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American Call Option Pricing with Poisson Jumps in Itô-Liu Financial Markets

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Abstract:
In practice, uncertainty and randomness are two common, basic forms of indeterminacy. These two forms of indeterminacy are logically modelled by two distinct mathematical systems. Randomness is modelled by probability theory and uncertainty is modelled by uncertainty theory. However, in some cases, uncertainty and randomness concurrently surface in a multifaceted mathematical system. This study develops an American call option pricing model with random jumps for Itô-Liu financial markets. Uncertain stochastic differential equations (USDEs) incorporating Poisson jumps are employed to model the underlying asset dynamics. These jump-diffiusion USDEs are driven by a Brownian motion, a canonical Liu process, and a Poisson jump process. To validate the proposed American call option pricing model, a numerical example is given. The findings of the study show that the developed model is capable of pricing American call options in Itô-Liu financial markets. This study adds a voice to the discourse of option pricing in Itô-Liu financial markets.

Keywords: Poisson jumps, American option, Uncertain stochastic differential equation, Stock model, Itô-Liu markets.
AMS 2000 subject Classification: C63; G12.

1 Introduction

Decisions, especially financial decisions, are commonly made in the state of indeterminacy [37]. They are rarely made with perfect foresight. Indeterminacy is a phenomena whose state cannot be foretold with precision in advance [36, 44]. Matenda et al. [39] described indeterminacy as the condition of being not predictable beforehand as far as the outcomes of events are concerned. Thus, indeterminacy is an intrinsic component of different processes and phenomena that are noticeable in nature and society.

Uncertainty and randomness are two basic, common forms of indeterminacy [37]. In practice, these two forms of indeterminacy are logically modelled by two different mathematical systems [36]. Randomness is modelled by probability theory [28]

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and uncertainty is modelled by uncertainty theory [31]. Probability theory is an axiom-premised branch of pure mathematics concerned with modelling the behaviour of dynamic random phenomena. Uncertainty theory is an axiom-based branch of pure mathematics devoted to modelling human uncertainty. Matenda and Chikodza [40] postulated that randomness is characterised as a property of phenomena whose behavior can be captured and explained through probabilistic principles. In the same vein, [36] highlighted that randomness denotes any phenomenon which can be measured by a probability measure. Probability theory is used in the representation of aleatory indeterminacy which rests on the unpredictability of outcomes of a random experiment. On the other hand, uncertainty is characterised as a property of phenomena whose nature can be described by degrees of belief [40]. Under the same line of reasoning, [36] opined that uncertainty denoted any phenomenon which can be measured by an uncertain measure. Uncertainty theory represents the epistemic form of indeterminacy which is driven by the insufficiency of accurate indisputable knowledge concerning a given reality. In sum, randomness lies at the core of probability theory and its applications, while uncertainty lies at the core of uncertainty theory and its applications.

Probability theory is applied when the sample size is sufficiently large to estimate the probability distribution from historical frequency data. In contrast, when the sample size is too small or entirely absent to support such estimation, uncertainty theory is employed instead. In this case, domain experts are consulted to assess their belief degrees regarding the likelihood of each event occurring [34]. However, human beings tend to overestimate unlikely events, which means that belief degrees may exhibit greater variance than actual observed frequencies. Importantly, belief degrees are distinct from subjective probability, and applying probability theory under such circumstances yields counter-intuitive results. The application of probability theory in finance theory gave rise to stochastic finance theory, while uncertainty theory serves as the foundational framework upon which uncertain finance theory is built.

To capture the dynamics of random phenomena that evolve over time, stochastic processes were introduced within the framework of probability theory. In general terms, a stochastic process is a series of random variables indexed by time or space. One of the most prominent examples of a stochastic process is Brownian motion, first observed by Robert Brown in 1827. A satisfactory mathematical explanation of this phenomenon was later provided by Einstein in 1905. Building on this foundation, Norbert Wiener further developed the concept in 1923 by constructing a rigorous mathematical framework known as the Wiener process. Consequently, Brownian motion is alternatively referred to as a Wiener process or a Wiener-Einstein process.

Brownian motion was first introduced into the field of finance by [3], who proposed that the behaviour of stock prices could be described by this process [10, 40]. Basically, [3] is one of the first researchers to bring the notion of aleatory indeterminacy to the subject area of mathematical finance by applying a stochastic process (i.e., Brownian motion) in financial modelling. The author introduced an option pricing formula premised on the belief that stock prices are powered by a Brownian motion with zero drift [41]. Samuelson [48], Black and Scholes [5], and Merton [41] further indicated that stock prices follow a geometric Brownian motion. With the establishment of the Chicago Options Exchange in 1973, options trading gained increasing prominence and became progressively more widespread [25]. Black and Scholes [5] developed the widely acclaimed Black-Scholes model (BSM), which has since become an indispensable cornerstone of stochastic finance theory. Since the publication of these foundational works, myriad researchers have contributed to the theory of option pricing in stochastic finance [2, 11, 38].

To account for the dynamics of uncertain phenomena that evolve over time, [32] introduced the concept of an uncertain process within the framework of uncertainty theory. An uncertain process is a series of uncertain variables indexed by time or space. Two prominent examples of uncertain processes are the renewal process, developed by [32], and the canonical Liu process, constructed by [33]. The canonical Liu process is a counterpart to the Brownian motion. Uncertain differential equations were first introduced into the field of finance by [33], who indicated that a stock price can be described by an uncertain differential equation. In addition, [33] developed the first uncertain stock model, i.e. Liu's stock model, and designed its European option pricing formulae.

Uncertainty and randomness can simultaneously occur in a mathematical system [34]. In this case, both uncertainty and randomness are supposed to be taken into account. Matenda and Chikodza [40] postulated that probability theory and uncertainty theory complement each other. Liu [34] introduced chance theory to deal with mathematical systems exhibiting both randomness and uncertainty. Chance theory is an axiom-based branch of pure mathematics concerned with modelling the behaviour of dynamic uncertain random phenomena.

In chance theory, [18] proposed an uncertain random process to model the dynamics of uncertain random phenomena that change with time. An uncertain random process is a series of uncertain random variables indexed by space or time. Interestingly, an uncertain random process is a generalisation of both an uncertain process and a stochastic process [18]. Furthermore, in chance theory, [18] introduced an uncertain random renewal process and a stationary increment uncertain random process.

The notion of uncertain random processes has emerged as a crucial aspect of modern finance theory, especially in the pricing and risk management of financial options. Financial options are usually classified into three types: American options, European options, and Asian options. An American option permits the holders of the contract to exercise their right to sell or buy an underlying asset at any given time before the expiration date of the option, while a European option can only be exercised at the expiration date of the option. The payoffs of European and American options are contingent on the price of the underlying asset at a specific point in time. Nonetheless, Asian options are a kind of exotic options. Like European options, Asian options are exercised on the expiration date. An Asian option payoff is contingent on the average price of the underlying asset over a certain period of time. The application of probability theory in option pricing led to the emergence of stochastic option pricing theory [21, 49], in which stochastic differential equations driven by Brownian motion serve as the foundational mathematical framework. Conversely, the application of uncertainty theory in option pricing theory gave birth to uncertain option pricing theory [25, 50], wherein uncertain differential equations driven by a canonical Liu process serve as the foundational mathematical framework. Furthermore, the application of chance theory in option pricing led to the emergence of uncertain stochastic option pricing theory [12, 13], wherein uncertain stochastic differential equations (USDEs) serve as the foundational mathematical framework. USDEs are driven by uncertain random processes, i.e., they are driven by both a Brownian motion and a canonical Liu process.

In practice, asset prices display time-varying jumps during extreme events. These jumps carry significant economic implications for financial market participants, particularly with respect to portfolio choice [25]. Merton [16] argued that market information arrivals could precipitate either marginal price movements, representing normal fluctuations in stock prices, or supramarginal price movements, which manifest as abnormal changes, commonly referred to as jumps. Mathematical models for pricing options, such as the BSM, that assume that the price of the underlying asset follows a continuous process fail to account for the sudden jumps in the underlying asset's price that can occur in real markets as a result of substantial economic events and unexpected news announcements, among other factors. For example, in Q3 of 2019, Tesla's shares spiked more than 20% after it has announced a surprise profit and stated that it was ahead of schedule with a new factory in Shanghai. Also, in 2016, when Microsoft declared that it was acquiring LinkedIn, LinkedIn's stock price jumped by 47%. To account for such sudden price jumps, jump-diffusion models, such as the Merton jump-diffusion model, were developed to incorporate both the continuous part of the asset returns and discontinuous jump dynamics. These models offer a more precise pricing framework under particular market conditions.

