\ell - \varpi_{\ell'}}{(\varpi_\ell - \varpi_{\ell'}) - (\kappa_\ell - \kappa_{\ell'})} \in (0,1). \tag{9}$$

If in addition, $\rho(R_T, L) \leq 1/2$, $\varpi_\ell - \varpi_{\ell'} > \kappa_{\ell'} - \kappa_\ell$ (partner ℓ bridges the funding gap for capital through his management contributions), then, $\gamma_\ell \geq \gamma_{\ell'}$.

(iii) If $\varpi_\ell < \varpi_{\ell'}$ and $\kappa_\ell > \kappa_{\ell'}$ (partner ℓ contribues less in terms of management than partner ℓ', but more in terms of capital), then,

$$\gamma_\ell \geq \gamma_{\ell'} \qquad \iff \qquad \rho(R_T, L) \geq \frac{\varpi_\ell - \varpi_{\ell'}}{(\varpi_\ell - \varpi_{\ell'}) - (\kappa_\ell - \kappa_{\ell'})} \in (0,1). \tag{10}$$

If furthermore, $\rho(R_T, L) \geq 1/2$, $\kappa_\ell - \kappa_{\ell'} > \varpi_{\ell'} - \varpi_\ell$ (partner ℓ closes the gap in management contributions through his capital contribution), then, $\gamma_\ell \geq \gamma_{\ell'}$.

Remark 4.8. We may also note that when $c_i = 1$, for any $\ell \in \{1, \dots, \}$, we have $\varpi_\ell = 1/d$. Equation (52) becomes

$$\gamma_\ell = \frac{1}{d} - \Big(\frac{1}{d} - \kappa_\ell\Big)\rho(R_T, L). \tag{58}$$

In this case, we have for any $\ell, \ell' \in \{1, \dots, d\}$, $\gamma_\ell - \gamma_{\ell'} = (\kappa_\ell - \kappa_{\ell'})\rho(R_T, L)$, so that the difference in the profit-sharing ratio between two partners comes from the difference in their share of the capital contribution. If all partners have contributed the same amount of capital ($\kappa_\ell = \kappa_{\ell'}$, for any $\ell \neq \ell'$), then the fair profit-sharing ratio will be equal for all partners, regardless of the value of $\rho(R_T, L)$: $\gamma_\ell = \frac{1}{d}$, for any $\ell \in \{1, \dots, d\}$. This also suggests that the best way to absorb investment risk in a *PL-musharakah* contract is to encourage equal participation in capital and labour.

Figure 1 depicts the (c_1, c_2, c_3, c_4)-fair profit sharing ratio as a function of the investment risk $\rho(R_T, L)$. We set $(c_1, c_2, c_3, c_4) = (3, 5, 4, 2)$ and $(\kappa_1,\ \kappa_2,\ \kappa_3,\ \kappa_4) = (1/8,\ 5/8,\ 2/8,\ 0)$. Depending on the values taken by the c_ℓ's and the κ_ℓ's, we see that the profit sharing ratios may be either non-increasing or non-decreasing functions of $\rho(R_T, L)$. This is in line with Corrolary 4.7.

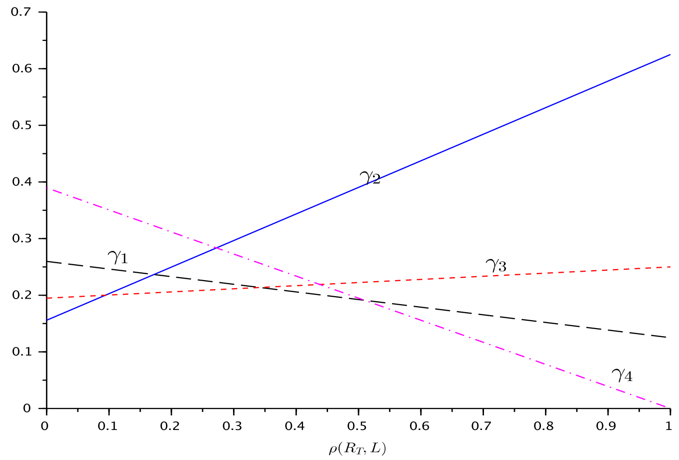

Figure 1: We represent the (c_1, c_2, c_3, c_4)-fair profit sharing ratios γ_ℓ (in ordinate axis) as functions of the investment risk $\rho(R_T, L)$ (in abscissa axis): $(c_1, c_2, c_3, c_4) = (3, 5, 4, 2)$ and $(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (1/8, 5/8, 2/8, 0)$.

