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Credit Risk Management Of Portfolio And Distance To Default Estimation Based On Firm Equity Approach

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Abstract:
In this paper, the credit risk of investment is managed by a new structural mean-reverting model. For this purpose, the utility maximization problem is expressed as an optimization problem with an expected logarithmic objective function and a mean-reverting constraint. The existence of a solution to the expressed problem is proved in a lemma. Moreover, one theorem for facilitating the optimal strategy design algorithm is proved. In this way, the optimal strategy is obtained for hedging the credit risk of investing in stocks of companies. Finally, the distance to default of investment is calculated by using the Black-Scholes formula for call options on wealth values obtained by optimal portfolio selection based on the proposed the optimal strategy. Also, the stability of the proposed method is proved in the second theorem. For demonstrating the applicable results, the portfolio allocation problem is simulated by using the proposed method and compared with the non-mean-reverting equity approach. According to the simulation result, the portfolio values don't fall and default doesn't occur during the investment period [0, T], approximately. Also, distance to default tends to zero.

Keywords: Credit risk, Mean-reverting model, Utility maximization.
Classification: 35F21; 35R60; 91G10.

1 Introduction

The continuous-time portfolio problem is allocation of wealth between stocks and a risk-free asset so as to maximize expected utility [1]. The problem is usually formulated as a stochastic optimization problem for logarithmic utility function to risk aversion and obtain local maximums, uniquely [2]. The log utility function exhibits decreasing risk aversion because the log utility function is strictly increasing, differentiable and strictly concave [3].

Risk aversion means investing with lower returns but with more security. An agent invests less due to risk aversion. If conditions such as an increase in the risk of bankruptcy or default of the company issuing the shares cause fear in investing, the risk goes to the highest and the agent invest less in that asset. Therefore, the portfolio problem must be formulated to obtain an optimal strategy for reducing risk aversion and this is guaranteed by log utility function.

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Active management of portfolio is a common strategy to reduce risk in financial markets where individual stocks are selected analyzing market trends, company fundamentals, and valuation metrics, actively [4].

One of the valuation metrics is credit risk modeling of the firm whose stock is to be selected. As Banks and financial institutions have a challenge in assessing a borrower's creditworthiness, we have a challenge in presenting the portfolio problem formulation with credit risk of the firm whose stock is to be selected [4].

The credit risk of the firm is the risk of loss due to the firm non-payment of a loan or other line of credit. When the firm is unable to pay debt, default occurs. Credit risk is associated with the risk of default of a firm [5].

The asset value must fall to lead to default. In default situation, the difference between the firm value and liabilities is negative. According to accounting equation, equity is negative [6].

Credit risk modeling helps to calculate how much capital the firm need to set aside to protect against risks such as non-payment of a loan, not paying an employee's earns wages and generally not meeting its obligations [5].

Two main classes of credit risk models are structural and intensity models models. Structural models are used to calculate the probability of default for a firm based on the value of its assets and liabilities. Intensity models are characterized by their hazard rate [5].

In this paper, a new structural model is present and used in two purposed. First, based on a new structural approach, an optimal strategy is obtained to select the number of stocks, optimally. Because if the firm value and debt value are calculated precisely then equity can be calculated. If the firm equity value is negative then stocks of firm must be in short position. In the proposed structural model, the dynamics of the firm equity value is expressed by mean-reverting stochastic volatility model. This model is used in the fields of quantitative finance and financial engineering to evaluate derivative securities, such as options and swaps [7]. Consequently, the optimal portfolio is obtained based on this strategy subject to firm equity be maximum. Therefore, credit risk portfolio is managed based on firm equity approach.

Second, the distance to default of investing on the company is calculated by using the Black-Scholes formula based on the structural model of credit risk. For the first time, Merton used the Black Scholes model to find the distance to default of the firms [8]. In Merton's model, the firm's equity is simply a European call option on the firm value with maturity T and the firm's debt value is consider as strike price. In [9], the actual effect on distance default and probability of default is analyzed. In this paper, the distance to default of investment is calculated by using the Black-Scholes formula for call options on wealth values obtained by optimal portfolio selection based on the proposed the optimal strategy.