Many jump-diffusion models are based distinctively on probability theory and uncertainty theory, even though in reality probability theory and uncertainty theory complement each other. For instance, [16] jump-diffusion model assumed that the returns of the underlying stock are produced by a combination of both a continuous process (i.e., Wiener process) and a jump process (i.e., a compound Poisson process). Ji and Zhou [23] suggested an uncertain stock model associated with both negative and positive jumps using uncertain differential equation with jumps. This uncertain stock model is powered by a combination of both a continuous process (i.e., a canonical Liu process) and a jump process (i.e., an uncertain renewal process). In the extant literature, a limited number of studies have priced options in uncertain stochastic financial markets (also known as Itô-Liu financial markets) in the presence of jump dynamics [12, 13]. Uncertain stochastic financial markets are financial markets in which asset processes are driven by USDEs. Hence, pricing options in uncertain stochastic financial markets in the presence of jumps is still a problem that needs to be investigated. Interestingly, existing studies have priced Asian and European options in uncertain stochastic financial markets, predominantly assuming the existence of uncertain jumps. To the best of the authors' knowledge, no study has yet examined the pricing of American options within this framework. Furthermore, no prior study has considered the possibility that such jumps may be random rather than certain.

This study develops an American call option pricing model with random jumps for Itô-Liu financial markets. USDEs incorporating Poisson jumps are employed to model the underlying asset dynamics. These jump-diffusion USDEs are driven by a Brownian motion, a canonical Liu process, and a Poisson jump process. To validate the proposed American call option pricing model, numerical examples are presented. The findings of the study show that the developed model is capable of pricing American call options in Itô-Liu financial markets. Our work contributes to the discourse on jump dynamics in option pricing in Itô-Liu financial markets. To the best of our knowledge, no such study has been conducted before.

The remainder of this paper is structured as follows. Section 2 presents the literature review. Section 3 outlines the study methodology. Section 4 presents the findings and discussion and Section 5 examines managerial implications. Conclusions are given in Section 6.

2 Literature Review

The canonical Liu process and the geometric Brownian motion have independently been applied to pricing options by many researchers [5, 7–9, 15, 16, 20, 29, 33, 41, 52, 57]. Black and Scholes [5] assumed that the geometric Brownian motion has a continuous sample path. While acknowledging the landmark contribution of [5] to option pricing, [41] noted that their proposed formula remains independent of investor preferences and beliefs regarding expected returns. Merton [16] opined that information pertaining to supply and demand imbalances and fluctuations in capitalisation rates gives rise to marginal changes in stock value. In this case, the stock price process is appropriately modelled by a standard geometric Brownian motion with a continuous sample path. However, [16] postulated that important information whose arrival time is discrete may also be experienced. This information has the effect of pushing the value of a stock to a position that exceeds marginal levels. Stock prices that exceed marginal levels are considered to be extreme values. The stock price process with extreme values experiences jumps at discrete time points. In modelling such processes, [16] incorporated a jump process that caters for extreme values.

The research work of [16] was extended together with the BSM by [7] to the case where two types of jumps are experienced, i.e., extreme upwards and downwards jumps. Camara and Heston [7] proffered that stock price return distributions are not log-normally distributed, as assumed by [5], and further noted that option prices exhibit excess kurtosis relative to the normal distribution. Further, [7] noted that the BSM fails to address problems due to volatility smiles, a finding consistent with [29]. Moreover, their empirical results closely approximated observed market prices, and the model was subsequently adopted by [27], whose findings demonstrated its superior performance relative to the BSM. Kou [29] proposed a jump-diffusion model incorporating a double exponential shift in option prices, which successfully captured the effects of skewness and the volatility smile in option pricing. Other researchers who looked at the effects of skewness and the volatility smile are [1, 14, 45, 47].

Poisson jumps have been widely adopted in financial modelling to account for jumping stochastic phenomena. When a system is simultaneously influenced by both continuous and discrete noise processes, Brownian motion and Poisson jumps can be jointly employed to simulate the resulting disturbances. Of late, the concept of Poisson jumps has been applied in the field of finance and has made many achievements [4, 6, 19, 26, 30, 43, 46, 54, 56]. Yu [54] examined a catastrophe option pricing model with double jump processes, wherein the stock price process of an insurance company issuing catastrophe options was described through an exponential jump-diffusion process, with all jump terms modelled using two compound Poisson processes. Further, explicit analytical formulas for the option price were derived and Monte Carlo simulation was used to validate the model. Rihan et al. [46] investigated the existence of solutions and optimal control results of fractional stochastic differential equations with a Hilfer fractional derivative under Poisson jumps. Liang et al. [30] analysed the optimal reinsurance and investment problem with jump-diffusion risky assets, assuming an insurance risk model driven by a compound Poisson process in which the jumps in the risky asset and insurance risk process were correlated through a common shock. Closed-form expressions for both the optimal strategy and the value function were derived under the expected value principle and the variance premium principle.

In 2020, [6] designed novel mispricing of markets under asymmetric information and jump dynamics for both informed and uninformed investors, named them m-double Poisson markets, powered by independent double Poisson processes. Further, the mean, variance, skewness, and kurtosis of instantaneous returns were stated implementing jump amplitudes and frequencies. Josa-Fombellida and López-Casado [26] examined the optimal management of an aggregated pension fund of defined benefit type using a differential game with two players, the firm and the participants. Assuming that fund wealth exceeds the actuarial liability, the manager builds a pension surplus. To account for sudden shifts in financial market conditions, this surplus is invested in a portfolio comprising a risk-free bond and multiple risky assets, with uncertainty modelled through both Brownian motions and Poisson processes. Zhao et al. [56] applied the contraction mapping principle to establish the approximate controllability and the existence of optimal control for a multi-delay stochastic fractional differential equation system, where the associated control function incorporates Poisson jumps, across a specified range of the Hurst parameter.

The above discussion indicates that Poisson jumps are foundational to stochastic finance since they model the discontinuous and extreme events that continuous Brownian motion fails to capture. Incorporating jumps corrects deficiencies in classical models like BSM, making option pricing much more accurate. Hlahla et al. [22] opined that models that integrate jump processes, such as Poisson processes, more accurately reflect market realities and enhance effectiveness of portfolio construction and hedging strategies. Consequently, jump modelling is indispensable for sound and robust financial decision-making.

In uncertain markets, the notion of option pricing was examined by several authors including [9, 10, 24, 25, 33, 58]. Liu [33] proposed the so-called Liu's stock model for pricing options. In this model the stock price process is driven by a geometric Liu process, which has proven indispensable in modern finance, especially within the theory of option pricing. Building on this stock model, [33] developed the European put and call option pricing formulas, while [9] derived American put and call option pricing formulas. Both sets of formulas were validated through numerical examples. Subsequently, [10] extended this work by pricing Asian call and put options.

Interestingly, the concept of jump processes was also studied under uncertainty theory by numerous researchers including [23, 52, 53, 55]. Yu [52] proposed a stock model with jumps for uncertain markets, thus laying the foundation of option pricing in uncertain markets with jumps. Yao [51] examined uncertain calculus with the renewal process. Yu [53] derived European put and call options pricing formulas with jumps where the jump component is a renewal process. Ji and Zhou [23] introduced an uncertain stock model incorporating both positive and negative jumps, and derived the corresponding European option pricing formulas for this model. Yu and Ning [55] suggested an interest-rate model with jumps for uncertain financial markets.

In uncertain stochastic financial markets, the literature concerning the pricing of options remains at its early stages of development. Using USDEs with uncertain jumps, [12] analysed the dynamics of a stock price process and designed a model to price American call options. Building on the concepts of probability, uncertainty, and USDEs with jumps, [13] applied an uncertain random stock model with a jump to derive Asian option pricing models in uncertain random markets characterized by jumps. The reviewed literature reveals that uncertain stochastic option pricing has predominantly been examined in the presence of uncertain jumps. However, in practice, such jumps may also manifest as random rather than purely uncertain phenomena. Furthermore, the reviewed studies focus largely on Asian and European options, leaving American options comparatively under explored. To address these gaps, the present study assumes the existence of random jumps and examines the pricing of American options within uncertain stochastic financial markets.