Now, let us prove Proposition 4.4.

Proof. (of Proposition 4.4)] Let $\ell \in \{1, \dots, d\}$. Since the contract is $(c_1, \dots, c_d)$-fair, we have in particular

$$\begin{cases} \mathrm{Pay}_\ell(\gamma_\ell, \kappa_\ell) \ / \ c_\ell = \mathrm{Pay}_1(\gamma_1, \kappa_1) \ / \ c_1 \\ \mathrm{Pay}_\ell(\gamma_\ell, \kappa_\ell) \ / \ c_\ell = \mathrm{Pay}_2(\gamma_2, \kappa_2) \ / c_2 \\ \quad \vdots \quad \vdots \quad \vdots \qquad \vdots \quad \vdots \\ \mathrm{Pay}_\ell(\gamma_\ell, \kappa_\ell) \ / \ c_\ell = \mathrm{Pay}_d(\gamma_d, \kappa_d) \ / c_d \\ \gamma_1 + \ldots + \gamma_d = 1. \end{cases}$$

Il follows that

$$\begin{cases} \mathrm{Pay}_\ell(\gamma_\ell, \kappa_\ell) \ / \ \left(c_\ell \prod_{j \neq 1} c_j\right) = \mathrm{Pay}_1(\gamma_1, \kappa_1) \ / \ \left(\prod_{j=1}^d c_j\right) \\ \mathrm{Pay}_\ell(\gamma_\ell, \kappa_\ell) \ / \ \left(c_\ell \prod_{j \neq 2} c_j\right) = \mathrm{Pay}_2(\gamma_2, \kappa_2) \ / \ \left(\prod_{j=1}^d c_j\right) \\ \quad \vdots \quad \vdots \quad \vdots \qquad \vdots \quad \vdots \\ \mathrm{Pay}_\ell(\gamma_\ell, \kappa_\ell) \ / \ \left(c_\ell \prod_{j \neq d} c_j\right) = \mathrm{Pay}_d(\gamma_d, \kappa_d) \ / \ \left(\prod_{j=1}^d c_j\right) \\ \gamma_1 + \ldots + \gamma_d = 1. \end{cases}$$

Adding the left and right hand sides of the d first equations of the previous system gives (setting $E_1 := \mathbb{E}(R_T - L)^+$ and $E_2 := \mathbb{E}(L - R_T)^+$)

$$\left(\gamma_\ell \, E_1 - \kappa_\ell \, E_2\right) \ \frac{1}{c_\ell} \left(\sum_{i=1}^d \frac{1}{\prod_{j \neq i} c_j} \right) = \left(E_1 - E_2\right) \ / \ \left(\prod_{i=1}^d c_i \right).$$

We deduce that

$$\big(\gamma_\ell\, E_1 - \kappa_\ell\, E_2\big)\ \frac{1}{c_\ell}\left(\sum_{i=1}^{d}\frac{c_i}{\prod_{j=1}^{d} c_j}\right) = \big(E_1 - E_2\big)\ /\ \Big(\prod_{i=1}^{d} c_i\Big).$$

Consequently,

$$\big(\gamma_\ell\, E_1 - \kappa_\ell\, E_2\big)\ \frac{1}{c_\ell}\sum_{i=1}^{d} c_i = \big(E_1 - E_2\big).$$

Therefore, $\gamma_\ell\, E_1 - \kappa_\ell\, E_2 = \varpi_\ell\big(E_1 - E_2\big)$, and this leads to Equation (52).

Now, the expected payoff $\mathrm{Pay}_\ell(\gamma_\ell, \kappa_\ell)$ of each partner ℓ is obtained by replacing in Equation (48), γ_ℓ by its value in (52). □

Example 4.9. Consider a viable *PL-sharing musharakah* contract of four partners, with time horizon T, based on the asset R. Suppose the partners agree to weight their contrubution to the venture's success by the coefficients $1/c_\ell$, with $\ell = 1, \dots, 4$. We determine c-fair profit sharing ratios for given values of c_ℓ, when $\rho(R_T) \in \{\frac{1}{8}, \frac{3}{4}\}$.