2 Credit Risk Portfolio Management Based on Firm Equity

2.1 Equity and Wealth Dynamics

In this section dynamics of Equity and Wealth Dynamics is presented.

Firm Value Dynamics

Consider a portfolio problem where an investor can put her/his wealth into the firm value and a money market account. So, according to Black-Scholes model, the firm value at time t follows a geometric Brownian motion. Indeed, for Brownian motion, (W_t) , defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t>0}, P)$, the dynamics of the firm value and a money market account are the following SDE, respectively:

$$dV_t = V_t\left(\mu dt + \sigma dW_t\right), \tag{1}$$

$$V(0) = V_0, \tag{2}$$

$$dM_t = M_t r dt, \tag{3}$$

$$M(0) = M_0, \tag{4}$$

(M_t) is the value of the money market. The variable (V_t) denotes the firm value which has a constant drift μ and a constant volatility σ. The filtration $\{\mathcal{F}_t\}_{t>0}$ is the usual natural filtration of (W_t) . In this paper, process (W_t) is one dimensional. The interest rate is denoted by the constant r.

Equity Dynamics

The debt of the company is modeled by a zero-coupon bond with maturity $T$and debt value $L > 0$. In my proposed structural model, the company defaults if, at time T, when the firm value is smaller than its debt. Beside, according to the accounting equation, the equity value of the firm at time t (denoted by E_t) is the difference between the firm value and the debt value:

$$E_t = \log(V_t) - \log(L_t) \tag{5}$$

where L_t stand for liability of company at time t. If $V_t < L_t$, this equity is negative (i.e., default occurs when the asset value falls). Then L_t/V_t is the firm debt ratio. The acceptable firm debt ratio is smaller than 0.5, usually. This acceptable firm debt ratio can be considered as an equilibrium level. Therefore, if the variable equity value of the firm, E_t is modeled by a mean-reverting stochastic volatility model [7] which reverts to the equilibrium level, then the firm debt ratio is smaller than 0.5 and the credit risk is hedged by optimal selecting strategy so as to maximize expected logarithmic utility, consequently. The general mean-reverting stochastic volatility model can be expressed by the following equation:

$$dX_t = (a - bX_t)dt + \sigma X_t^{\theta} dB_t, \tag{6}$$

where X_t presents the price or the variance of the price returns of a stock, a,b are positive constants, σ is the volatility rate, θ is a non-negative constant, B_t is a Brownian motion defined on a complete filtration probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t>0}, P)$. The variance of the price returns of stock and the option price is expressed by equation (6) [10]. Indeed, for different value of θ, equation (6) has been applied for different purposes. For example, when $\theta = 0$, it reduces to the mean-reverting Ornstein-Uhlenbeck model, Hwang [11] discussed the limit distribution of its solution as time t goes to infinity. According to explanation above, in this paper the equity value is modeled by the following stochastic volatility equation:

$$dE_t = (\bar{E} - E_t)dt + \sigma_E dB_t, \qquad E(0) = \ln\left(\frac{V_0}{L}\right) \tag{7}$$

where E_t is the equilibrium level or threshold of default that is set 0.7 in implementing the proposed method. Also, σ_E is the volatility of the equity value. If we set

$$E_t^* = E_t - \bar{E}$$

We see that E_t^* is the unique solution of the stochastic differential equation

$$dE_t^* = -E_t^* dt + \sigma_E dB_t, \qquad E^*(0) = \frac{V_0}{L} - \bar{E}, \tag{8}$$

which means that (E_t^*) is the Ornstein-Uhlenbeck process that can be written explicitly as follows:

$$E_t^* = e^{-t}\Big(E_0^* + \sigma_E \int_0^t e^s dB_s\Big) \tag{9}$$

Wealth Dynamics

Actually, the equity value of firm is a non-traded asset. However, the elasticity approach tells us that we can solve the portfolio problem with the two investment opportunities E_t and M and then compute how the optimal wealth process can be achieved by investing in the tradable assets, i.e. in the stock issued by the company and money market account. As a consequence, we look at the following wealth equation:

$$\frac{dW_t}{W_t} = \alpha dE_t + (1-\alpha)\frac{dM}{M}, \qquad W(0) = W_0, \tag{10}$$

where W_t denotes the investor's total wealth and α stands for the percentage of the total wealth put into the firm value. By using equations (4) and (7), equation (10) can be rewritten as follows:

$$\begin{aligned}\frac{dW_t}{W_t} &= \alpha((\bar{E} - E_t)dt + \sigma_E dB_t) + (1-\alpha)rdt, \\ &= (\alpha(\bar{E} - r) - \alpha E_t + r)dt + \alpha\sigma_E dB_t. \end{aligned} \tag{11}$$