3 Methodology

This section examines some preliminary definitions and the Itô-Liu stock model with Poisson jumps.

3.1 Preliminaries

This subsection summarizes some important definitions and theorems which are essential in this study. The concepts of Brownian motion, Itô integral, Poisson process, compound Poisson process, uncertain process, Liu integral, and Merton's jump-diffusion model are introduced.

Definition 3.1. [37] To provide a firm mathematical foundation, we assume that $(\Omega, \mathcal{F}, P)$ is a probability space and T is a totally ordered set. A stochastic process is a function $X_t(\omega)$ from $T \times (\Omega, \mathcal{F}, P)$ to the set of real numbers such that $\{X_t(\omega) \in B\}$ is an event for any Borel set B of real numbers at each time t.

Definition 3.2. [36] A stochastic process W_t is said to be a standard Brownian motion if

(i) $W_0 = 0$ and almost all sample paths are continuous,

(ii) W_t has stationary and independent increments,

(iii) every increment $(W_{s+t} - W_s) \sim N(0, t)$.

An integral concerning a stochastic process is termed a stochastic integral. The Itô integral is a special type of a stochastic integral.

Definition 3.3. [36] Let X_t be a stochastic process and let W_t be a standard Brownian motion. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1<i\leq k} |t_{i+1} - t_i|.$$

The Itô integral of X_t with respect to W_t is given by

$$\int_a^b X_t \, dW_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}(W_{t_{i+1}} - W_{t_i}),$$

if the limit exists in the mean square and is a random variable.

Another special stochastic process is the Poisson process which plays an important role in modelling stock prices in the presence of jumps.

Definition 3.4. [36] A Poisson process with rate $\lambda > 0$ is a counting process $\{N^p(t) : t \geq 0\}$ which satisfies the following five properties:

(i) $N^p(0) = 0$,

(ii) $\forall s, t > 0$, $N^p(t+s) - N^p(t)$ has independent increments,

(iii) $\forall s, t > 0$, $N^p(t+s) - N^p(t)$ has stationary increments,

(iv) $P(N^p(t+h) - N^p(t) = 1) = \lambda h + o(h)$ (unit jumps for time interval h), and

(v) $P(N^p(t+h) - N^p(t) \geq 2) = o(h)$.

Definition 3.5. [16] A compound Poisson process with rate λ and distribution G is a continuous time stochastic process $\{D(t) : t \geq 0\}$ given by:

$$D(t) = \sum_{i=1}^{N^p(t)} Y_i,$$

where $\{Y_i, i \geq 1\}$ are independent and identically distributed random variables with distribution G. The Y_i's are independent of $\{N^p(t) : t \geq 0\}$.

Definition 3.6. [35] An uncertain random variable is a function ξ from a probability space $(\Omega, \mathcal{F}, P)$ to the set of uncertain variables such that $\mathcal{M}\{\xi(\omega) \in B\}$ is a measurable function of ω for any Borel set B of $\mathbb{R}$.

Definition 3.7. [35] For a measurable function ψ, if $F(x)$ is the cumulative distribution function of a random variable η, τ is an uncertain variable and $\psi(x, \tau)$ has an uncertainty distribution $\Psi(x, y)$, then the chance distribution of $\psi(\eta, \tau)$ is given by

$$\Phi(y) = \int_{-\infty}^{\infty} \Psi(x, y)\, dF(x),$$

where x and y are the realizations of η and τ, respectively.

Definition 3.8. [33] An uncertain process C_t is said to be a Liu process if

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,

(ii) C_t has stationary and independent increments, and

(iii) every increment $(C_t - C_s) \sim N(0, t^2)$.

The uncertainty distribution of C_t is given by

$$\Phi(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}\, t}\right)\right)^{-1}, \qquad x \in \mathbb{R}.$$

An integral concerning an uncertain process is called an uncertain integral. The Liu integral is a special uncertain integral that plays a fundamental role in uncertain calculus.

Definition 3.9. [36] Let X_t be an uncertain process and let C_t be a Liu process. For any partition of closed interval $[a, b]$ with

$$a = t_1 < t_2 < \cdots < t_{k+1} = b,$$

the mesh is written as

$$\Delta = \max_{1<i\leq k} |t_{i+1} - t_i|.$$

Then the Liu integral of X_t with respect to C_t is defined as

$$\int_a^b X_t\, dC_t = \lim_{\Delta\to 0} \sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i}),$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process X_t is said to be integrable.

Definition 3.10. [17] Let $X_t = (Z_t, Y_t)^{\mathsf{T}}$ be an uncertain stochastic process. For any partition

$$P = \{a = t_1, t_2, \ldots, t_{k+1} = b\}$$

of the closed interval $[a, b]$, with

$$a = t_1 < t_2 < \cdots < t_{k+1} = b,$$

the mesh is written as

$$\Delta = \max_{1<i\leq k} |t_{i+1} - t_i|.$$

Then the Itô-Liu integral of X_t with respect to an Itô-Liu process $G_t = (W_t, C_t)$ is defined as follows:

$$\int_a^b X_s\, dG_s = \lim_{\Delta\to 0} \sum_{i=1}^{N} Z_{t_i}(W_{t_{i+1}} - W_{t_i}) + \lim_{\Delta\to 0} \sum_{i=1}^{N} Y_{t_i}(C_{t_{i+1}} - C_{t_i}),$$

given that the first limit is a random variable that exists in mean square and the second limit is a finite uncertain variable which exists almost surely. In this case, X_t is called Itô-Liu integrable. In particular, when $Y_t \equiv 0$, X_t is called Liu integrable.

3.2 Stock Price Models

Consider the Black-Scholes stochastic differential equation (SDE) which assumes that the dynamics of the underlying stock is driven by a geometric Brownian motion which has a continuous sample path. If the price of a stock is S_t at time t, then S_t satisfies the SDE

$$dS_t = eS_t\, dt + \sigma_1 S_t\, dW_t,$$

with solution

$$S_t = S_0 \exp\left(\left(e - \frac{\sigma_1^2}{2}\right) t + \sigma_1 W_t\right),$$

where e, σ_1 are constants and W_t is a one-dimensional Brownian motion. The stock model in this context can be expressed as

$$\begin{cases} dX_t &= rX_t\,dt \\ dS_t &= eS_t\,dt + \sigma_1 S_t\,dW_t, \end{cases} \tag{1}$$

where X_t is the bond price and r is the riskless interest rate. In the context of uncertainty theory, we assume that the dynamics of the underlying stock is driven by a geometric Liu process. If the price of a stock is S_t at time t, then S_t satisfies an uncertain differential equation (UDE)

$$dS_t = eS_t\,dt + \sigma_2 S_t\,dC_t,$$

with solution

$$S_t = S_0 \exp(et + \sigma_2 C_t)\,,$$

where e, σ_2 are constants and C_t is a Liu canonical process. The Liu stock model is given by

$$\begin{cases} dX_t &= rX_t\,dt \\ dS_t &= eS_t\,dt + \sigma_2 S_t\,dC_t. \end{cases} \tag{2}$$

Merton's extension to the Black-Scholes equation is of the form

$$dS_t = (e - \lambda k)S_t\,dt + \sigma_1 S_t\,dW_t + (y_t - 1)S_t\,dN^p(t).$$

The model differs from the Black-Scholes formula due to an additional compound Poisson process

$$D(t) = \sum_{i=1}^{N^p(t)} Y_i,$$

with two sources of randomness. The source of random noise is $dN^p(t)$ which is a Poisson process with mean number of jumps per unit of time λ and the size of the price jump Y_t. Note that in 1976, Merton [16] used the geometric Brownian motion together with a jump coefficient $N^p(t)$ in pricing options. In this case the stock model can be expressed as

$$\begin{cases} dX_t &= rX_t\,dt \\ dS_t &= (e - \lambda k)S_t\,dt + \sigma_1 S_t\,dW_t + (y_t - 1)S_t\,dN^p(t), \end{cases} \tag{3}$$

where N^p is a counting process, e is the instantaneous expected return on the asset, σ_1 is the instantaneous volatility of the asset return conditional on no occurrence of the jumps and k is the expected value of the relative price jump size of S_t, i.e.,

$$k = \mathbb{E}_\eta[y_t - 1] = \exp\left(\mu + \frac{\upsilon^2}{2}\right) - 1,$$

for the absolute price jump size y_t, drift log-jump μ and log jump standard deviation υ.