(i) Assumung that all partners contribute the same share of $\kappa_\ell = 25\%$ of the capital, we have:

- ◇ If $(c_1, c_2, c_3, c_4) = (1, 1, 1, 1)$, then $(\varpi_1,\ \varpi_2,\ \varpi_3,\ \varpi_4) = (1/4,\ 1/4,\ 1/4,\ 1/4)$ and $\gamma_\ell = 25\%$, $\ell \in \{1, \dots, 4\}$, .
- ◇ If $(c_1, c_2, c_3, c_4) = (4, 2, 4, 1)$, then $(\varpi_1,\ \varpi_2,\ \varpi_3,\ \varpi_4) = \frac{1}{11}\,(4,\ 2,\ 4,\ 1)$ and

$$\begin{array}{rrcll} & (\gamma_1, \gamma_2, \gamma_3, \gamma_4) & = & (35\%,\ 19\%,\ 35\%,\ 11\%) & \quad \text{if } \rho(R_T) = 1/8 \\ \text{and} & (\gamma_1, \gamma_2, \gamma_3, \gamma_4) & \simeq & (29\%,\ 20\%,\ 29\%,\ 22\%) & \quad \text{if } \rho(R_T) = 2/3. \end{array}$$

Since $\kappa_\ell = \kappa_{\ell'}$, for all $\ell \neq \ell'$, $\ell, \ell' \in \{1, 2, 3, 4\}$, we deduce from Corollary 4.7 that $\gamma_\ell \geq \gamma_{\ell'}$ whenever $\varpi_\ell \geq \varpi_{\ell'}$.

(ii) Suppose now that $(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (\frac{1}{8}, \frac{3}{8}, \frac{1}{8}, \frac{3}{8}) = (12.5\%, 37.5\%, 12.5\%, 37.5\%)$. In this case, we have:

- ◇ If $(c_1, c_2, c_3, c_4) = (1, 1, 1, 1)$, then $(\varpi_1,\ \varpi_2,\ \varpi_3,\ \varpi_4) = (1/4,\ 1/4,\ 1/4,\ 1/4)$ and

$$\begin{array}{rrcll} & (\gamma_1, \gamma_2, \gamma_3, \gamma_4) & = & (23\%,\ 27\%,\ 23\%,\ 27\%) & \quad \text{if } \rho(R_T) = 1/8 \\ \text{and} & (\gamma_1, \gamma_2, \gamma_3, \gamma_4) & = & (17\%,\ 33\%,\ 17\%,\ 33\%) & \quad \text{if } \rho(R_T) = 2/3. \end{array}$$

Since $\varpi_\ell = \varpi_{\ell'}$, for all $\ell \neq \ell'$, $\ell, \ell' \in \{1, 2, 3, 4\}$, then $\gamma_\ell \geq \gamma_{\ell'}$ whenever $\kappa_\ell \geq \kappa_{\ell'}$ (see Corollary 4.7).

- $\diamond$ If $(c_1, c_2, c_3, c_4) = (1, 2, 1, 4)$, then $(\varpi_1,\ \varpi_2,\ \varpi_3,\ \varpi_4) = \frac{1}{11}\,(4,\ 2,\ 4,\ 1)$ and

$$\begin{array}{rrcll} & (\gamma_1, \gamma_2, \gamma_3, \gamma_4) & = & (33\%,\ 21\%,\ 33\%,\ 13\%) & \text{if } \rho(R_T) = 1/8 \\ \text{and} & (\gamma_1, \gamma_2, \gamma_3, \gamma_4) & \simeq & (20\%,\ 32\%,\ 20\%,\ 28\%) & \text{if } \rho(R_T) = 2/3. \end{array}$$

  We remark from the foregoing that $\rho(R_T) = 1/8$ and $\gamma_1 \geq \gamma_4$ whereas $\kappa_1 \leq \kappa_4$. This is because $\varpi_1 - \varpi_4 > \kappa_4 - \kappa_1$ (see Corollary 4.7).

(iii) Now suppose that the company's manager (lthe general partner) is Partner 4 and that this one is bound to the entrepreneurs (shareholders) by a *PL-sharing mudarabah* contract. He does not contribute capital, but brings his expertise to help grow the business. Assume that $(\kappa_1, \kappa_2, \kappa_3) = (1/3, 1/3, 1/3)$ and $\rho(R_T, L) = 1/2$.