The Gaussian process (W_t) is the solution of Black-Scholes SDE(11). It is written as:

$$W_t = W_0 \exp\Big(\int_0^T \Big(\mu - \frac{\alpha^2\sigma_E^2}{2}\Big)dt + \int_0^T \alpha\sigma_E dB_t\Big) \tag{12}$$

where

$$\mu = \alpha(\bar{E} - r) - \alpha E_t + r \tag{13}$$

2.2 Optimal Portfolio Based on Firm Equity Approach

In this section optimal strategy based on firm equity approach is obtained and simulated for a given firm with determined parameters.

Optimal Strategy

In this section, under assumption mentioned above, the optimal portfolio problem is formulated as follows:

$$\max_{\alpha\in A(0,W_0)} E(\ln(W^{\alpha_t}(T))) \tag{14}$$

subject to

$$\begin{aligned} &W_t = W_0 \exp\Big(\int_0^T \Big(\mu - \frac{\alpha^2\sigma_E^2}{2}\Big)dt + \int_0^T \alpha\sigma_E dB_t\Big) \\ &\mu = \alpha(\bar{E} - r) - \alpha E_t + r \end{aligned}$$

where $A(0, W_0)$ denotes the set of all admissible strategy given the initial condition $(0, W_0)$. The variable T denotes the investment horizon. Also, $U(W) = E[\ln(W)]$ is applied as the utility function and $E(.)$ stands for expectation function.

In the following Lemma, the existence of a solution to the utility maximization problem (14) is proved.

Lemma 2.1. *The utility maximization problem (14) has a solution.*

Proof. The utility function $E[\ln(W)]$ is continuous. Because it is a composition function of two continuous functions $\ln(.)$ and expectation function $E(.)$ also, the process (W_t) is Gaussian with continuous paths. The mean-reverting Ornstein-Uhlenbeck model (7) can be expressed in risk-neutral frame; because according to Girsanov theorem [5], $d\tilde{B} = B_t + qt$, for constant q , is a standard Brownian motion. Thus, there is an equivalent risk–neutral probability. According to first fundamental theorem [12], there is no arbitrage. Consequently, the utility maximization problem (14) has a solution. □

Theorem 2.2. *consider the portfolio problem (13) and the firm equity value (7). Then the optimal attainable portfolio is as follows:*

$$\alpha^*(t) = \frac{(\bar{E} - E_t) - r}{\sigma_E^2} \tag{15}$$

Proof. The explicit solution to the wealth equation (11) is given by (12). By using (13) and (14), the equation (12) can be written as:

$$\begin{aligned}
\ln(W_T) &= \ln(W_0) + \ln\Big(exp\Big(\textstyle\int_0^T \Big(\mu - \dfrac{\alpha^2\sigma_E^2}{2}\Big)dt + \textstyle\int_0^T \alpha\sigma_E dB_t\Big)\Big) \\
&= \ln(W_0) + \textstyle\int_0^T \Big(\mu - \dfrac{\alpha^2\sigma_E^2}{2}\Big)dt + \textstyle\int_0^T \alpha\sigma_E dB_t \\
&= \ln(W_0) + \textstyle\int_0^T \Big(\alpha(\bar{E} - r) - \alpha E_t + r - \dfrac{\alpha^2\sigma_E^2}{2}\Big)dt + \textstyle\int_0^T \alpha\sigma_E dB_t \\
&= \ln(W_0) + rT + \textstyle\int_0^T \Big(\alpha(\bar{E} - E_t) - \alpha_t r - \dfrac{\alpha^2\sigma_E^2}{2}\Big)dt + \textstyle\int_0^T \alpha\sigma_E dB_t
\end{aligned}$$

therefore, the investor's expected terminal utility reaches as

$$E[\ln(W_T)] = \ln(W_0) + rT + E\Big[\int_0^T (\alpha(\bar{E} - E_t - r) - \frac{\alpha^2\sigma_E^2}{2})dt\Big]. \tag{16}$$

