Journal of Mathematics and Modeling in Finance (JMMF)
Vol. 7, No. 1, Winter & Spring 2027
Research paper

Implementing an Optimized Neural Network to Price American Option under Uncertain Volatility Model with Stochastic Bounds

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Abstract:
The purpose of this paper is to investigate the asymptotic behavior of the price of an American-style put option under an uncertain volatility model that takes into account the risk associated with market volatility. We propose a novel representation for the uncertain volatility model and examine its convergence. An approximation method for the pricing problem is developed by adding conditions to the Black-Scholes-Barenblatt equation and splitting it into two equations. To improve the accuracy of our results, we employ an optimized Neural Network approach that combines finite difference and radial basis function network.

Keywords: American Option, Black-Scholes-Barenblatt, Uncertain Volatility, Stochastic bounds, Volatility Risk, Neural Network, RBF.
MSC2020 Classification: 91G20; 35Q91; 68T07.

1 Introduction

Option is an essential tool for investors to purchase benefits and minimise risks in financial derivatives. One of the key drivers of market-making in options and derivatives is the unpredictability of forward volatility, since it is considered as a critical factor in trading and risk management.

The pricing of American-style contingent claims has been a major area of research for many years, due to their widespread use in financial markets. Unlike European options that have predetermined expiration dates, American options can be exercised at any time before they expire. As a result, determining the optimal exercise strategy or timing to maximize payoff is a difficult task, given the random fluctuations of the underlying asset.

The Black-Scholes model [4] is the most widely used model for pricing options, but it assumes that volatility is constant. This assumption, however, is not able

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Received: 08/02/2026 Accepted: 31/08/2026
https://doi.org/10.22054/jmmf.2026.90314.1261

to accurately explain the observed market prices for options and contradicts price movements. To address this issue, several modifications have been proposed for the Black-Scholes model. Academics have suggested various stochastic volatility models in a series of studies. For instance, the Hull-White stochastic volatility model [12] and the Heston stochastic volatility model [11] have been introduced. Additionally, some scholars have explored option pricing with the jump diffusion model [1] and [15] or the uncertain volatility model [2] and [14]. These models aim to incorporate uncertainty in volatility, which can better explain the observed prices of options in the market.

The uncertain volatility model presents an alternative method for representing non-uniform volatility in financial markets. This approach, originally proposed by Lyons [14] and further developed by Avellaneda et al. [2], characterizes volatility as a range of possible values or an interval. This deviation from traditional models, which assume a fixed level of volatility, acknowledges the inherent unpredictability of financial markets and allows for a more nuanced representation of market behaviour.

As the uncertain volatility model adheres to the no-arbitrage principle, which states that it should not be possible to make risk-free profits by trading in financial markets, prices under this model are no longer unique. Instead, the model predicts the best-case and worst-case scenario prices for a given asset, depending on the level of volatility within the represented interval. This allows for a more accurate representation of potential profits and losses, and can provide valuable insights into market behaviour during times of volatility.

Various issues and outcomes related to the uncertain volatility model have been addressed in the literature. One notable study by Fouque and Ren (2014) focused on European derivatives and determined the worst-case scenario prices under the uncertain volatility model, using wavelet fluctuation intervals[8]. Building on this work, Fouque and Ning (2018) examined uncertain volatility models with stochastic bounds, where the volatility varies between two stochastic bounds produced by a stochastic volatility process, as opposed to using two deterministic bounds[9].

Another study by Zhou and Li (2019) explored the asymptotic behavior of vulnerable option prices in the worst-case scenario under an uncertain volatility model that includes both corporate assets and underlying assets [21]. Moreover, Mezdoud et al. (2021) introduced an uncertain volatility framework and connected the corresponding worst-case pricing problem to HJB-type nonlinear PDEs and second-order backward stochastic differential equations, while also validating the analysis numerically through a deep-learning-based approximation method [17]. These various studies provide important insights into the uncertain volatility model and can aid in developing a more accurate understanding of market behaviour and potential outcomes.

Recent work has also explored neural-network approaches for American option pricing from a free-boundary perspective. For example, Nwankwo et al. (2022) proposed a deep learning framework for American options in which the early exercise boundary is extracted using a Landau transformation and the neural network is trained within an auxiliary-function formulation that enforces the main boundary conditions [18].

The pricing of American-style options under uncertain volatility is a relatively new area of research with many potential approaches. Although several methods have been proposed for pricing American options under uncertain volatility, our literature review indicates that no published papers utilize a neural network approach for American-style option valuation under uncertain volatility with stochastic bounds while considering market volatility risk. This gap motivates the hybrid methodology proposed in this paper. To address this, we propose a framework that employs an Artificial Neural Network to calculate the price of an American put option under uncertainty. Our framework consists of multiple Radial Basis Function Neural Networks, where each network utilizes the Gaussian basis function to learn the price function's difference. To optimize our method, we train the Neural Network on a dataset generated by an advanced financial model.

The primary objective of this research is to analyze the American put option within an uncertain volatility model. The volatility is subject to fluctuations in the market and is dependent on various states and parameters. Our approach involves modeling the stochastic bounds of the volatility using the Heston model. Additionally, we have taken into account the market price of volatility risk, which we assume to be non-zero. While some previous studies have assumed that the market price of volatility risk is zero due to difficulties in estimation [13] and [22], other researches suggests that this assumption may not be valid in various practical scenarios([5]).

Our initial obstacle in determining the estimated American option prices for the worst case scenario was to acquire the Hamilton-Jacobi-Bellman (HJB) equation governing the prices. Within financial mathematics, this particular equation is referred to as the Black-Scholes-Barenblatt (BSB) equation. Using the stochastic control theory, we get the BSB equation. The next issue is to prove the convergence of the price value and the second derivative. After that, we present the main result of this work, which is the approximation of the option price, obtained by controlling the error term. Thus, when analysing the error term, we get its expectation and find the conditions we should impose on the payoff function and derivatives. Then, we adapt the radial basis function neural network combined with finite difference into our proposed model since that the use of neural network, leads to a refinement and sophisticated model which will be a useful tool to study the problem of pricing contingent claims in finance [6]. Finally, we obtain the approximation procedure for the American option prices.

The structure of the rest of this document is as follows. In Section 2, we present the mathematical formulation of the problem and provide an overview of the uncertain volatility model. Following that, Section 3 outlines the BSB equations for option prices, which are broken down into two partial differential equations of

Black-Scholes. We also discuss the convergence of the worst-case scenario price and its second derivative. Moving on to Section 4, we introduce our main result and include an analysis of the convergence of the error term, along with a proof of the main result. Section 5 presents our proposed method for pricing put options under uncertain volatility, which involves utilizing RBF-NN in conjunction with an explicit finite difference method. Lastly, in Section 6, we provide numerical results to demonstrate the performance, accuracy, and robustness of our model. Finally, we conclude our paper with a summary of our findings in Section 7.

2 Setting of the Problem

The objective of this paper is to determine the price of an American put option in the presence of uncertain volatility of the underlying asset. This is because the market price movement is not constant, as presumed in the standard Black-Scholes model. The uncertain volatility model, which assumes that the volatility is bounded between σ_{max} and σ_{min}, is an appealing and commonly used approach.