However, in this study we assume that the stock price process is driven by an Itô-Liu process mentioned in Definition 3.10 and Poisson jumps as seen in the Merton [16] model. To adequately develop option-pricing models in uncertain stochastic markets with jumps, we start by defining an uncertain stochastic differential equation with Poisson jumps. The Itô integral is extended to the case of uncertain random processes and a chain rule for solving USDES is also laid down. A new option pricing model in the context of an uncertain stochastic differential equation with Poisson jumps is sought and used to price American call options. The option pricing formula is developed using the solution to an uncertain stochastic differential equation with Poisson jumps. Numerical examples are given where results from the proposed model are compared to those obtained using Merton's 1973 jump-diffusion model.

4 Findings and Discussion

In this section, we focus on the important findings of this work. We propose an uncertain stochastic differential equation with Poisson jumps and an uncertain stochastic stock model with Poisson jumps and use these concepts in pricing American call options.

4.1 Uncertain Stochastic Differential Equation with Poisson Jumps

The focus of this section is on the development of the chain rule for solving USDES with Poisson jumps. If Poisson jumps are introduced in an uncertain stochastic differential equation, the method for solving this DE has to be improved to cater for the Poisson jump component.

Definition 4.1. Suppose W_t is a standard Brownian motion, C_t is a Liu process,

$$J_t = \sum_{i=1}^{N^p(t)} \Delta X_i$$

is a compound Poisson process and X_t is an uncertain random diffusion process with Poisson jumps. If an uncertain stochastic process

$$X_t = X_0 + \int_0^t \upsilon_s \, ds + \int_0^t \eta_s \, dC_s + \int_0^t \mu_s \, dW_s + \sum_{i=1}^{N^p(t)} \Delta X_i \tag{4}$$

exists for the drift υ_t, uncertain diffusion η_t and stochastic diffusion μ_t given that $t \geq 0$, then X_t has an uncertain stochastic differential equation (USDE) with Poisson jumps

$$dX_t = \upsilon_t \, dt + \eta_t \, dC_t + \mu_t \, dW_t + J_t \, dN^p(t). \tag{5}$$

The process X_t is a differentiable uncertain random diffusion process with drift υ_t, uncertain diffusion η_t, stochastic diffusion μ_t and jump coefficient J_t.

Theorem 4.2 (Itô Formula for USDES with Poisson Jumps)**.** *Let W_t be a standard Brownian motion and C_t be a Liu process. For any C^2 function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, the uncertain random process $Y_t = f(t, X_t)$ satisfies the following uncertain stochastic differential:*

$$\begin{aligned} df(t, X_t) = {} & \frac{\partial f}{\partial t}\, dt + \upsilon_t \frac{\partial f}{\partial X}\, dt + \eta_t \frac{\partial f}{\partial X}\, dC_t + \mu_t \frac{\partial f}{\partial X}\, dW_t \\ & + \frac{\mu_t^2}{2} \frac{\partial^2 f}{\partial X^2}\, dt + \big(f(X_t + J_t) - f(X_t)\big)\, dN^p(t). \end{aligned} \tag{6}$$

All derivatives of f are evaluated just before any possible jump in the interval $(t, t+dt]$. Theorem 4.2 extends the Itô formula for solving USDES proposed by Fei [17] to the case of USDES with Poisson jumps.

Proof. Suppose $f(t, x)$ is a C^2 function of t and x. Also, let

$$d(t, X_t) = \upsilon_t\, dt + \eta_t\, dC_t + \mu_t\, dW_t + J_t\, dN^p(t),$$

as given in Equation (5) which is equivalent to the uncertain integral equation in Equation (4). If the diffusion part of $d(t, X_t)$ is

$$dZ_t = \upsilon_t\, dt + \eta_t\, dC_t + \mu_t\, dW_t,$$

then $d(t, X_t)$ can be expressed as

$$d(t, X_t) = dZ_t + J_t\, dN^p(t).$$

This means,

$$\begin{aligned} df(t, X_t) &= f(X_{t+dt}) - f(X_t) \\ &= \begin{cases} f(X_t + dZ_t) - f(X_t), & \text{with probability of no jump } 1 - \lambda\, dt \\ f(X_t + J_t + dZ_t) - f(X_t), & \text{with probability of one jump } \lambda\, dt. \end{cases} \end{aligned}$$

The function $f(X_t + dZ_t) - f(X_t)$ can be expressed as

$$\frac{\partial f}{\partial X}\, dZ_t + \frac{1}{2} \frac{\partial^2 f}{\partial X^2} (dZ_t)^2$$

and $f(X_t + J_t + dZ_t) - f(X_t)$ as

$$f(X_t + J_t + dZ_t) - f(X_t + J_t) + f(X_t + J_t) - f(X_t),$$

thus

$$\begin{aligned} df(t, X_t) = {} & \frac{\partial f}{\partial X}\, dZ_t + \frac{1}{2} \frac{\partial^2 f}{\partial X^2} (dZ_t)^2 \\ & + \big(f(X_t + J_t + dZ_t) - f(X_t + J_t)\big)\, dN^p(t) \\ & + \big(f(X_t + J_t) - f(X_t)\big)\, dN^p(t), \end{aligned}$$

but

$$dW_t\,dN^p(t) = dC_t\,dN^p(t) = dt\,dN^p(t) = dC_t\,dW_t = dC_t\,dC_t$$
$$= dC_t\,dt = dW_t\,dt = 0$$

and also, the quantities $dZ_t\,dN^p(t)$ and $(dZ_t)^2\,dN^p(t)$ are also equal to 0. This leads us to the expression

$$df(t,X_t) = \frac{\partial f}{\partial X}\,dZ_t + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}(dZ_t)^2 + \big(f(X_t+J_t) - f(X_t)\big)\,dN^p(t).$$

Since $(dZ_t)^2 = \mu_t^2\,dt$, we have

$$\begin{aligned} df(X_t) &= \frac{\partial f}{\partial X}(\upsilon_t\,dt + \eta_t\,dC_t + \mu_t\,dW_t) + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}(\mu_t^2\,dt) \\ &\quad + \big(f(X_t+J_t) - f(X_t)\big)\,dN^p(t) \\ &= \upsilon_t\frac{\partial f}{\partial X}\,dt + \eta_t\frac{\partial f}{\partial X}\,dC_t + \mu_t\frac{\partial f}{\partial X}\,dW_t + \frac{\mu_t^2}{2}\frac{\partial^2 f}{\partial X^2}\,dt \\ &\quad + \big(f(X_t+J_t) - f(X_t)\big)\,dN^p(t). \end{aligned}$$

This leads to

$$\begin{aligned} df(t,X_t) &= \frac{\partial f}{\partial t}\,dt + \upsilon_t\frac{\partial f}{\partial X}\,dt + \eta_t\frac{\partial f}{\partial X}\,dC_t + \mu_t\frac{\partial f}{\partial X}\,dW_t \\ &\quad + \frac{\mu_t^2}{2}\frac{\partial^2 f}{\partial X^2}\,dt + \big(f(X_t+J_t) - f(X_t)\big)\,dN^p(t), \end{aligned}$$

which is identical to Equation (6). □

4.2 Uncertain Stochastic Stock Model with Poisson Jumps

In this section, we propose an uncertain stochastic stock model with Poisson jumps and use it to price American call options. The stock price process evolves as an uncertain stochastic differential equation with Poisson jumps. This form of DE captures the presence of uncertainty and randomness and enables us to adequately capture stock price fluctuations.

Let S_t be the stock price, e be the expected return on the asset, r be the riskless interest rate, σ_1 be the stochastic volatility, σ_2 be the uncertain volatility, λ be the intensity of the Poisson process, $y_t - 1$ be the relative jump price size of S_t, $k = \mathbb{E}_\eta[y_t - 1]$, C_t be a one-dimensional Liu process and W_t a one-dimensional Brownian motion.