Note that due to the boundedness of α_t the expectation of the Ito integral vanishes. The equation 16 can be written as:

$$E[\ln(W_T)] = \ln(W_0) + rT + \frac{1}{2}E\Big[\Big(\frac{\lambda}{\sigma_E}\Big)^2\Big] - \frac{1}{2}E\Big[\int_0^T \Big(\alpha_t\sigma_E - \frac{\lambda}{\sigma_E}\Big)^2 dt\Big] \tag{17}$$

where $\lambda = \bar{E} - E_t - r$. The utility reaches a maximum when $\int_0^T \Big(\alpha_t\sigma_E - \dfrac{\lambda}{\sigma_E}\Big)^2 dt$ is minimum. Since, integrand is square thus if

$$\Big(\alpha_t\sigma_E - \frac{\lambda}{\sigma_E}\Big)^2 = 0.$$

Then utility reaches a maximum. Therefore, $\alpha^*(t) = \dfrac{(\bar{E} - E_t) - r}{\sigma_E^2}$. □

Remark 2.3. According to (15), the proposed optimal strategy based on firm equity approach relays on the equilibrium level of equity $\bar{E}$, interest rate r and volatility of the firm equity σ_E. From equation (7), $\bar{E} - E_t$ is the instant return of equity. So if instant return of equity is bigger than interest rate $((\bar{E} - E_t) > r)$, we can put α_t percentage of the total wealth into the firm value. Otherwise, we sell the stock of the firm and put the total wealth into money market account.

3 The Distance to Default Estimation Based on Firm Equity Approach

In this section, a model based on firm equity approach is applied for estimating the distance to default of investing on a firm with liability L and maturity T. The proposed model estimates distance to default of the firm by applying the Merton's model with the firm equity value at any time t to maturity T.

3.1 Estimating the Distance to Default by BSM

First, the role of BSM for European Call option in Merton for credit risk is investigated. Since, the firm's equity value is just the firm value minus the debt value, $E = \log(V) - \log(L)$ and its total face value of debt is the "strike" that must be paid to retire debt and own the firm's assets. Therefore, the firm's equity is simply European call option on the firm value V with maturity T and strike price L as follows:

$$C(V,T) = V * N(d_1) - L * e^{-T} N(d_2), \tag{18}$$

where,

$$d_1 = \frac{\ln(\frac{V}{L}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \tag{19}$$

$$d_2 = \frac{\ln(\frac{V}{L}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \tag{20}$$

r is risk free rate and σ is asset volatility. In (20), if $V < L$ then $\ln\left(\frac{V}{L}\right) < 0$ and default occurs. Thus $\ln\left(\frac{V}{L}\right)$ is compounded return on the assets that is necessary to lead to default. Also, the term $(r - \frac{\sigma^2}{2})$ is the expected value of the compounded return (usually positive). Therefore, the numerator of d_2 is the "surprise", or unexpected component of the rate of return necessary for realizing default. The denominator is the standard deviation of the rate of return. Therefore, the ratio d_2 (again typically negative) measures the number of standard deviations of return necessary to lead to default at time T. The negative of ratio (that is a positive number) is called the distance-to default. The higher volatility and the longer maturity, the smaller distance to default [6]. Now, BSM for European Call option on the wealth values obtained by optimal portfolio selection based on the proposed the optimal strategy at time t (denoted by W_t) is considered where initial capital W_0 stands for the "strike" price. Therefore, the difference: $R_t = \log(W_t) - \log(W_0)$ is the compounded return on the portfolio. Thus, distance to default at any time t is:

$$d_2 = \frac{R_t + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \tag{21}$$

where (W_t) is the wealth value of the portfolio at time t that is obtained by the optimal strategy proposed in section 2.