2.1 Mathematical Formulation

We start by assuming a set of probability spaces $(\Omega, \mathbb{F}, Q)$. In this paper, we consider a market that is incomplete due to uncertainty in volatility. Furthermore, we make the assumptions that the stock S does not pay any dividends and there are no transaction costs involved in trading.
According to the model proposed by Avellaneda et al. (1995)[2], the underlying price process is a member of a set of stochastic processes that satisfy the following stochastic differential equation:

$$dS_t = rS_t dt + \sigma_t S_t dW_t^Q \qquad where \quad \sigma_{min} \leqslant \sigma_t \leqslant \sigma_{max} \quad , \tag{1}$$

where Q is a risk-neutral probability measure, and W_t^Q is a standard Brownian motion under Q. We assume that the Brownian motion which governs the movement of the stock prices is $1-$dimensional, and r is the interest rate. In addition, σ_t is non-anticipative and positive, with σ_{min} and σ_{max} are volatility bounds, which will be considered in the next as a stochastic processes of the underlying price S and time t.

Consider an American put option with the payoff function $g(S) = (K - S)^+$ at the expiry time T . Under an assumed uncertain volatility model 1 with fixed bounds σ_{min} and σ_{max}, there are a pair of option values V_- and V^+ associated with this option, such that $V_- \leqslant V^+$ and:

$$V_-(t, S_t) := \exp^{(-r(T-t))} \operatorname*{ess\,inf}_{\sigma \in [\sigma_{min}, \sigma_{max}]} \mathbb{E}_t(g(S_T)), \tag{2}$$

and

$$V^{+}(t, S_t) := \exp^{(-r(T-t))} \operatorname*{ess\,sup}_{\sigma \in [\sigma_{min}, \sigma_{max}]} \mathbb{E}_t(g(S_T)), \tag{3}$$

where $\mathbb{E}_t[.]$ is the conditional expectation given $\mathbb{F}_t$ with respect to the risk neutral measure, and ess sup (resp ess inf) is the essential supremum (resp essential infimum). Based on the arguments in stochastic control theory, $V(t, S)$ is the viscosity solution to the following HJB equation, which is the generalized BSB nonlinear equation in financial mathematics. According to [2][19], the value $V(t, S_t)$ of an option with an uncertain volatility is given by:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} - rV + \sup_{\sigma \in [\sigma_{min}, \sigma_{max}]} \left[\frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2}\right] = 0 \quad , \tag{4}$$

with the final condition:

$$V(T, S_T) = g(S_T). \tag{5}$$

In addition, the solution to Equation4 with σ provided, determines the maximum potential loss for an investor who holds a long position in the option.

$$\sigma = \sigma^{-} = \begin{cases} \sigma_{max} & if \quad \dfrac{\partial^2 V}{\partial S^2} < 0 \\ \sigma_{min} & if \quad \dfrac{\partial^2 V}{\partial S^2} \geqslant 0 \end{cases} .$$

On the other hand, the best case value for an investor with a long position is determined from the solution to Equation4 with σ given by:

$$\sigma = \sigma^{+} = \begin{cases} \sigma_{max} & if \quad \dfrac{\partial^2 V}{\partial S^2} \geqslant 0 \\ \sigma_{min} & if \quad \dfrac{\partial^2 V}{\partial S^2} < 0 \end{cases} .$$

In reality, assuming that the bounds are constant for derivatives with longer maturities, is no longer compatible with observed volatility in the reality.
As a result, rather than modeling σ_t between two deterministic boundaries, we will consider, in the next part, a new representation of the uncertain volatility such that σ_t moves between two stochastic bound.

2.2 Representation of the Uncertain Volatility Model with Stochastic Bounds and the Market Price of Volatility Risk

Traders often find volatility the most challenging input to comprehend when evaluating options, yet it often has the most significant impact on trading decisions. Any alteration in assumptions regarding volatility can drastically affect an option's worth. Additionally, the way the market perceives volatility can have an equally substantial impact on the price of an option. To address this, we introduce a novel

model for uncertain volatility.
We examine a scenario where the volatility is uncertain and fluctuates between two stochastic bounds. To characterize this situation, we utilize the subsequent equation:

$$\sigma(t) = \sigma_{min}(t)\mathbb{1}_{\left\{\frac{\partial^2 V}{\partial S^2} \geqslant 0\right\}} + \sigma_{max}(t)\mathbb{1}_{\left\{\frac{\partial^2 V}{\partial S^2} < 0\right\}}, \tag{6}$$

with:

$$\sigma_{min}(t) = d\sqrt{\nu(t)}, \qquad \sigma_{max}(t) = u\sqrt{\nu(t)}, \qquad for \quad t \in [t_0, T], \tag{7}$$

where u (upper parameter) and d (down parameter) are two constants such that $0 < d < 1 < u$, and $\nu(t)$ can be any positive stochastic process. In this paper, we consider that the stochastic volatility bounds follows a Heston model, thus we obtain the second stochastic differential equation for volatility:

$$d\nu(t) = \xi[\kappa(\theta - \nu(t)) - \lambda(t, S, \nu)]dt + \sqrt{\xi}\sqrt{\nu(t)}dB(t), \tag{8}$$

with $\lambda(t, S, \nu)$ is the price of volatility risk. A linear function of volatility is used to calculate the price of volatility risk using Breeden's (1979) consumption model, so that $\lambda(t, S, \nu) = \lambda_f = \lambda\beta\sqrt{\nu}$, where λ is a constant, and we note $\lambda(t, S, \nu) = \lambda_f$, moreover, we suppose that λ is contained in some interval $\Omega \subset \mathbb{R}$ and consider the set $\mathbf{\Omega}$ of all measurable mappings from $[t_0, T]$ to Ω.
We assume that κ and θ are strictly positive parameters with the Feller condition $\theta\kappa \geqslant \xi^2/2$. ξ represent the volatility of volatility, and it is a small positive parameter that characterizes the slow variation of the process ν_t, and B_t is a Brownian motion. We denote the correlation coefficient between W and B as ρ, with $d < W, B >_t = \rho dt$ for $|\rho| < 1$.

Denoting $\sigma_t = \beta_t\sqrt{\nu_t}$, with $d \leqslant \beta \leqslant u$, for all $t \in [t_0, T]$, and we have:

$$\beta_t = \begin{cases} d \quad if \quad \dfrac{\partial^2 V}{\partial S^2} \geqslant 0 \\ u \quad if \quad \dfrac{\partial^2 V}{\partial S^2} < 0 \end{cases}.$$

Remark 1. If $g(.)$ is a convex (or concave) function, the worst-case scenario price can be shown to be convex (or concave) because the supremum and expectation operations preserve convexity (or concavity). This means that we can analyze perturbations around Black-Scholes prices, which have already been studied in [7]. However, our analysis in this paper extends to general terminal payoff functions that may not be convex or concave.