Definition 4.3 (Itô-Liu Process with Poisson Jumps)**.** Let $(\Gamma,\mathcal{L},\mathcal{M})$ be an uncertainty space and $(\Omega,\mathcal{F},P)$ be a probability space. The product space $(\Gamma\times\Omega,\mathcal{L}\otimes\mathcal{F},\mathcal{M}\times P)$ is called a chance space. The stock price process S_t is governed by the Itô-Liu process with Poisson jumps defined on this chance space as follows:

$$dS_t = (e-\lambda k)S_t\,dt + \sigma_2 S_t\,dC_t + \sigma_1 S_t\,dW_t + (y_t - 1)S_t\,dN^p(t), \tag{7}$$

where W_t is a Brownian motion representing stochastic randomness, C_t is a Liu process representing epistemic uncertainty, and $N^p(t)$ is a Poisson process with intensity λ representing random jump arrivals. The processes W_t, C_t, and $N^p(t)$ are mutually independent.

Note. Existence and uniqueness of the solution to (7) follow from [12] for the USDE framework, with the Poisson jump component satisfying the standard integrability and Lipschitz conditions required for stochastic differential equations (SDEs) with jumps.

If the stock price follows a geometric Itô-Liu process with Poisson jumps, then we can define a stock model where the bond price X_t and stock price S_t are determined by

$$\begin{cases} dX_t &= rX_t\,dt \\ dS_t &= (e-\lambda k)S_t\,dt + \sigma_2 S_t\,dC_t + \sigma_1 S_t\,dW_t + (y_t - 1)S_t\,dN^p(t). \end{cases} \tag{8}$$

The bond price is given by

$$X_t = X_0 \exp(rt).$$

If there is no jump, i.e., $dN^p(t) = 0$ in the price of the asset for the time interval dt, then the jump diffusion process is given by

$$dS_t = (e-\lambda k)S_t\,dt + \sigma_2 S_t\,dC_t + \sigma_1 S_t\,dW_t$$

and if there is a jump in the interval dt, i.e., $dN^p(t) = 1$, then the jump diffusion is given by

$$dS_t = (e-\lambda k)S_t\,dt + \sigma_2 S_t\,dC_t + \sigma_1 S_t\,dW_t + (y_t - 1).$$

The stock price is a solution to Equation (7) and can be expressed as

$$S_t = S_0 \exp\left(\left(e - \frac{\sigma_1^2}{2} - \lambda k\right)t + \sigma_1 W_t + \sigma_2 C_t + \sum_{i=1}^{N^p(t)} Y_i\right). \tag{9}$$

Applying Theorem 4.2 to Equation (7), $d\ln S_t$ can be expressed as

$$\begin{aligned} d\ln S_t &= (e-\lambda k)S_t \frac{1}{S_t}\,dt - \frac{\sigma_1^2 S_t^2}{2S_t^2}\,dt + \sigma_2 S_t \frac{1}{S_t}\,dC_t + \sigma_1 S_t \frac{1}{S_t}\,dW_t \\ &\quad + \left[\ln y_t + \ln S_t - \ln S_t\right] \\ &= (e-\lambda k)\,dt - \frac{\sigma_1^2}{2}\,dt + \sigma_2\,dC_t + \sigma_1\,dW_t + \ln y_t. \end{aligned}$$

Integrating both sides yields

$$\ln S_t - \ln S_0 = \left(e - \frac{\sigma_1^2}{2} - \lambda k\right)t + \sigma_2 C_t + \sigma_1 W_t + \sum_{i=1}^{N^p(t)} \ln y_i$$

and

$$S_t = S_0 \exp\left(\left(e - \frac{\sigma_1^2}{2} - \lambda k\right) t + \sigma_2 C_t + \sigma_1 W_t + \sum_{i=1}^{N^p(t)} \ln y_i\right)$$
$$= S_0 \exp\left(\left(e - \frac{\sigma_1^2}{2} - \lambda k\right) t + \sigma_2 C_t + \sigma_1 W_t\right) \prod_{i=1}^{N^p(t)} Y_i.$$

Let the log price jump size be $\ln y_t = Y_t$, where $\ln y_t$ are independent identically distributed normal random variables with mean μ and standard deviation υ. Then the stock price becomes

$$S_t = S_0 \exp\left(\left(e - \frac{\sigma_1^2}{2} - \lambda k\right) t + \sigma_1 W_t + \sigma_2 C_t + \sum_{i=1}^{N^p(t)} Y_i\right),$$

which is the same as Equation (9).

In the following subsection, we examine American Call Options for a non-dividend-paying stock.

4.3 American Call Options

A type of option that can be exercised on or before its expiration time is called an American option. Its payoff is given by

$$\sup_{0 \le \tau \le T} e^{-rt}\big((S_t - K)^+\big),$$

where T is the time to expiration, S_t is the stock price at time t and K is the strike price. The expected present value of the payoff

$$f_c = \mathbb{E}_{ch}\left[\sup_{0 \le \tau \le T} e^{-rt}\big((S_t - K)^+\big)\right], \tag{10}$$

is called the option price or premium. The stock price S_t in Equation (9) can be expressed as

$$S_t = S_0 \exp\left(\left(-\frac{\sigma_1^2}{2} - \lambda k\right) t + \sigma_1 W_t + \sum_{i=1}^{N^p(t)} Y_i\right) \exp(et + \sigma_2 C_t), \tag{11}$$

which is a function of a stochastic process W_t, a random process with jumps $\sum_{i=1}^{N^p(t)} Y_i$ and an uncertain process C_t. In this case, the stock price is comprised of a random part and an uncertain part. According to Example A.2 in Liu [37], the stock price is an uncertain random variable. Let the random part from Equation (11) be represented by

$$\eta(t) = \exp\left(\left(-\frac{\sigma_1^2}{2} - \lambda k\right) t + \sigma_1 W_t + \sum_{i=1}^{N^p(t)} Y_i\right)$$

and the uncertain part be

$$\tilde{S} = \exp(et + \sigma_2 C_t).$$

This means $\eta(t)$ itself is a random variable. Let us consider a function of an uncertain variable

$$S_t = \eta(t) S_0 \exp(et + \sigma_2 C_t).$$

From Equation (10), the price becomes

$$f_c = \mathbb{E}_\tau \left[\sup_{0 \leq \tau \leq T} e^{-rt} \big((\eta(t) S_0 \exp(et + \sigma_2 C_t) - K)^+ \big) \right]. \tag{12}$$

To evaluate the expected value in Equation (12), we find the uncertainty distribution of

$$\sup_{0 \leq \tau \leq T} e^{-rt} \big((\eta(t) S_0 \exp(et + \sigma_2 C_t) - K)^+ \big).$$

Given $t \in (0, T]$, $\Phi_t(x) = 0$ when $x < 0$. For $x > 0$, the distribution can be expressed as

$$\begin{aligned}
\Phi_t(x) &= \mathcal{M}\Big\{ e^{-rt} \big(\eta(t) S_0 \exp(et + \sigma_2 C_t) - K \big)^+ \leq x \Big\} \\
&= \mathcal{M}\Big\{ \big(\eta(t) S_0 \exp(et + \sigma_2 C_t) - K \big)^+ \leq x e^{rt} \Big\} \\
&= \mathcal{M}\big\{ \eta(t) S_0 \exp(et + \sigma_2 C_t) \leq K + x e^{rt} \big\} \\
&= \mathcal{M}\big\{ \eta(t) S_0 e^{et} \exp(\sigma_2 C_t) \leq K + x e^{rt} \big\} \\
&= \mathcal{M}\left\{ \exp(\sigma_2 C_t) \leq \frac{K + x e^{rt}}{\eta(t) S_0 e^{et}} \right\} \\
&= \mathcal{M}\left\{ C_t \leq \frac{1}{\sigma_2} \ln \frac{K + x e^{rt}}{\eta(t) S_0 e^{et}} \right\} \\
&= \mathcal{M}\left\{ C_t \leq \frac{1}{\sigma_2} \ln \frac{K + x e^{rt}}{\eta(t) S_0} - \frac{et}{\sigma_2} \right\} \\
&= \left(1 + \exp\left(\frac{\pi e}{\sqrt{3}\sigma_2} + \frac{\pi}{\sqrt{3}\sigma_2 t} \ln \frac{\eta(t) S_0}{K + x e^{rt}} \right) \right)^{-1},
\end{aligned}$$

which is an uncertainty distribution of an uncertain process.