4 Stability

Theorem 4.1. *Consider the portfolio wealth modeled by SDE (10) where optimal strategy is* $\alpha_t^* = \dfrac{(\bar{E} - E_t) - r}{\sigma_E^2}$. *Then distance to default of the portfolio,* d_{2_t}, *(21) is bounded.*

Proof. In the formula of the distance to default of the portfolio, d_{2_t}, (21), the difference determining the propagation of the error is, R_t. Therefore the optimal proposed method will be stable when the norm of R_t is bounded. Thus,

$$\begin{aligned} \|d_2\| &= \|\frac{R_t + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\| \\ &\leq \|R_t\| + \|(r - \frac{\sigma^2}{2})T\| \end{aligned} \tag{22}$$

where $\|.\|$ denotes the norm of the space $\mathcal{L}^2$. According to equation (11)-(17) and by substituting the optimal strategy $\alpha_t^* = \dfrac{(\bar{E} - E_t) - r}{\sigma_E^2}$, we have:

$$\|R_t\| = rT + \frac{1}{2}E\Big[\Big(\frac{(\bar{E} - E_t) - r}{\sigma_E}\Big)^2\Big] \tag{23}$$

where the equity process, E_t is a mean- reverting process, $\lim_{t\to\infty} E_t = \bar{E}$. Thus we have

$$\|R_t\| = rT + \frac{r^2}{2\sigma^2}$$

Therefore,

$$\|d_2\| \leq \frac{r^2 + 4rT\sigma^2 + T\sigma^4}{2\sigma^2}$$

□

5 Simulation

5.1 Simulation of the Optimal Strategy

In this section, the stochastic optimal strategy based on the proposed mean-reverting equity approach for a given firm is simulated and compared with the non mean-reverting equity approach. In the non mean-reverting equity approach, is E_t modeled by the following stochastic differential equation

$$dE_t = E_t(\mu dt + \sigma dB_t),$$

where, μ is the instant return of E_t. All parameters of this simulation are chosen same as [6] so as to compare the simulation results. Thus, the equilibrium level, $\bar{E}$ is set 0.7, drift μ is set 0.05 and a constant volatility σ is set 0.2. Also, initial firm value $V_0 = 130\$$ and firm debt value ($L = 100\$$). The Brownian motion (B_t) is standard normal random processes.

For obtaining the equity value paths (E_t) for the known firm, the above simulations are implemented for one trading year (51 days). The results are shown in the Figures 1-4.

Now, according to strategy rule (15), the optimal strategy hedging the credit risk of portfolio based on firm equity approach can be computed via the equity value of firm with the corresponding parameter values. The simulation result is shown in the Figure 1.

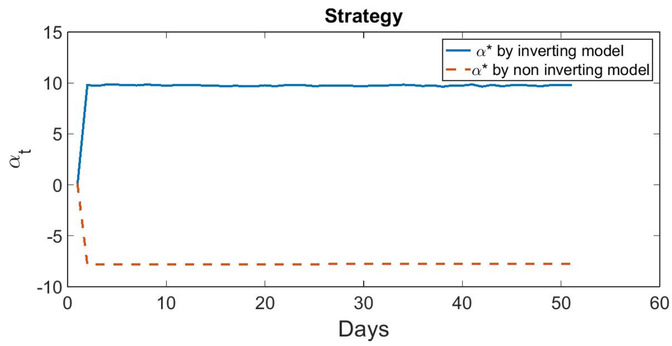

Figure 1: The optimal strategy hedging the credit risk of portfolio based on mean-reverting equity approach (Solid line) and strategy by non mean-reverting equity (Dashed line)

As it is observed from the Figure 1, strategy of optimal portfolio selection is obtained by the proposed method, quickly. Also, the strategy signal converges to an equilibrium level, asymptotically. While, the strategy values obtained by non mean-reverting are negative and don't converge. Now, according to the above simulated data the mean-reverting and non mean-reverting equity values simulation is illustrated in the Figure 2.

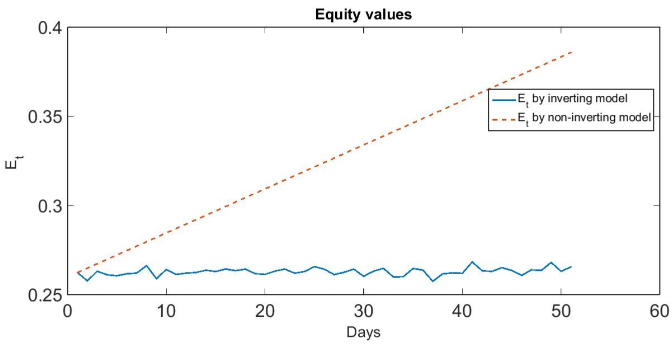

Figure 2: The mean-reverting equity path (Solid line) and non mean-reverting equity path (Dashed line).