- $\diamond$ If $(c_1, c_2, c_3, c_4) = (1, 1, 1, 1)$, then $(\varpi_1,\ \varpi_2,\ \varpi_3,\ \varpi_4) = (1/4,\ 1/4,\ 1/4,\ 1/4)$ and $(\gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4) = (29\%,\ 29\%,\ 29\%,\ 13\%)$.
- $\diamond$ If $(c_1, c_2, c_3, c_4) = (2, 2, 2, 3)$, then $(\varpi_1,\ \varpi_2,\ \varpi_3,\ \varpi_4) = \frac{1}{9}(2,\ 2,\ 2,\ 3)$ and $(\gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4) = (28\%,\ 28\%,\ 28\%,\ 16\%)$.

Remark 4.10. (About a *diminishing musharakah* contract) Consider a viable T-term and R-asset-based *musharakah* contract. Equation (52) provides the profit-sharing ratios of the stockholders of the *musharakah* contract at time t = 0, so we develop a one-period model in this work.

Consider a *diminishing musharakah* contract in which some stockholders wish to sell a portion of their stocks at time t, where $0 < t < T$, and suppose these stocks can be traded on a secondary stock market. In this situation, we need to evaluate the profit-sharing ratios at time t, a problem which we propose to investigate separately from the present paper, both because of the need for more sophisticated tools from modern financial modelling, such as backward stochastic differential equations, and to avoid making the paper any longer.

Next, we consider a generalisation of Proposition 4.1 2. to d partners, where the $(d-1)$ partners of a *PL sharing - musharakah* contract appoint a manager (the d'th partner) outside of their group and enter into a *wakalah* (an agency) contract with them.

Proposition 4.11. *Consider a viable T-term and R-asset based* PL - musharakah *contract involving $d-1$ partners. Suppose the partners appoint an external manager (partner d) and enter into a* wakalah *contract with him, which provides a fixed remuneration (a amount p that is paid out k times up to the maturity T). If the partners agree to rate their respective expected payoff by the coefficients $1/c_\ell$, $\ell \in \{1, \ldots, d\}$ then the $(c_1, \ldots, c_d)$-fair profit sharing ratios and periodic remuneration p satisfy*

$$\theta(r, T, k)\, p = \varpi_d\, \Delta(R_T, L) \tag{59}$$

$$\gamma_\ell = \left(\frac{\varpi_d}{d-1} + \varpi_\ell\right)\big(1 - \rho(R_T, L)\big) + \kappa_\ell\, \rho(R_T, L),\ \ \ell = 1, \ldots, d-1. \tag{60}$$

Here, $\theta(r,T,k) = (1+r)^{\frac{T}{k}} - 1\,/\,(1+r)^{T} - 1$, and the weights $(\varpi_\ell)_{\{1\le\ell\le d\}}$ is still defined by $\varpi_\ell = c_\ell\,/\,(c_1+\ldots+c_d)$, for every $\ell \in \{1,\ldots,d\}$.

Furthermore, the expected payoff of each partner ℓ (even for the manager) is given by

$$Pay_\ell(\gamma_\ell,\kappa_\ell) = \varpi_\ell\,\Delta(R_T,L), \qquad \ell = 1,\ldots,d. \tag{61}$$

Before presenting the proof, it is worth noting that $\gamma_1+\ldots+\gamma_{d-1} = 1$. A typical example where Proposition 4.11 applies is the situation where we have three partners:

- ◇ an investor (Partner 1) who provides all the capital ($\kappa_1 = 1$),
- ◇ a bank (Partner 2) acting as an agent for the investor via a *wakalah* contract and arranges a fixed remuneration p that is paid out k times up to the maturity T; and
- ◇ a company (Partner 3) in need of funding, bound to the bank by a *PL-mudarabah* contract ($\kappa_3 = 0$).