3 American Option Pricing under Uncertain Volatility

In this section, focusing on the worst case scenario, we study the Lipschitz continuity of the price V^{ε}, with respect to the parameter $\varepsilon = (\xi, \lambda)$. After that, we present the main BSB equation that the price should follow. In order to study the asymptotic behavior, we highlight the importance of ξ and λ, and reparameterize the stochastic differential equation 1 of the risky asset price process as:

$$dS_t^{\varepsilon} = rS_t^{\varepsilon}dt + \beta_t\sqrt{\nu_t}S_t^{\varepsilon}dW_t \quad , \tag{9}$$

where $\varepsilon = (\xi, \lambda)$.

Assumption 1. We assume that the payoff function g is lipschitz, continuous and differentiable up to the fourth order and polynomial growth conditions on the first four derivatives of g:

$$\begin{cases} \mid g'(s) \mid \leqslant C_1 \\ \mid g''(s) \mid \leqslant C_2(1+ \mid s \mid^m) \\ \mid g'''(s) \mid \leqslant C_3(1+ \mid s \mid^n) \\ \mid g^4(s) \mid \leqslant C_4(1+ \mid s \mid^q) \\ \mid g(s) - g(y) \mid \leqslant C_0 \mid s - y \mid \end{cases} , \tag{10}$$

with m, n and q are positive constants.

3.1 Pricing Non-linear PDE

The value of the American option in the worst case scenario is the highest option price which can occur with market prices of volatility risk λ_f, and the volatility of volatility ξ:

$$V^{\varepsilon}(t, S, \nu) := \exp^{(-r(T-t))} \operatorname*{ess\,sup}_{\lambda_f \in \Omega} \operatorname*{ess\,sup}_{\beta \in [d,u]} \mathbb{E}_{(t,s,\nu)}(g(S_T^{\varepsilon})), \tag{11}$$

where $\mathbb{E}_{(t,s,\nu)}[.]$ is the conditional expectation given $\mathbb{F}$ with $S_t^{\varepsilon} = s$.
By using the ambiguity of the uncertain volatility model, we get the price definition as equation (11). Furthermore, the issue of model ambiguity in financial mathematics has piqued many people's interest, for this reason, we emphasize the importance of pricing options in the worst case scenario. We know that V^{ε} satisfies the BSB equation according to the stochastic control theory [20]:

Proposition 1. $V^{\varepsilon}(t, S, \nu)$ satisfies the following BSB equation

$$\begin{aligned} \frac{\partial V^{\varepsilon}}{\partial t} + rS\frac{\partial V^{\varepsilon}}{\partial S} \quad &- \quad rV^{\varepsilon} + \sup_{\beta \in [d,u]} \left[\frac{1}{2}\beta^2\nu S^2\frac{\partial^2 V^{\varepsilon}}{\partial S^2} + \sqrt{\xi}\beta\rho\nu S\frac{\partial^2 V^{\varepsilon}}{\partial S \partial \nu}\right] \\ &+ \quad \frac{1}{2}\xi\nu\frac{\partial^2 V^{\varepsilon}}{\partial \nu^2} + \xi\left[\kappa(\theta - \nu) - \lambda_f\right]\frac{\partial V^{\varepsilon}}{\partial \nu} = 0, \end{aligned} \tag{12}$$

with terminal condition:

$$V(T, S, \nu) = K - S_T, \tag{13}$$

and boundary conditions:

$$\begin{aligned} V(t, b(\nu,t), \nu) &= K - b(\nu,t), \\ \lim_{S \to b(\nu,t)} \frac{\partial V}{\partial S} &= -1, \\ \lim_{S \to b(\nu,t)} \frac{\partial V}{\partial \nu} &= 0, \end{aligned} \tag{14}$$

where $b(\nu, t)$ represents the early exercise boundary at time t for the Heston volatility ν.

Proof. For all $(t, s, \nu) \in [t_0, T] \times \mathbb{R}^+ \times \mathbb{R}^+$, we establish the stochastic control system given by:

$$\begin{cases} dS_t^\varepsilon &= rS_t^\varepsilon dt + \beta_t \sqrt{\nu_t} S_t^\varepsilon dW_t \\ d\nu(t) &= \xi[\kappa(\theta - \nu(t)) - \lambda(t, S, \nu)]dt + \sqrt{\xi}\sqrt{\nu}(t)dB(t) \end{cases}. \tag{15}$$

The cost function is:

$$J^\varepsilon(t, s, \nu) = \mathbb{E}_{(t,s,\nu)} \left[\exp^{-r(T-t)} g(S_T^\varepsilon) \right], \tag{16}$$

and since the value function depend on the parameter ε, we can write V^ε as follow:

$$V^\varepsilon(t, s, \nu) = \operatorname*{ess\,sup}_{\lambda_f \in \Omega} \operatorname*{ess\,sup}_{\beta \in [d,u]} J^\varepsilon(t, s, \nu). \tag{17}$$

For convenience, we use g to denote $g(S_T^\varepsilon)$, and for all $t_0 \leqslant t \leqslant \hat{t} \leqslant T$, $\beta \in [d, u]$ *and* $\lambda_f \in \Omega$, we obtain:

$$\begin{aligned} V^\varepsilon(t, s, \nu) &\geqslant \mathbb{E}_{(t,s,\nu)} \left[\exp^{-r(T-t)} g \right] \\ &= \mathbb{E}_{(t,s,\nu)} \left[\int_t^{\hat{t}} -r \exp^{-r(T-\tau)} g d\tau + \exp^{-r(T-\hat{t})} g \right]. \end{aligned} \tag{18}$$

Then we get:

$$0 \geqslant \mathbb{E}_{(t,s,\nu)} \left[\int_t^{\hat{t}} -r \exp^{-r(T-\tau)} g d\tau \right] + V^\varepsilon(\hat{t}, s, \nu) - V^\varepsilon((t, s, \nu)).$$

Dividing by $\hat{t} - t$ on both sides of the inequality, we have that:

$$0 \geqslant \mathbb{E}_{(t,s,\nu)} \left[\frac{\left[\int_t^{\hat{t}} -r \exp^{-r(T-\tau)} g d\tau \right]}{\hat{t} - t} \right] + \frac{V^\varepsilon(\hat{t}, s, \nu) - V^\varepsilon(t, s, \nu)}{\hat{t} - t}.$$

Assuming that Assumption 1 holds and using Equation 15, we can apply the Itô formula. Based on established arbitrage arguments outlined in works such as Black and Scholes (1973) and Merton (1973) [11], we arrive at the following partial differential equation:

$$\begin{aligned} \partial V^{\varepsilon} &= \partial_t V^{\varepsilon} + rS\partial_s V^{\varepsilon} + \frac{1}{2}\beta^2 \nu S^2 \partial^2_{ss} V^{\varepsilon} + \beta\rho\nu S\sqrt{\xi}\partial^2_{s\nu} V^{\varepsilon} \\ &\quad + \frac{1}{2}\xi\nu\partial^2_{\nu\nu} V^{\varepsilon} + \xi\left[\kappa(\theta-\nu) - \lambda_f\right]\partial_\nu V^{\varepsilon}. \end{aligned} \tag{19}$$