In stochastic and uncertain finance, for a non-dividend-paying stock, it is well-established that early exercise of an American call option is never optimal [9, 41]. Consequently, the American call price equals the European call price, and the supremum over exercise times is attained at maturity T. Therefore, we need only consider $t = T$ in the uncertainty distribution.

Since we require the uncertainty distribution of

$$\sup_{0 \leq \tau \leq T} \left(1 + \exp\left(\frac{\pi e}{\sqrt{3}\sigma_2} + \frac{\pi}{\sqrt{3}\sigma_2 t} \ln \frac{\eta(t) S_0}{K + x e^{rt}} \right) \right)^{-1},$$

and the supremum is attained at $t = T$ [9, 41], we have

$$\Psi(x) = \Phi_T(x) = \left(1 + \exp\left(\frac{\pi e}{\sqrt{3}\sigma_2} + \frac{\pi}{\sqrt{3}\sigma_2 T} \ln \frac{\eta(T)S_0}{K + xe^{rT}}\right)\right)^{-1}.$$

Theorem 4.4 (American Call Option Pricing Formula)**.** *The price of an American call option with strike price K which expires on or before the expiration time T for stocks in uncertain stochastic markets with Poisson jumps is given by*

$$f_c = e^{-rT}\rho(T)S_0 \int_{\frac{K}{\rho(T)S_0}}^{+\infty} \left(1 + \exp\left(\frac{\pi(\ln y - eT)}{\sqrt{3}\sigma_2 T}\right)\right)^{-1} dy, \tag{13}$$

where $\rho(T)$ is the realization of the random process $\eta(T)$ at maturity T.

Note. The quantity $\rho(T)$ in Equation (13) is an estimate of the random process $\eta(T)$ defined by

$$\eta(T) = \exp\left(\left(-\frac{\sigma_1^2}{2} - \lambda k\right) T + \sigma_1 W_T + \sum_{i=1}^{N^p(T)} Y_i\right)$$

at the maturity time T. In practice, $\eta(T)$ is estimated from market data via Monte Carlo simulation at the maturity date. Conditional on this estimate, $\eta(T)$ is treated as deterministic for pricing purposes. The expectation over the random component is then obtained by averaging the resulting option prices over multiple simulated paths of $\eta(T)$.

Proof. The expected value of an uncertain variable is given in Theorem 2.23 in Liu [37]. Applying this theorem, we have

$$\begin{aligned} f_c &= \mathbb{E}_\tau\left[\sup_{0\le\tau\le T} e^{-rt}\left((S_t - K)^+\right)\right] = \int_0^{+\infty} \left(1 - \Psi(x)\right) dx \\ &= \int_0^{+\infty} \left(1 - \left(1 + \exp\left(\frac{\pi e}{\sqrt{3}\sigma_2} + \frac{\pi}{\sqrt{3}\sigma_2 T} \ln \frac{\eta(T)S_0}{K + xe^{rT}}\right)\right)^{-1}\right) dx. \end{aligned}$$

Using the notion of inverse uncertainty distribution, we have

$$\Phi^{-1}(\alpha) = \exp\left(eT + \frac{\sqrt{3}\sigma_2 T}{\pi} \ln \frac{\alpha}{1 - \alpha}\right)$$

and

$$\Phi_T(y) = \left(1 + \exp\left(\frac{\pi(eT - \ln y)}{\sqrt{3}\sigma_2 T}\right)\right)^{-1}.$$

Replacing $\eta(T)$ by its realizations $\rho(T)$,

$$\begin{aligned} f_c &= \int_0^{+\infty} \left(1 - \left(1 + \exp\left(\frac{\pi e}{\sqrt{3}\sigma_2} + \frac{\pi}{\sqrt{3}\sigma_2 T} \ln \frac{\rho(T)S_0}{K + xe^{rT}}\right)\right)^{-1}\right) dx \\ &= e^{-rT}\rho(T)S_0 \int_{\frac{K}{\rho(T)S_0}}^{+\infty} \left(1 + \exp\left(\frac{\pi(\ln y - eT)}{\sqrt{3}\sigma_2 T}\right)\right)^{-1} dy, \end{aligned}$$

by letting $y = \dfrac{K + xe^{rT}}{\rho(T)S_0}$.

Alternatively, using Example 14 in Liu [34], if $F(\rho(T))$ is the distribution function of the random variable $\eta(T)$ with realizations $\rho(T)$, then

$$\begin{aligned}\Phi(x) &= \int_{-\infty}^{\infty} \mathcal{M}\Big\{e^{-rT}\big(\rho(T)S_0\exp(eT+\sigma_2 C_T)-K\big)^+ \le x\Big\}\, dF(\rho(T))\\ &= \int_{-\infty}^{\infty}\left(1+\exp\left(\frac{\pi e}{\sqrt{3}\sigma_2}+\frac{\pi}{\sqrt{3}\sigma_2 T}\ln\frac{\rho(T)S_0}{K+xe^{rT}}\right)\right)^{-1} dF(\rho(T)).\end{aligned}$$

Let

$$g(\rho(T)) = \left(1+\exp\left(\frac{\pi e}{\sqrt{3}\sigma_2}+\frac{\pi}{\sqrt{3}\sigma_2 T}\ln\frac{\rho(T)S_0}{K+xe^{rT}}\right)\right)^{-1}.$$

Then

$$\Phi(x) = \int_{-\infty}^{\infty} g(\rho(T))\, dF(\rho(T)) = \int_{-\infty}^{\infty} g(\rho(T))f(\rho(T))\, d\rho(T),$$

where $f(\rho(T))$ is the probability density function of $\rho(T)$. From the definition of the expected value of a function of a random variable,

$$\Phi(x) = \mathbb{E}_\eta[g(\eta)] = \mathbb{E}_\eta\left[\left(1+\exp\left(\frac{\pi e}{\sqrt{3}\sigma_2}+\frac{\pi}{\sqrt{3}\sigma_2 T}\ln\frac{\eta(T)S_0}{K+xe^{rT}}\right)\right)^{-1}\right].$$

Thus, replacing $\eta(T)$ by its realization $\rho(T)$ and effecting a change of variable gives

$$\begin{aligned}f_c &= \int_0^{+\infty}\left(1-\mathbb{E}_\eta\left[\left(1+\exp\left(\frac{\pi e}{\sqrt{3}\sigma_2}+\frac{\pi}{\sqrt{3}\sigma_2 T}\ln\frac{\eta(T)S_0}{K+xe^{rT}}\right)\right)^{-1}\right]\right)dx\\ &= e^{-rT}\rho(T)S_0\int_{\frac{K}{\rho(T)S_0}}^{+\infty}\left(1+\exp\left(\frac{\pi(\ln y - eT)}{\sqrt{3}\sigma_2 T}\right)\right)^{-1} dy,\end{aligned}$$

which is the same as Equation (13). □

In the following subsection, we examine Theorem 4.4 using some numerical examples.

4.4 Numerical Examples

In this subsection, we examine some numerical examples in order to establish how Theorem 4.4 performs in pricing American calls under different scenarios. We check the sensitiveness of numerical option prices to changes in jump intensity, jump size distribution, stochastic stock diffusion, time to maturity and uncertain stock diffusion. This process enables us to assess the robustness of the proposed model. A comparative study using different volatility regimes is carried out in order to strengthen the validation of our results.

Example 4.5. Let the stock drift be $e = 0.06$, the risk-less interest rate $r = 0.08$, and the initial stock price $S_0 = 40$. Our goal is to find American call option prices at $T = 1$ for strike prices ranging from $K = 20$ to $K = 75$ using Theorem 4.4 and Merton's jump-diffusion model for a non-dividend-paying stock under different economic regime specifications given in Table 1. We also carry out sensitivity analysis using the information presented in Table 2.