Proof. The $(c_1,\ldots,c_d)$-fair assumption implies for any $\ell = 1,\ldots,d-1$,

$$\mathrm{Pay}_\ell(\gamma_\ell,\kappa_\ell,p,k)\;/\;c_\ell = \mathrm{Pay}_3(p,k)\;/\;c_d, \tag{62}$$

where

$$\begin{aligned} \mathrm{Pay}_\ell(\gamma_\ell,\kappa_\ell,p,k) &= (1+r)^{-T}\left(\gamma_\ell\,\mathbb{E}(R_T-L)^+ - \kappa_\ell\,\mathbb{E}(L-R_T)^+\right) - \frac{\mathrm{Pay}_3(p,k)}{d-1} \quad (63)\\ \text{and}\quad \mathrm{Pay}_3(p,k) &= p\left(1+r\right)^{-T}\theta(r,T,k), \quad \text{for all } \ell = 1,\ldots,d-1. \end{aligned}$$

It follows from (62) that for any $\ell = 1,\ldots,d-1$,

$$\left(\gamma_\ell\,\mathbb{E}(R_T-L)^+ - \kappa_\ell\,\mathbb{E}(L-R_T)^+ - \frac{p\,\theta(r,T,k)}{d-1}\right)\;/\;c_\ell = p\,\theta(r,T,k)\;/\;c_d.$$

Set, for all $\ell,d \in \{1,\ldots,d\}$, $\mathcal{P}_d = \{1,\ldots,d\}$ and $\bar{\mathcal{P}}_d(\ell) = \mathcal{P}_d\backslash\{\ell\}$. Multiplying both sides of the previous equation by $\prod_{i\in\bar{\mathcal{P}}_{d-1}(\ell)}(1/c_i)$ gives

$$\left(\gamma_\ell\,\mathbb{E}(R_T-L)^+ - \kappa_\ell\,\mathbb{E}(L-R_T)^+ - \frac{p\,\theta(r,T,k)}{d-1}\right)\prod_{i=1}^{d-1} c_i^{-1} = p\,\theta(r,T,k)\prod_{i\in\bar{\mathcal{P}}_d(\ell)} c_i^{-1}.$$

Since $\gamma_1+\ldots+\gamma_{d-1} = 1$ and $\kappa_1+\ldots+\kappa_{d-1} = 1$, taking the sum over the ℓ's (from 1 to $d-1$) yields:

$$\Big(\Delta(R_T,L) - p\,\theta(r,T,k)\Big)\prod_{i\in\mathcal{P}_{d-1}} c_i^{-1} = p\,\theta(r,T,k)\sum_{\ell\in\mathcal{P}_{d-1}}\prod_{i\in\bar{\mathcal{P}}_d(\ell)} c_i^{-1}.$$

We deduce that

$$\begin{aligned}\Delta(R_T, L) \prod_{i\in\mathcal{P}_{d-1}} c_i^{-1} &= p\,\theta(r,T,k)\Big(\prod_{i\in\mathcal{P}_{d-1}} c_i^{-1} + \sum_{\ell\in\mathcal{P}_{d-1}} \prod_{i\in\bar{\mathcal{P}}_d(\ell)} c_i^{-1}\Big)\\ &= p\,\theta(r,T,k)\Big(\prod_{i\in\bar{\mathcal{P}}_d(d)} c_i^{-1} + \sum_{\ell\in\mathcal{P}_{d-1}} \prod_{i\in\bar{\mathcal{P}}_d(\ell)} c_i^{-1}\Big)\\ &= p\,\theta(r,T,k)\Big(\sum_{\ell\in\mathcal{P}_d} \prod_{i\in\bar{\mathcal{P}}_d(\ell)} c_i^{-1}\Big),\end{aligned}$$

and, finally, that,

$$\theta(r,T,k)\,p = \varpi_d\,\Delta(R_T, L).$$

Now, replacing $\theta(r,T,k)\,p$ by its value in Equation (62) gives

$$c_\ell\left(\gamma_\ell\,\mathbb{E}(R_T - L)^+ - \kappa_\ell\,\mathbb{E}(L - R_T)^+ - \frac{\varpi_d\,\Delta(R_T, L)}{d-1}\right) = c_d\,p\,\theta(r,T,k).$$

This implies that

$$\begin{aligned}\gamma_\ell &= \left(\frac{\varpi_d}{d-1} + \frac{c_\ell}{c_d}\varpi_d\right)\big(1-\rho(R_T,L)\big) + \kappa_\ell\,\rho(R_T,L)\\ &= \left(\frac{\varpi_d}{d-1} + \varpi_\ell\right)\big(1-\rho(R_T,L)\big) + \kappa_\ell\,\rho(R_T,L).\end{aligned}$$

Now, the expected payoff of each partner ℓ is obtained by replacing $\theta(r,T,k)\,p$ and γ_ℓ by their values in Equation (62) □

Example 4.12. Consider the following situation involving three partners. The first partner is an investor who provides all the capital ($\kappa_1 = 1$). The second partner is a bank acting as an agent for the investor via a *wakalah* contract, arranging fixed remuneration p to be paid k times up to maturity (T). The third partner is a company requiring funding, bound to the bank by a *PL-mudarabah* contract ($\kappa_3 = 0$).