Let $\hat{t} \longrightarrow t$, for all $\beta \in [d, u]$, then we have:

$$\begin{aligned} 0 &\geqslant -r\mathbb{E}_{(t,s,\nu)}\left[\exp^{-r(T-\tau)} g\right] + \partial V^{\varepsilon} \\ &\geqslant -rV^{\varepsilon} + \partial_t V^{\varepsilon} + rS\partial_s V^{\varepsilon} + \frac{1}{2}\beta^2\nu S^2\partial^2_{ss}V^{\varepsilon} + \beta\rho\nu S\sqrt{\xi}\partial^2_{s\nu}V^{\varepsilon} \\ &\quad + \frac{1}{2}\xi\nu\partial^2_{\nu\nu}V^{\varepsilon} + \xi\left[\kappa(\theta-\nu)-\lambda_f\right]\partial_\nu V^{\varepsilon}, \end{aligned} \tag{20}$$

which implies that:

$$\begin{aligned} 0 &\geqslant -rV^{\varepsilon} + \partial_t V^{\varepsilon} + rS\partial_s V^{\varepsilon} + \sup_{\beta\in[d,u]}\left[\frac{1}{2}\beta^2\nu S^2\partial^2_{ss}V^{\varepsilon} + \beta\rho\nu S\sqrt{\xi}\partial^2_{s\nu}V^{\varepsilon}\right] \\ &\quad + \frac{1}{2}\xi\nu\partial^2_{\nu\nu}V^{\varepsilon} + \xi\left[\kappa(\theta-\nu)-\lambda_f\right]\partial_\nu V^{\varepsilon}. \end{aligned} \tag{21}$$

On the other hand, according to supremum characteristics, for any $\chi > 0$, we have:

$$\begin{aligned} V^{\varepsilon}(t,s,\nu) - \chi(\hat{t}-t) &\leqslant \mathbb{E}_{(t,s,\nu)}\left[\exp^{-r(T-t)} g\right] \\ &= \mathbb{E}_{(t,s,\nu)}\left[\int_t^{\hat{t}} -r\exp^{-r(T-\tau)} g\,d\tau + \exp^{-r(T-\hat{t})} g\right]. \end{aligned} \tag{22}$$

Then:

$$-\chi \leqslant \mathbb{E}_{(t,s,\nu)}\left[\frac{\left[\int_t^{\hat{t}} -r\exp^{-r(T-\tau)} g\,d\tau\right]}{\hat{t}-t}\right] + \frac{V^{\varepsilon}(\hat{t},s,\nu) - V^{\varepsilon}(t,s,\nu)}{\hat{t}-t}.$$

We obtain

$$\begin{aligned} 0 &\leqslant -rV^{\varepsilon} + \partial_t V^{\varepsilon} + rS\partial_s V^{\varepsilon} + \sup_{\beta\in[d,u]}\left[\frac{1}{2}\beta^2\nu S^2\partial^2_{ss}V^{\varepsilon} + \beta\rho\nu S\sqrt{\xi}\partial^2_{s\nu}V^{\varepsilon}\right] \\ &\quad + \frac{1}{2}\xi\nu\partial^2_{\nu\nu}V^{\varepsilon} + \xi\left[\kappa(\theta-\nu)-\lambda_f\right]\partial_\nu V^{\varepsilon}. \end{aligned} \tag{23}$$

We conclude from 21 and 23, that V^{ε} satisfies the Black-Scholes-Barenblatt equation given by Equation 12. $\square$

Remark 2. It should be noted that the PDE described by (12) is completely nonlinear and lacks a solution, unlike the Black-Scholes equation. Therefore, we opt to tackle this issue by breaking it down into two Black-Scholes PDEs.

The fact that the option price in the worst-case scenario is higher than any Black-Scholes price with uncertain volatility may seem evident. Nevertheless, our subsequent demonstration establishes that the worst-case scenario price for American options will ultimately approach its Black-Scholes counterpart. Let $V_0 = V^\varepsilon |_{\varepsilon=0}$, $V_1 = \partial_\nu V^\varepsilon |_{\varepsilon=0}$. When $\varepsilon = 0$ (i.e $\xi = 0$ and $\lambda = 0$), we notice that the risky asset price process follows the dynamic:

$$dS_t^0 = rS_t^0 dt + \beta_t \sqrt{\nu} S_t^0 dW_t, \tag{24}$$

and the selling option price in the worst case scenario is given by:

$$V_0(t, S, \nu) := \exp^{(-r(T-t))} \operatorname*{ess\,sup}_{\lambda_f \in \Omega} \operatorname*{ess\,sup}_{\beta \in [d,u]} \mathbb{E}_{(t,s,\nu)}(g(S_T^0)). \tag{25}$$

In order to obtain an approximation for the worst-case scenario price V^ε, which represents the solution to the HJB equation linked to the control problem specified by the generalized BSB nonlinear Equation 12, we calculate V_0 and V_1.
Thus, we employ the regular perturbation approach as shown below:

$$V^\varepsilon = V_0 + \sqrt{\xi} V_1 + \xi V_2 + ... \quad . \tag{26}$$

In the rest of the paper, we suppose that the interest rate $r = 0$ for analytical convenience.

Convergence of V^ε

In the next theorem, we will demonstrate the convergence of the price option V^ε to V_0, controlled by the volatility of volatility (ξ), and the risk price of the market volatility (λ).

Theorem 3.1. *Under Assumption 1, $V^\varepsilon(t, s, \nu)$ is uniformly converge to $V_0(t, s, \nu)$ with rate ε, for all $(t, s, \nu) \in [t_0, T] \times \mathbb{R}^+ \times \mathbb{R}^+$.*

Proof. For V^ε given by Equation(11) and V_0 given by Equation(25), using the

Lipschitz continuous of $g(.)$ and by the Cauchy-Schwartz inequality, we have:

$$\begin{aligned} \mid V^{\varepsilon} - V_0 \mid &= \exp^{-r(T-t)} \mid \operatorname*{ess\,sup}_{\lambda_f \in \Omega} \operatorname*{ess\,sup}_{\beta \in [d,u]} \mathbb{E}(g(S_T^{\varepsilon})) - \operatorname*{ess\,sup}_{\lambda_f \in \Omega} \operatorname*{ess\,sup}_{\beta \in [d,u]} \mathbb{E}(g(S_T^{0})) \mid \\ &\leqslant \exp^{-r(T-t)} \mid \operatorname*{ess\,sup}_{\lambda_f \in \Omega} \operatorname*{ess\,sup}_{\beta \in [d,u]} \mathbb{E}(g(S_T^{\varepsilon})) - \mathbb{E}(g(S_T^{0})) \mid \\ &\leqslant \exp^{-r(T-t)} \operatorname*{ess\,sup}_{\lambda_f \in \Omega} \operatorname*{ess\,sup}_{\beta \in [d,u]} \mid \mathbb{E}[g(S_T^{\varepsilon}) - g(S_T^{0})] \mid \\ &\leqslant K \exp^{-r(T-t)} \operatorname*{ess\,sup}_{\lambda_f \in \Omega} \operatorname*{ess\,sup}_{\beta \in [d,u]} \mathbb{E}[\mid S_T^{\varepsilon} - S_T^{0} \mid] \\ &\leqslant K \exp^{-r(T-t)} \operatorname*{ess\,sup}_{\lambda_f \in \Omega} \operatorname*{ess\,sup}_{\beta \in [d,u]} [\mathbb{E}(S_T^{\varepsilon} - S_T^{0})^2]^{1/2}, \end{aligned} \tag{27}$$

or, we have from the proposition (4) given in Appendix (2):

$$\mathbb{E}(S_T^{\varepsilon} - S_T^{0})^2 \leqslant K_0 \xi. \tag{28}$$

Then, we obtain:

$$\mid V^{\varepsilon} - V_0 \mid \leqslant K_1 \sqrt{\xi}, \tag{29}$$

where K_1 is a positive constant independent of ξ and λ. □

Convergence of $\partial_{ss}^2 V^{\varepsilon}$

Suppose that the second derivative of the payoff function satisfies the polynomial growth condition,