Table 1: Economic regime specifications for randomness and uncertainty

Regime	**Type**	Parameters				
		σ_1	σ_2	λ	μ	υ
TRANQUIL	Low-Low	0.12	0.10	0.4	−0.05	0.08
KNOWN CRISIS	High-Low	0.55	0.10	1.2	−0.12	0.15
AMBIGUOUS	Low-High	0.12	0.65	0.8	−0.08	0.12
UNPRECEDENTED	High-High	0.55	0.65	2.8	−0.18	0.22
BASE CASE	Moderate	0.29	0.32	1.0	−0.10	0.10

Table 2: Parameter Values for Sensitivity Analysis

Parameters	Test Values				
σ_1	0.10	0.20	0.29	0.45	0.65
σ_2	0.15	0.25	0.32	0.50	0.75
λ	0.2	0.6	1.0	2.5	5.0
μ	−0.30	−0.15	−0.05	0.00	0.15
υ	0.05	0.10	0.20	0.35	0.50
T	0.25	0.5	1.0	1.5	2.0

In this study, we price American options using Theorem 4.4 and Merton's jump-diffusion model. We apply the Monte Carlo simulation approach in estimating $\eta(T)$. We use $N = 50000$ iterations to simulate this random component at maturity. After examining Theorem 4.4 and Merton's jump-diffusion model using Python and the parameters outlined in Example 4.5, we obtain different scenarios of option prices. This is also the same when we use economic regime specifications information in Table 1 and parameter values for sensitivity analysis in Table 2.

As shown in Table 3, Monte Carlo estimates converged rapidly, yielding relative standard errors between 0.06% and 0.34% across all regimes. This high precision confirms that sampling variability does not materially affect our results and that the chosen simulation size is more than adequate. Importantly, the substantial price differentials reaching 13.0850 in the Ambiguous regime reflect true epistemic uncertainty rather than numerical artifacts. These findings validate the Itô-Liu

Table 3: Monte Carlo Convergence Summary for at the money(ATM) call price.

Regime	$\rho(T)$ estimate	% Relative Error	Itô-Liu	Merton	Differences
1. TRANQUIL	0.999234	0.06	3.0144	3.9688	0.9544
2. KNOWN CRISIS	0.997926	0.28	2.9745	10.7526	7.7781
3. AMBIGUOUS	0.999642	0.08	17.6484	4.5634	13.0850
4. UNPRECEDENTED	1.003670	0.34	17.8003	13.5270	4.2733
5. BASE CASE	0.998778	0.15	7.2385	6.7636	0.4749

framework's ability to capture model uncertainty while maintaining computational precision.

Figure 1 shows that in calm markets, the extra uncertainty component from the proposed model in Theorem 4.4 adds little value, so the pure random Merton model prices higher. The two models agree on intrinsic value deep in the money (ITM). The Merton model price rises sharply with increasing stochastic volatility, while the Itô-Liu price remains almost flat. This indicates that the Itô-Liu model is largely insensitive to randomness when epistemic uncertainty is low. The Itô-Liu price explodes as σ_2 increases. The Merton price is unchanged since it lacks a Liu process component. This confirms that σ_2 is the key driver of the Itô-Liu model. The Merton price increases with jump intensity, whereas the Itô-Liu price exhibits a slight decrease or remains flat. This surprising negative sensitivity may be attributed to the $-\lambda k$ drift adjustment in the Itô-Liu framework.

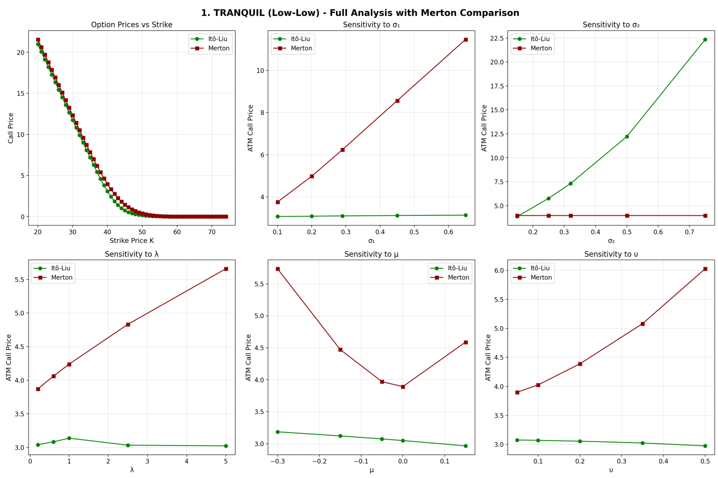

Figure 1: Complete analysis for TRANQUIL: low-low volatility regime

Negative jumps lower both prices, but the Merton model shows greater sensitivity. This suggests that the random component responds more strongly to adverse jump directions. The Merton price rises sharply with jump volatility, while the Itô-Liu price slightly falls. This again demonstrates negative sensitivity to jump risk in the

Tranquil regime.

Figure 2 shows that when only randomness is high but uncertainty is low, the probability-based model captures more risk. The new model appears to undervalue relative to the Merton approach in this known crisis scenario. The negative difference between the models widens as σ_1 increases. The Merton model captures more random risk, leading to systematically higher prices.

For low values of σ_2, the Merton prices exceed Itô-Liu prices. However, when σ_2 exceeds approximately 0.5, the Itô-Liu model surpasses Merton. Across all jump parameters, Merton prices remain consistently higher than Itô-Liu prices. The Itô-Liu prices stay low and relatively flat, indicating that jump risk contributes little to pricing when randomness dominates uncertainty.

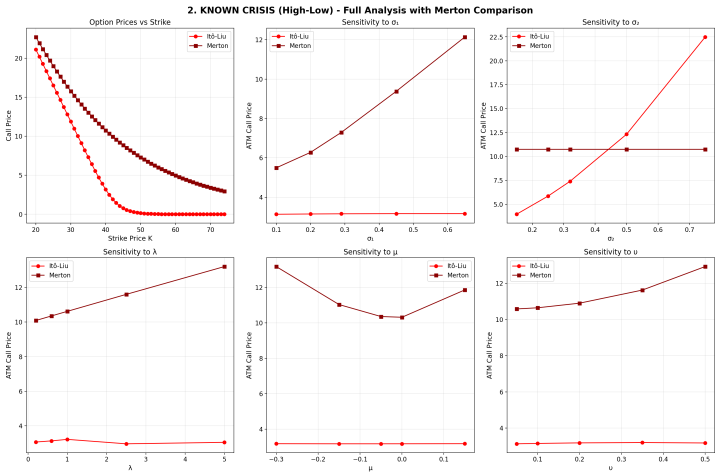

Figure 2: Complete analysis for KNOWN CRISIS: high-low volatility regime

In figure 3, we have a regime where the the Liu process captures epistemic indeterminacy that probability cannot. The high price reflects ambiguity aversion, that is the model prices in worst-case scenarios. The Itô-Liu price remains very high and stable at approximately \$17.8 across all σ_1 values. The Merton price increases with σ_1 but remains far lower. The same explosive pattern observed in previous regimes occurs, confirming that uncertain volatility dominates pricing when epistemic uncertainty is high. The Itô-Liu price remains high across all jump parameter values. This confirms that in the Ambiguous regime, uncertainty drives the price, not jumps or randomness. The epistemic component completely overshadows jump risk and stochastic volatility.

Figure 4 depicts a regime where both randomness and uncertainty are extreme, the new model still prices above Merton, but the gap is smaller than in the Ambiguous regime. This suggests the two forms of indeterminacy partially offset each other in pricing. The difference between models narrows as σ_1 rises because the Merton model catches up due to higher random volatility. The two models converge toward