Suppose they invest an amount L in company asset R for a maturity period T. Given the vector of weighting coefficients $c = (c_1, c_2, c_3)$, the respective shares of the bank, the investor and the company are given by:

$$\begin{aligned}\theta(r,T,k)\,p &= \varpi_3\,\Delta(R_T, L),\\ \gamma_1 &= \left(\frac{\varpi_3}{2} + \varpi_1\right)\big(1-\rho(R_T,L)\big) + \rho(R_T,L),\\ \gamma_2 &= \left(\frac{\varpi_3}{2} + \varpi_2\right).\end{aligned}$$

For example, if $c = (1,1,1)$ and $\rho(R_T, L) = 1/2$, then

$$\theta(r,T,k)p = \frac{1}{3}\Delta(R_T, L), \qquad \gamma_1 = \frac{3}{4}, \qquad \gamma_2 = \frac{1}{4}.$$

This leads to the expected investment profit being sharing equally between the three partners, i.e. $\frac{1}{3}\Delta(R_T, L)$ each.

4.4 How it Applies to Central Banking?

Consider an interbank investment market in which banks can create and sell securities or investment notes backed by the financing of major government or corporate projects. These instruments can be structured under various types of contract, including *PL-sharing-style joint venture agreements*. If a bank is unable to raise the necessary funds to finance a project, it can it can turn to the central bank. The central bank can then use deposits from other banks - which it is already mandated to invest in such projects (then, the central bank acts as an agent) - to invest under a *PL-sharing-style joint venture* agreement. The central bank may also invest under a *PL-sharing-style limited partnership* agreement when necessary.

We know that controlling the money supply is one of the central bank's missions. The question arises as to how this can be achieved in an interbank system based on an interbank investment market. This paper provides an answer regarding the money supply in relation to investment. Indeed, the central bank can adjust its monetary policy by setting a minimum profit-sharing rate (MPS rate), possibly sector-specific depending on its monetary policy guidelines, on *PL-sharing-style joint venture* contracts. This allows it to implement (we denote by the FPS rate the c-fair profit-sharing rate on PL-sharing-style joint venture contracts, including *PL-sharing-style limited partnership* contracts):

(i) Either *a macroeconomic stimulus policy through a reduction in the MPS rate*: when economic activity slows down, the central bank can set the MPS rate below the FPS rate to try to revitalize it by encouraging innovation and investment. Thus, commercial banks will reduce their margin on investments to the benefit of businesses, which can pass this reduction on to prices. This will increase household purchasing power.

(ii) *A deterrent macroeconomic policy through an increase in the MPS rate*: when investment is too high, risking deflation, the central bank can set the MP rate > FPS rate to discourage investment and encourage consumption by economic agents, in order to rebalance supply and demand levels.

5 Modelisation Perspectives

First, note that following our approach, the computation of the profit-sharing ratios and all the also quantities of interest depend on both $\rho(R_T, L)$ and $\Delta(R_T, L)$. In practice, to compute these quantities, we need to specify an adequate model (dependent on the observations) for the dynamics of the investment asset R. Among the possible models, we may enumerate the usual econometrics models, such as autoregressive (AR) models, moving average (MA) models, autoregressive integrated moving average (ARIMA) models, autoregressive conditional heteroskedasticity (ARCH) models, as well as their generalizations, such as GARCH models. We may also use diffusion model governed by stochastic differential equations, as used in finance. Of these finance-based models, we will consider the famous Black-Scholes model, for which all the quantities of interest can be computed explicitly.