Assumption 2. Throughout the paper, we make the following assumptions on $V^{\varepsilon}(t,.,.)$:
i) $V^{\varepsilon}(t,.,.)$ belongs to $\mathbb{C}_p^{1,2,2}$ (p for polynomial growth), for ε fixed.
ii) $\partial_s V^{\varepsilon}(t,.,.)$ and $\partial_{ss}^2 V^{\varepsilon}(t,.,.)$ are uniformly bounded in ε.
Then, we have the following theorem:

Theorem 3.2. *Under Assumptions 1 and 2, the family $\partial_{ss}^2 V^{\varepsilon}(t,.,.)$ of functions of S and ν indexed by ε, converges to $\partial_{ss}^2 V_0(t,.,.)$ as ε tends to 0 with rate ξ, uniformly on compact sets in S and ν, for $t \in [t_0, T]$.*

Proof. Under Assumptions 1 and 2, and by Theorem 3.1 , the proof of the convergence can be obtained by following the arguments in the Arzela-Ascoli Theorem (see Appendix 3). □

4 Main Result

In this section, we try to control the error term to prove that we can compute V^{ε} with its estimation $V_0 + \sqrt{\xi} V_1$ cited in the next theorem 4.1. Additionally, from the conditions imposed on g mentioned in assumptions 1, we have the following process of proof.

4.1 Main Theorem

Assumption 3. We assume polynomial growth for the following derivatives of V_0 and V_1

$$\begin{cases} \mid \partial_\nu V_0(p,s,\nu) \mid & \leqslant & A_{01}(1+s^{m_{(01)}}+\nu^{n_{(01)}}) \\ \mid \partial^2_{s\nu} V_0(p,s,\nu) \mid & \leqslant & A_{02}(1+s^{m_{(02)}}+\nu^{n_{(02)}}) \\ \mid \partial^2_{\nu\nu} V_0(p,s,\nu) \mid & \leqslant & A_{03}(1+s^{m_{(03)}}+\nu^{n_{(03)}}) \\ \mid \partial_\nu V_1(p,s,\nu) \mid & \leqslant & A_{11}(1+s^{m_{(11)}}+\nu^{n_{(11)}}) \\ \mid \partial^2_{s\nu} V_1(p,s,\nu) \mid & \leqslant & A_{12}(1+s^{m_{(12)}}+\nu^{n_{(12)}}) \\ \mid \partial^2_{\nu\nu} V_1(p,s,\nu) \mid & \leqslant & A_{13}(1+s^{m_{(13)}}+\nu^{n_{(13)}}) \end{cases},$$

where $m_{(ij)}$ and $n_{(ij)}$ are positive integers, with $i \in \{0,1\}$ and $j \in \{1,2,3\}$.

Theorem 4.1. *We define the residual function $R^\varepsilon(t,s,\nu)$ as follow:*

$$R^\varepsilon(t,s,\nu) = V^\varepsilon(t,s,\nu) - V_0(t,s,\nu) - \sqrt{\xi}V_1(t,s,\nu). \tag{30}$$

Under Assumptions 1, 2 and 3, $\forall (t,s,\nu) \in [t_0,T] \times \mathbb{R}^+ \times \mathbb{R}^+$ there exists a positive constant $\mathcal{C}$, such that $\mid R^\varepsilon(t,s,\nu) \mid \leqslant \mathcal{C}\xi$, where $\mathcal{C}$ may depend on (t,s,ν) but not on ξ or λ. Moreover, we can say that R^ε is of order $\mathcal{O}(\xi)$.

4.2 The Proof of the Main Theorem

Expectation Form of the Error term

In this part, we analyse the error term and give its expectation form as preparation work before proving the convergence of $V_0 + \sqrt{\xi}V_1$.
We suppose that $r = 0$ for analytical convenience.

Proof. To prove Theorem 4.1, we define the following operator:

$$\begin{aligned} \mathcal{L}^\varepsilon(\beta) &= \partial_t + \frac{1}{2}\beta^2\nu s^2\partial^2_{ss} + \sqrt{\xi}\beta\rho\nu s\partial^2_{s\nu} + \frac{1}{2}\xi\nu\partial^2_{\nu\nu} + [\kappa(\theta-\nu) - \xi\lambda_f]\partial_\nu \\ &= \mathcal{L}_0(\beta) + \sqrt{\xi}\mathcal{L}_1(\beta) + \xi\mathcal{L}_2(\beta), \end{aligned} \tag{31}$$

where the operators $\mathcal{L}_0, \mathcal{L}_1$, and $\mathcal{L}_2$ are defined by:

$$\begin{aligned} \mathcal{L}_0 &= \partial_t + \frac{1}{2}\beta^2\nu s^2\partial^2_{ss}, \\ \mathcal{L}_1 &= \beta\rho\nu s\partial^2_{s\nu}, \\ \mathcal{L}_2 &= \frac{1}{2}\nu\partial^2_{\nu\nu} + (\kappa(\theta-\nu) - \lambda_f)\partial_\nu. \end{aligned}$$

Applying the operator $\mathcal{L}^{\varepsilon}(\beta)$ to the error term, it follows that