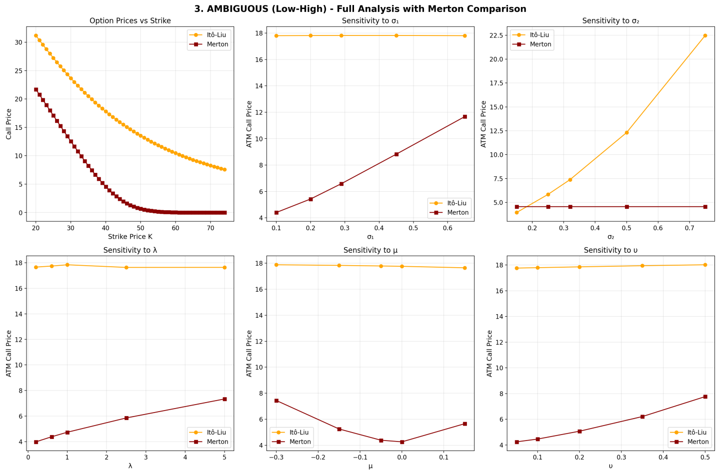

Figure 3: Complete analysis for AMBIGUOUS: low-high uncertainty regime

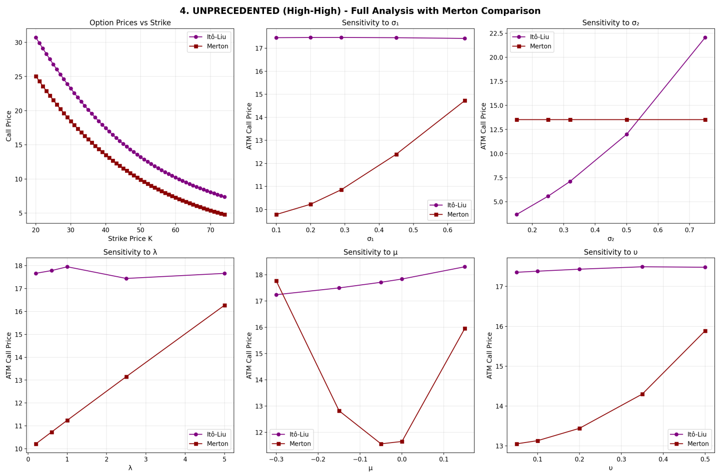

Figure 4: Complete analysis for UNPRECEDENTED: high-high regime

each other under extreme randomness. At low σ_2 values, the Merton model prices exceed the Itô-Liu prices. However, at $\sigma_2 = 0.75$, the Itô-Liu price becomes higher than Merton. At $\lambda = 5.0$, the two models almost converge. Extreme jumps cause randomness to dominate, thereby reducing the uncertainty premium. An interesting crossover occurs at $\mu = -0.30$, where the Merton price actually exceeds the Itô-Liu price. Negative jumps hurt the purely random model more than the mixed uncertainty model.

For the base case in Figure 5, that is in normal market conditions, the new model produces similar prices to Merton, with a small premium for uncertainty. This validates that the model does not introduce arbitrary bias. The Itô-Liu price remains

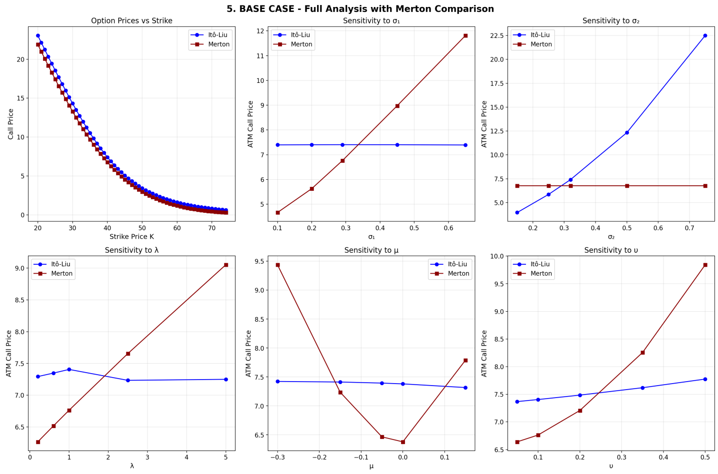

Figure 5: Complete analysis for BASE CASE: regime

flat while the Merton price rises. A crossover occurs near $\sigma_1 = 0.45$, after which Merton becomes higher. The Itô-Liu price rises sharply with increasing uncertain volatility. At $\sigma_2 = 0.75$, the Itô-Liu price is higher than the Merton price. At low jump intensities, Itô-Liu prices are higher and at high jump intensities, Merton prices become higher. The crossover occurs near $\lambda = 2.5$. Differences between the models remain small. The two models agree most closely under normal market conditions, which validates the Itô-Liu framework as a reasonable extension that does not introduce arbitrary bias.

Overall, the results exhibit a clear pattern: the Itô-Liu model dominates when epistemic uncertainty is high, the Merton model dominates when aleatory randomness is high, and the two models converge when both forms of indeterminacy are either both high or both low. However, these patterns are modulated by the jump parameters; for example, increasing jump intensity or jump volatility can shift the balance between the models, particularly in regimes where randomness is already dominant. This confirms that the proposed framework captures the dual sources of indeterminacy without introducing arbitrary bias, while remaining sensitive to the underlying jump dynamics.

5 Managerial Implications

The findings of this study carry significant implications for financial managers, portfolio strategists and risk controllers who operate in Itô-Liu financial markets. The proposed uncertain stochastic option pricing model with Poisson jumps equips decision-makers with a more realistic and comprehensive tool for valuing American options compared to continuous-path models. By integrating both randomness (aleatory risk) and epistemic uncertainty (Liu's indeterminacy), the model allows managers to capture the dual sources of unpredictability inherent in asset prices, thereby improving the accuracy of option valuation and hedging strategies.

For practitioners in investment banking and asset management, the ability to incorporate sudden market jumps triggered by news shocks, mergers, policy changes or earnings surprises provides a superior foundation for pricing and risk assessment. Ignoring jumps can result in systematic mispricing and underestimated risk exposure, potentially leading to significant financial losses. Managers can therefore use this model to structure more effective hedging instruments, design option contracts that better reflect market realities and optimise portfolio allocation in environments characterised by discontinuities.

From a corporate finance perspective, the framework has implications for firms engaging in strategic risk management and capital structure planning. For instance, treasury managers and chief financial officers dealing with convertible bonds, employee stock options or risk-linked financing instruments can benefit from pricing models that recognise both continuous volatility and discontinuous shocks. This improves financial forecasting, enhances market confidence and supports more informed investment, financing and dividend decisions under indeterminacy.

Moreover, the model's closeness in performance to Merton's jump-diffusion model, while also capturing epistemic uncertainty, suggests that it can serve as a robust benchmark for regulators, policymakers and rating agencies when evaluating systemic risk. Financial regulators concerned with stability can encourage adoption of such models to better stress-test institutions' option exposures under scenarios of extreme volatility and uncertainty.

Finally, as markets become increasingly influenced by technological disruptions, geopolitical risks and climate-related events, the managerial value of option pricing frameworks that explicitly incorporate jump dynamics will grow. Adoption of this approach fosters resilience, strengthens strategic adaptability and positions firms to respond proactively to unforeseen shocks. The insights from this study therefore extend beyond academic theory and provide financial managers with practical, world-class tools for decision-making in today's complex market landscape.

6 Conclusion

In this study, we designed a novel American option pricing model in uncertain stochastic financial markets with Poisson jumps. An uncertain stochastic differential equation with Poisson jumps was proposed. The approach assumes that the stock price process is driven by an uncertain random process in the presence of Poisson jumps. Thus, the proposed model is powered by a canonical Liu process, a standard Brownian motion and a compound Poisson process. To demonstrate the effectiveness of the designed model, an illustrative practical example was analysed. The results indicated that the designed uncertain stochastic model with Poisson jumps can precisely describe the price dynamics of American options in Itô-Liu financial markets. The proposed model has the advantage of adequately capturing stock price fluctuations since it incorporates jumps and both forms of indeterminacy, i.e., randomness and uncertainty. Further, we compared the performance of Merton's jump-diffusion model and the designed uncertain random model with Poisson jumps. Compared to Merton's jump-diffusion model, the Itô-Liu model overprices in high-uncertainty, low-randomness environments and under-prices in high-randomness, low-uncertainty environments, with near-parity under normal conditions. This is not surprising since the uncertain stochastic model with Poisson jumps incorporates both forms of indeterminacy. After successfully pricing American options in Itô-Liu financial markets with Poisson jumps, we recommend that researchers price other types of options using this robust framework. In addition, we recommend incorporating features such as multi-asset dependencies or time-varying jump intensities into the proposed framework. We also recommend the use of machine learning to enhance model accuracy.

While the present study assumes a non-dividend-paying stock, a natural extension is to price American call options on dividend-paying stocks. In the presence of dividends, early exercise may become optimal just before ex-dividend dates, as the option holder forfeits the dividend by holding rather than exercising. Extending the proposed uncertain stochastic framework with Poisson jumps to accommodate dividend payments would require a modified optimal stopping problem, where the supremum over exercise times must consider discrete dividend dates. This remains an open problem for future investigation.

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