5.1 The Black-Scholes Model

The Model and the Main Result

In the Black-Scholes model, we suppose that the investment asset $(R_t)_{t\in[0,T]}$ evolves following the SDE:

$$\mathrm{d}R_t = \mu\, R_t\, \mathrm{d}t + \sigma R_t\, \mathrm{d}W_t, \qquad R_0 = x_0, \quad t \in [0,T], \tag{64}$$

where μ is the instantaneous return, σ is the instantaneous volatility, and, the process $(W_t)_{t\in[0,T]}$ is a standard brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We have the following easy result.

Proposition 5.1. *If the dynamics of R is that of Black-Scholes, then, an investment on the asset R is viable if and only if $\mu > 0$. Furthermore, we have*

$$\rho(R_T, L) = -\frac{\Phi(-\vartheta(\mu,\sigma)) - e^{\mu T}\Phi(-\vartheta(\mu,\sigma) - \sigma\sqrt{T})}{\Phi(\vartheta(\mu,\sigma)) - e^{\mu T}\Phi(\vartheta(\mu,\sigma) + \sigma\sqrt{T})} \tag{65}$$

$$\text{and} \qquad \Delta(R_T, L) = L(e^{\mu T} - 1), \tag{66}$$

where $\vartheta(\mu,\sigma) = (2\mu - \sigma^2)T \,/\, (2\sigma\sqrt{T})$ and $\Phi(\cdot)$ is the cdf of the standard gaussian distribution.

Proof. The viability result follows from (66). In fact

$$\Delta(R_T, L) > 0 \quad \Longleftrightarrow \quad \mu > 0.$$

For the rest, we just need to show that $\mathbb{E}((L-R_T)^+) = L\big(\Phi(-\vartheta(\mu,\sigma)) - e^{\mu T}\Phi(-\vartheta(\mu,\sigma) - \sigma\sqrt{T})\big)$ and $\mathbb{E}((R_T - L)^+) = L\big(\Phi(\vartheta(\mu,\sigma)) - e^{\mu T}\Phi(\vartheta(\mu,\sigma) + \sigma\sqrt{T})\big)$, and use the relation $\Delta(R_T, L) = \mathbb{E}((R_T - L)^+) - \mathbb{E}((L - R_T)^+)$. □

The Investment Risk as Functions of the Volatility and the Return

To understand how the investment risk $\rho(R_T, L)$ behaves with respect to volatility and return parameters (σ and μ), Figure 2 shows how $\rho(R_T, L)$ varies with σ and μ: with σ ranging from 5% to 90% in steps of 1%, and $\mu \in \{0.05, 0.07, 0.09, 0.11, 0.13\}$.

As expected, we see that

- ◇ for a fixed level of return, the investment risk is a nondecreasing function of the volatility: the greater the volatility of the investment asset, the high is the investment risk;
- ◇ for a fixed level of volatility, the investment risk is a non-increasing function of the return: the greater the yield, the smaller it is.

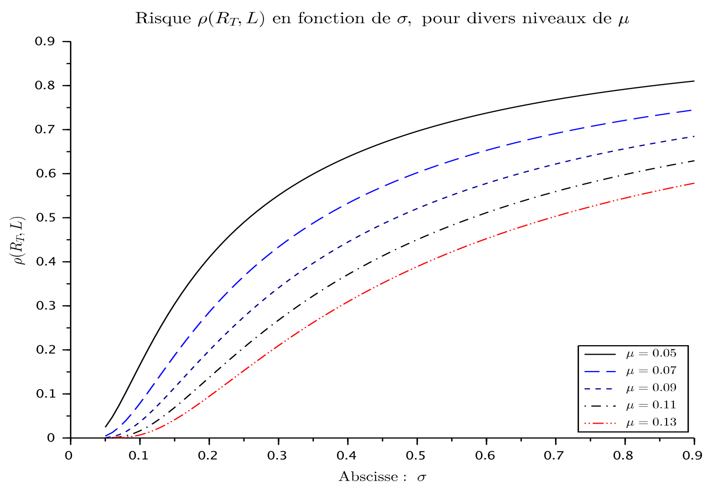

Figure 2: *Black-Scholes model*: $\rho(R_T, L)$ as a function of the volatility σ, for different values of μ.

Practical Use of the Previous Results

Consider a bank that aims to invest in a specific sector of the economy (fintechs, for example) through a *mudarabah* contract with a maturity period of T. Suppose the bank wants to capture two-fifths of the sector's development potential. The following are some steps the bank can take to achieve this:

(i) Assume that we have access to a dataset for the investment sector and that we can model the dynamics of the investment asset R up to maturity T.