$$\begin{aligned}
\mathcal{L}^{\varepsilon}(\beta)R^{\varepsilon} &= \mathcal{L}^{\varepsilon}(\beta)\left(V^{\varepsilon} - V_0 - \sqrt{\xi}V_1\right) \\
&= \mathcal{L}^{\varepsilon}(\beta)V^{\varepsilon} - \mathcal{L}^{\varepsilon}(\beta)\left(V_0 + \sqrt{\xi}V_1\right) \\
&= -\left(\mathcal{L}_0 + \sqrt{\xi}\mathcal{L}_1 + \xi\mathcal{L}_2\right)\left(V_0 + \sqrt{\xi}V_1\right) \\
&= -\sqrt{\xi}\left[\mathcal{L}_1 V_0 + \mathcal{L}_0 V_1\right] - \xi\left[\mathcal{L}_1 V_1 + \mathcal{L}_2 V_0\right] - \xi^{3/2}\mathcal{L}_2 V_1 \\
&= -\sqrt{\xi}\left[\beta\rho\nu s\partial^2_{s\nu}V_0 + \partial_t V_1 + \frac{1}{2}\beta^2\nu s^2\partial^2_{ss}V_1\right] \\
&\quad -\xi\left[\beta\rho\nu s\partial^2_{s\nu}V_1 + \frac{1}{2}\nu\partial^2_{\nu\nu}V_0 + (\kappa(\theta-\nu) - \lambda_f)\partial_\nu V_0\right] \\
&\quad -\xi^{3/2}\left[\frac{1}{2}\nu\partial^2_{\nu\nu}V_1 + (\kappa(\theta-\nu) - \lambda_f)\partial_\nu V_1\right] \\
&= -\sqrt{\xi}\beta\rho\nu s\partial^2_{s\nu}V_0 \\
&\quad -\xi\left[\beta\rho\nu s\partial^2_{s\nu}V_1 + \frac{1}{2}\nu\partial^2_{\nu\nu}V_0 + (\kappa(\theta-\nu) - \lambda_f)\partial_\nu V_0\right] \\
&\quad -\xi^{3/2}\left[\frac{1}{2}\nu\partial^2_{\nu\nu}V_1 + (\kappa(\theta-\nu) - \lambda_f)\partial_\nu V_1\right].
\end{aligned} \tag{32}$$

The terminal condition of R^{ε} is given by:

$$R^{\varepsilon}(T,s,\nu) = V^{\varepsilon}(T,s,\nu) - V_0(T,s,\nu) - \sqrt{\xi}V_1(T,s,\nu) = 0. \tag{33}$$

Otherwise, we have the following probabilistic representation of $R^{\varepsilon}(t,s,\nu)$ by Feynman–Kac formula:

$$R^{\varepsilon}(t,s,\nu) = \sqrt{\xi}I_0 + \xi I_1 + \xi^{3/2}I_2, \tag{34}$$

where

$$\begin{aligned}
I_0 &= \mathbb{E}_{(t,s,\nu)}\left[\int_t^T \beta\rho\nu_p S_p\partial^2_{s\nu}V_0(p,S,\nu)dp\right], \\
I_1 &= \mathbb{E}_{(t,s,\nu)}\left[\int_t^T \beta\rho\nu_p S_p\partial^2_{s\nu}V_1(p,S,\nu) + \frac{1}{2}\nu_p\partial^2_{\nu\nu}V_0(p,S,\nu) + (\kappa(\theta-\nu_p) - \lambda_f)\partial_\nu V_0(p,S,\nu)dp\right], \\
I_2 &= \mathbb{E}_{(t,s,\nu)}\left[\int_t^T \frac{1}{2}\nu_p\partial^2_{\nu\nu}V_1(p,S,\nu) + (\kappa(\theta-\nu_p) - \lambda_f)\partial_\nu V_1(p,S,\nu)dp\right].
\end{aligned}$$

Thus, by controlling $\mid I_0 \mid, \mid I_1 \mid$ and $\mid I_2 \mid$. we can conclude that R^{ε} is of order $\mathcal{O}(\varepsilon)$ Then, we can estimate V^{ε}.

Control of the Error Term

The process S and ν has finite moments of any order uniformly in $0 \leqslant \xi$ for $t \in [t_0, T]$. Moreover,

$$\mathbb{E}(t,s,\nu)[\int_t^T \mid S_p \mid^k dp] \leqslant M_k(T,s,\nu), \tag{35}$$

and

$$\mathbb{E}(t,\nu)[\int_t^T \mid \nu_p \mid^k dp] \leqslant N_k(T,\nu), \tag{36}$$

where $M_k(T,s,\nu)$ and $N_k(T,\nu)$ may depend on (k,T,s,ν) but not ε (see Appendix 2).

As denoted in section 2, we have $\lambda(t,s,\sigma) = \sigma\lambda = \lambda\beta\sqrt{\nu}$. Using theorem 3.2, and assumptions 3 we have the following results:

- Control of the term I_0

$$\begin{aligned}
\mid I_0 \mid &\leqslant \mathbb{E}_{(t,s,\nu)}\left[\int_t^T \mid \beta\rho\nu_p S_p \partial^2_{s\nu} V_0(p,S,\nu) \mid dp\right] \\
&\leqslant \beta\rho\mathbb{E}_{(t,s,\nu)}\left[\int_t^T \mid \nu_p S_p \partial^2_{s\nu} V_0(p,S,\nu) \mid dp\right] \\
&\leqslant \beta\rho\mathbb{E}^{1/2}_{(t,s,\nu)}\left[\int_t^T (\nu_p S_p)^2 dp\right]\mathbb{E}^{1/2}_{(t,s,\nu)}\left[\int_t^T (\partial^2_{s\nu} V_0(p,S,\nu))^2 dp\right] \\
&\leqslant \beta\rho\mathbb{E}^{1/4}_{(t,s,\nu)}\left[\int_t^T (\nu_p)^4 dp\right]\mathbb{E}^{1/4}_{(t,s,\nu)}\left[\int_t^T (S_p)^4 dp\right] \\
&\quad .A^2_{02}\mathbb{E}^{1/2}_{(t,s,\nu)}\left[\int_t^T (1+ \mid S_p \mid^{m_{02}} + \mid \nu_p \mid^{n_{02}})^2 dp\right] \\
&\leqslant \beta\rho N_4(T,\nu)^{1/4} M_4(T,s,\nu)^{1/4} A^*_{02}(M_{2m_{02}} + N_{2n_{02}})^{1/2} \\
&\leqslant C_{02}A^*_{02},
\end{aligned} \tag{37}$$

where A^*_{02} is a positive constant. Hence, the error $\mid I_0 \mid$ is bounded.
For I_1, we recall that

$$I_1 = \mathbb{E}_{(t,s,\nu)}\left[\int_t^T \beta\rho\nu s\partial^2_{s\nu}V_1(p,S,\nu) + \frac{1}{2}\nu\partial^2_{\nu\nu}V_0(p,S,\nu) + (\kappa(\theta-\nu) - \lambda_f)\partial_\nu V_0(p,S,\nu)dp\right],$$

and denote:

$$I_1 = I_1^{(0)} + I_1^{(1)} + I_1^{(2)}.$$

Then we have:

$$
\begin{aligned}
\mid I_1^{(0)} \mid \quad & \leqslant \quad \mathbb{E}_{(t,s,\nu)}\left[\int_t^T \mid \beta\rho\nu_p S\partial^2_{s\nu}V_1(p,S,\nu)\mid dp\right] \\
& \leqslant \quad \beta\rho\mathbb{E}^{1/4}_{(t,s,\nu)}\left[\int_t^T(\nu_p)^4dp\right]\mathbb{E}^{1/4}_{(t,s,\nu)}\left[\int_t^T(S_p)^4dp\right] \\
& \qquad .A_{12}^2\mathbb{E}^{1/2}_{(t,s,\nu)}\left[\int_t^T(1+\mid S_p\mid^{m_{12}}+\mid \nu_p\mid^{n_{12}})^2dp\right] \\
& \leqslant \quad \beta\rho N_4(T,\nu)^{1/4}M_4(T,s,\nu)^{1/4}A^*_{12}(M_{2m_{12}}+N_{2n_{12}})^{1/2} \\
& \leqslant \quad C_{12}A^*_{12},
\end{aligned}
$$