Figure 2 shows that the values of equity modeled by the proposed mean-inverting near to an equilibrium level. But the values of non mean inverting equity deviate and increase. That is not acceptable in emprical investment.

Also, according to the above simulated data the wealth values simulation is illustrated in the Figure 3.

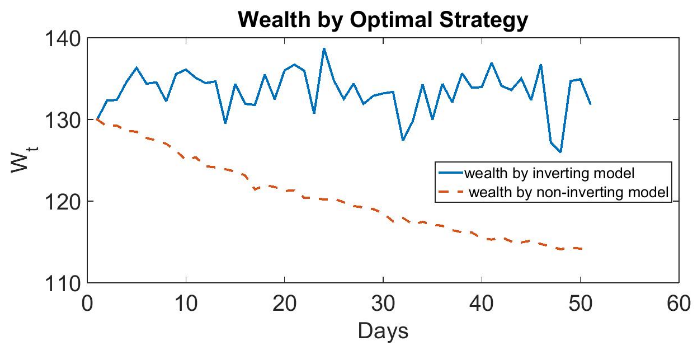

Figure 3: Wealth by proposed strategy based on mean-inverting model (Solid lin) and strategy based on non mean-inverting model (Dashed line).

Figure 3 shows that, the values of the portfolio obtained by strategy based on mean-inverting model remain above its initial value of 130, at any time approximately. Once the equity value of the firm is smaller than 130, the portfolio value come back to above 130 by implementing the proposed optimal strategy. While the values of the portfolio obtained by strategy based on non mean-inverting model fall.

5.2 Simulation of the Distance to Default Values

In this section, the distance to default process is estimated and simulated based on mean-inverting model and non mean-inverting model with the same parameters mentioned in the simulation 5.1. The simulation result is shown in the Figure 4.

Figure 4 shows that, by implementing the proposed optimal strategy based on the proposed mean-inverting equity approach, the value of distance to default tends to zero, approximately. But, the values of distance to default obtained by non mean-inverting equity approach increase exponentially.

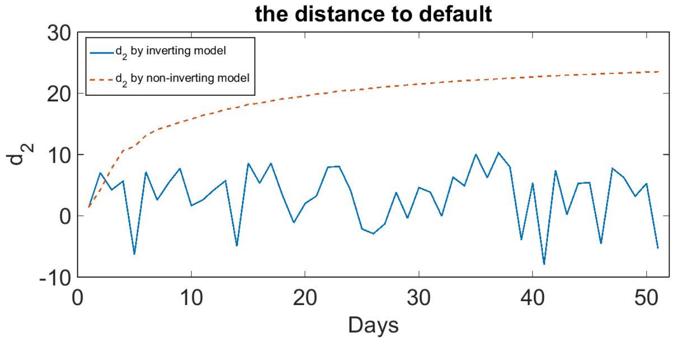

Figure 4: The distance to default of investment by optimal strategy based on mean-inverting equity approach (Solid line) and non mean-inverting equity approach (Dashed line).

6 Conclusion

In this paper, a structural mean-reverting model was proposed to manage the credit risk of investment and estimate the distance to default of investing. To this end, the expected logarithmic mean-reverting constrained portfolio allocation problem was expressed. The existence of a solution to the expressed problem was proved, by a lemma (2.1). The optimal strategy was obtained in theorem (2.2). Also, the stability of the proposed optimal method is proved in theorem (4.1). For demonstrating the applicable results of the proposed method, in section 5, the stochastic optimal strategy based on the proposed mean reverting equity approach for a given firm is simulated and compared with the non-mean-reverting equity approach. According to the simulation result, the portfolio values did not fall and default does not occur by implementing the proposed optimal strategy, approximately. Also, distance to default of the investment based on the proposed optimal strategy, tends to zero.

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