(ii) Ensure that the investment is viable by checking (from the model) that $\Delta(R_T, L) > 0$ and compute (or approximate) $\rho(R_T, L)$.

(iii) Since we want to capture two-fifths of the $\Delta(R_T, L)$, we may invest the amount L in a large number of independent projects (say, more than 30, for example, bearing in mind the law of large number). Now, if for example,

$$\rho(R_T, L) = 1/4, \text{ we negotiate a } \gamma_1 = 55\% \text{ share of the profits from each contract.}$$
$$\rho(R_T, L) = 1/2, \text{ we negotiate a } \gamma_1 = 70\% \text{ share of the profits from each contract.}$$

6 Conclusion

In this work, we introduce a new concept of a c-fair profit-sharing ratio, which is based on Islamic investment contracts such as *musharakah* or *mudarabah*, and can be combined with a *wakalah* contract. The c-fair concept allocates profits based on each partner's contribution to the overall success of the undertaking.

We demonstrate that the profit-sharing ratio is proportional to investment risk and opportunity, weighted by capital and labour contributions, respectively.

To calculate the quantities of interest, we must model the dynamics of the investment asset. The feasibility of this step depends on the nature of the available dataset and must be thoroughly investigated for any given dataset. Of the possible models used in finance and econometrics, we considered the well-known Black-Scholes model, for which all the quantities of interest can be computed explicitly. This model also enables us to observe how investment risk is expected to behave: it increases with volatility and decreases with return.

Finally,this work sheds new light on these Islamic contracts and on the engineering and industry of Islamic finance. It may also help Islamic finance institutions value Islamic investment contracts and related financial instruments such as Sukuk instruments based on these contracts, as well as manage *Takaful* (Islamic insurance) via a *wakalah* or *wakalah-moudharabah* models.

Acknowledgement

The author would like to thank Stephen Koffler for his relevant and useful review of the manuscript.

Bibliography

[1] AAOIFI: *The Accounting and Auditing Organization for Islamic Financial Institutions*, Sharia'ah Standards, 2015.

[2] I. Abdeljawad, M. Rashid, T. Mari, and D. Salameh, *Challenges and solutions for Mudarabah financing: an exploratory study of Islamic banks in Palestine*, Journal of Banking Regulation, 26, 958-971, 2025.

[3] H. Askari, I. Zamir, and A. Mirakhor, *Introduction to Islamic Economics - Theory and Application*, Wiley Finance series, Singapore, 2015.

[4] M. Ayub, *Understanding Islamic Finance*, Wiley, 2007.

[5] O. I. Bacha, *Adapting Mudarabah (profit sharing) Financing to Contemporary Realities: A Proposed Financing Structure*, Munich Personal RePEc Archieve, 1997.

[6] T. Bengarai, *Comprendre la Finance Islamique: Principes, Pratiques et Ethique*, Les deux Orients, 2012.

[7] G. Causse-Broquet, *La finance islamique*, La Revue Banque, 2012.

[8] T. Demirtaş, *Evaluation of Musharakah Financing, Standards, Country Cases, Major Obstacles and Solutions - A Study of Islamic Finance Partnership Methods*, Modern Economy, 17(1), 1-13, 2026.

[9] F. Guéranger, *Finance islamique: une illustration de la finance éthique*, Dunod, 2009.

[10] Z. Iqbal, and A. Mirakhor, *Introduction to Islamic Economics - Theory and Practice*, Wiley, 2011.

[11] S. Kunhibava, et al, *Islamic Social Finance: law and Practice in Malaysia*, Springer, 2024.

[12] D. LESSY, *Modèles mathématiques pour le microcrédit et la finance islamique*, PHD THESIS, UNIVERSITÉ CÔTE D'AZUR, France, 2021.

[13] N. M. SAPUAN, *An Evolution of Mudarabah Contract: A Viewpoint From Classical and Contemporary Scholars*, EM PROCEDIA ECONOMICS AND FINANCE, 35, 249-358. 2016.

How to Cite: Abass Sagna[1], *Fair Profit Sharing Ratios of Islamic Investment Contracts*, Journal of Mathematics and Modeling in Finance (JMMF), Vol. 6, No. 2, Pages:211–240, (2026).