$$
\begin{aligned}
\mid I_1^{(1)} \mid \quad & \leqslant \quad \frac{1}{2}\mathbb{E}_{(t,s,\nu)}\left[\int_t^T \mid \nu_p\partial^2_{\nu\nu}V_0(p,S,\nu)\mid dp\right] \\
& \leqslant \quad \frac{1}{2}\mathbb{E}^{1/2}_{(t,s,\nu)}\left[\int_t^T(\nu_p)^2dp\right]\mathbb{E}^{1/2}_{(t,s,\nu)}\left[\int_t^T(\partial^2_{\nu\nu}V_0(p,S,\nu))^2dp\right] \\
& \leqslant \quad \frac{1}{2}N_2(T,\nu)^{1/2}A^*_{03}(M_{2m_{03}}+N_{2n_{03}})^{1/2} \\
& \leqslant \quad C_{03}A^*_{03},
\end{aligned}
$$

$$
\begin{aligned}
\mid I_1^{(2)} \mid \quad & \leqslant \quad \mathbb{E}_{(t,s,\nu)}\left[\int_t^T \mid (\kappa(\theta-\nu)-\lambda\beta\sqrt{\nu})\partial_\nu V_0(p,S,\nu)\mid dp\right] \\
& \leqslant \quad \kappa\mathbb{E}^{1/2}_{(t,s,\nu)}\left[\int_t^T(\theta^2+\nu_p^2)dp\right]\mathbb{E}^{1/2}_{(t,s,\nu)}\left[\int_t^T(\partial_\nu V_0)^2dp\right] \\
& \qquad -\lambda\beta\mathbb{E}^{1/2}_{(t,s,\nu)}\left[\int_t^T\nu_p dp\right]\mathbb{E}^{1/2}_{(t,s,\nu)}\left[\int_t^T(\partial_\nu V_0)^2dp\right] \\
& \leqslant \quad \left[\kappa\left[\theta^2(T-t)+N_2(T,\nu)\right]^{1/2}-\lambda\beta N_1(T,\nu)^{1/2}\right]A^*_{01}\left[M_{2m_{01}}+N_{2n_{01}}\right] \\
& \leqslant \quad C_{01}A^*_{01}.
\end{aligned}
$$

Thus, the error $\mid I_1 \mid$ is bounded, with A^*_{01}, A^*_{03} and A^*_{12} are positive constants. Following the same process, we get:

$$I_2 = I_2^{(0)} + I_2^{(1)}.$$

Then we have

$$
\begin{aligned}
\mid I_2^{(0)} \mid \quad & \leqslant \quad C_{13}A^*_{11}, \\
\mid I_2^{(1)} \mid \quad & \leqslant \quad C_{11}A^*_{11},
\end{aligned}
$$

where $A^*_{11}$and A^*_{13} are positive constants.

After controlling the errors expectation terms, we conclude that

$$R^{\varepsilon}(t,s,\nu) = \sqrt{\xi} I_0 + \xi I_1 + \xi^{3/2} I_2, \tag{38}$$

is of order $\mathcal{O}(\xi)$, which completes the proof of the main Theorem 4.1. □

5 Artificial Neural Network for Pricing American Option under Uncertainty

Artificial neural networks (ANNs) have recently gained significant attention due to their success in diverse industries and applications, particularly in finance and pricing options. ANNs are models for information processing that evolved from mathematical models of human knowledge and were inspired by the functioning of the human brain. These networks consist of artificial neurons, which are fundamental building blocks that can perform computations and process inputs to generate output.

The main concept is to combine machine learning with the BSB model using optimization techniques. By determining the hidden unit values in a neural network's hidden layers, we can more accurately estimate the model's parameters compared to the conventional approach (using the Black Scholes model). We chose the RBF network because setting accurate initial states is crucial in this method, unlike other techniques such as MLP networks that use randomly generated parameters at the start. Additionally, since our inputs are obtained from explicit finite difference results and are not random, the RBF neural network appears to be more suitable for our situation.

5.1 Radial Basis Function Neural Network-Finite Difference Method

One of the most widely used methods for numerically solving partial differential equations is the finite difference method. Our improved method is a hybrid of artificial intelligence and the explicit finite difference method. Volatility is considered uncertain with stochastic bounds, as discussed in the preceding section, so there is no exact solution to our pricing problem. Although some numerical results can be used to approximate and find a closed solution to such problems.

The explicit finite difference method was chosen because it can be modified to allow for the pricing of American puts, which necessitates a reformulation of the boundary conditions. As a second step, we input the approximated prices into the radial basis function neural network, which will be used to find the closed real prices.
After approximating the solution by using the numerical operators of the function's

derivatives and finding the solution at specific grids, we will employ our modified method which based on radial basic function neural network algorithm presented in [6], then we will use a separate RBF neural networks for approximating the different solution components of option price. More precisely, one RBF neural network is trained for each of the functions (V_0), (V_1), and (V^ε). Each network is trained independently using its corresponding finite difference solution as the input data and parameters and the associated benchmark solution as the target output.

5.2 Pricing Algorithm

The primary challenge is to find the worst-case scenario price by solving Equation (12), which is fully non-linear and not solvable analytically in practice, especially in two dimensions. To tackle this problem, we first employ the finite difference method to solve V_0 in one dimension as a preliminary step. Then, to extend this approach to our two-dimensional problem, we use discretization grids on time and two state variables.

Denote $u^n_{i,j} = V^\varepsilon(t_n, s_i, \nu_j)$, $v^n_{i,j} = V_0(t_n, s_i, \nu_j)$ and $w^n_{i,j} = V_1(t_n, s_i, \nu_j)$, $n = 0, 1, ...NT$, represents the index of time, $i = 0, 1, ..., NS$ represents the index of the asset price process, and $j = 0, 1, ..., NV$ represents the index of the volatility process. In the following, we build a uniform grid of size $NT = 300, NS = 79$ and $NV = 39$. We use the classical discrete approximations to the continuous derivatives:

$$
\begin{aligned}
\partial_s(u^n_{i,j}) &= \frac{u^n_{i+1,j} - u^n_{i-1,j}}{2\Delta s}, \\
\partial_\nu(u^n_{i,j}) &= \frac{u^n_{i,j+1} - u^n_{i,j-1}}{2\Delta \nu}, \\
\partial^2_{ss}(u^n_{i,j}) &= \frac{u^n_{i+1,j} + u^n_{i-1,j} - 2u^n_{i,j}}{\Delta s^2}, \\
\partial^2_{s\nu}(u^n_{i,j}) &= \frac{u^n_{i+1,j+1} + u^n_{i-1,j-1} - u^n_{i-1,j+1} - u^n_{i+1,j-1}}{4\Delta s \Delta \nu}, \\
\partial^2_{\nu\nu}(u^n_{i,j}) &= \frac{u^n_{i,j+1} + u^n_{i,j-1} - 2u^n_{i,j}}{\Delta \nu^2}, \\
\partial_t(u^n_{i,j}) &= \frac{u^{n+1}_{i,j} - u^n_{i,j}}{\Delta t}.
\end{aligned}
$$

Then we denote the following operators to simplify our algorithms and make matrix operations easier to implement:

$$
\mathcal{L}_{ss} = \nu S^2 \partial^2_{ss}, \qquad \mathcal{L}_{s\nu} = \rho \nu S \partial^2_{s\nu}, \qquad \mathcal{L}_{\nu\nu} = \nu \partial^2_{\nu\nu}, \qquad \mathcal{L}_\nu = (\kappa(\theta - \nu) - \lambda_f)\partial_\nu. \tag{39}
$$

In order to find the value of our option we use the θ-method. This method combines the forward and backward scheme with the help of a scaling parameter and it is defined via the relationship

$$\frac{V^{n+1} - V^n}{dt} = L(\theta V^{n+1} + (1-\theta)V^n). \tag{40}$$

Since we based on an explicit finite difference framework, we set $\theta = 0$, then the equation (40) become

$$V^{n+1} = V^n + dtLV^n. \tag{41}$$

5.3 Simulation of V_0 and V_1

Back to equations 12 and 26, and by theorem 3.1, V_0 is the solution to the following system:

$$\begin{cases} \partial_t V_0 + \sup\limits_{\beta\in[d,u]} \left[\frac{1}{2}\beta^2 \nu S^2 \partial^2_{ss} V_0\right] & = & 0 \\ V_0(T, s, \nu) & = & g(s) \end{cases}. \tag{42}$$

In the other hand, V_1 is the first order derivative of V^ε with respect to ε, and let $\varepsilon = 0$, then we obtain the equation of V_1 as follows:

$$\begin{cases} \partial_t V_1 + \frac{1}{2}\beta^2 \nu S^2 \partial^2_{ss} V_1 + \beta\rho\nu S \partial^2_{s\nu} V_0 & = & 0 \\ V_1(T, s, \nu) & = & 0 \end{cases}. \tag{43}$$

Algorithm: Radial Basic Function Neural Network-FD for V_0 and V_1

• Initialize the parameters of Black Scholes Barenblatt equation
$T, K,\ S_0, S_{max},\ S_{min}, \kappa, \theta, \rho, \nu_0, d, u, \xi, \lambda, NT, NS, NV$

• Calculate initial conditions $log(\frac{S_{min}}{K}), log(\frac{S_{max}}{K})$ and semi analytic solution, taking on consideration the boundary condition (14), and final conditions in (42)(43)

• Using explicit finite difference method we calculate the results $v_{i,j}$ and $w_{i,j}$ of each level in, by solving the two following equations:

$$v^{n+1}_{i,j} = v^n_{i,j} - \sup_{\beta\in[d,u]} (\frac{1}{2}\beta^2)\mathcal{L}_{ss}.v^n_{i,j}dt,$$
$$w^{n+1}_{i,j} = w^n_{i,j} - \left[\frac{1}{2}\beta^2\mathcal{L}_{ss}.w^n_{i,j} + \beta\mathcal{L}_{s\nu}.w^n_{i,j}\right] dt.$$

• Set $v_{i,j}$ as input and semi analytic solution as target vector of neural network, (we do the same for $w_{i,j}$)

• Initialize RBF-NN parameters (weight and bias) randomly

• Training input data by using RBF-NN to find the optimal weight and bias

• Simulate the input $v_{i,j}$ data with target by using optimal weight and bias (we do the same for $w_{i,j}$)

5.4 Simulation of V^{ε}

Back to equations 12, and by theorem 3.1, V^{ε} is the solution to the following system:

$$\left\{\begin{array}{rcl} \dfrac{\partial V^{\varepsilon}}{\partial t}+rS\dfrac{\partial V^{\varepsilon}}{\partial S} & - & rV^{\varepsilon}+\sup\limits_{\beta\in[d,u]}\left[\dfrac{1}{2}\beta^2\nu S^2\dfrac{\partial^2 V^{\varepsilon}}{\partial S^2}+\sqrt{\xi}\beta\rho\nu S\dfrac{\partial^2 V^{\varepsilon}}{\partial S\partial\nu}\right] \\ & + & \dfrac{1}{2}\xi\nu\dfrac{\partial^2 V^{\varepsilon}}{\partial \nu^2}+\xi\left[\kappa(\theta-\nu)-\lambda_f\right]\dfrac{\partial V^{\varepsilon}}{\partial \nu}=0 \\ V^{\varepsilon}(T,s,\nu) & = & g(s) \end{array}\right. . \quad (44)$$

Algorithm: Radial Basic Function Neural Network-FD for V^{ε}

- Initialize the parameters of Black Scholes Barenblatt equation
$T, K, S_0, S_{max}, S_{min}, \kappa, \theta, \rho, \nu_0, d, u, \xi, \lambda, NT, NS, NV$
- Calculate initial conditions $log(\dfrac{S_{min}}{K}), log(\dfrac{S_{max}}{K})$ and semi analytic solution,
taking on consideration the boundary condition (14), and final conditions in (44)
- Using explicit finite difference method we calculate the results $v_{i,j}$ and $w_{i,j}$
of each level in, by solving the following equation:
$$u_{i,j}^{n+1}=u_{i,j}^{n}-[\sup_{\beta\in[d,u]}\left[\frac{1}{2}\beta^2\mathcal{L}_{ss}+\sqrt{\xi}\beta\mathcal{L}_{s\nu}\right]+\frac{1}{2}\xi\mathcal{L}_{\nu\nu}+\xi\mathcal{L}_{\nu}]dt\ .$$
- Set $u_{i,j}$ as input and semi analytic solution as target vector of neural network,
- Initialize RBF-NN parameters (weight and bias) randomly
- Training input data by using RBF-NN to find the optimal weight and bias
- Simulate the input $u_{i,j}$ data with target by using optimal weight and bias

Implementation Details of the RBF Neural Network

The RBF neural network implementation employed in this work follows the algorithm presented in our earlier publication([6]), with appropriate modifications to address the pricing of American options under the uncertain volatility model with stochastic bounds.

After obtaining the numerical solution of the finite difference scheme, the computed option prices were used as the input data for the Radial Basis Function Neural Network (RBF-NN), while the corresponding semi analytic option prices were used as the target outputs. This semi-analytic solution of an American option is a pricing method where the option value is expressed analytically once a numerically determined early-exercise boundary has been found[3]. A common semi-analytic approach begins with the partial differential equation governing the price of the American option, which is a free-boundary problem due to the unknown optimal exercise boundary. To simplify the analysis, the PDE is transformed into a Heat equation, through appropriate changes of variables and analytical transformations which enable the derivation of semi-closed-form solutions.

The RBF neural network was constructed using MATLAB's 'newrbe' function, which generates an exact radial basis function network. Instead, it automatically determines the hidden layer and computes the output-layer weights analytically, allowing the network to exactly interpolate the training data. The hidden layer consists of Gaussian radial basis functions defined by

$$[\phi(x) = \exp\left(-\frac{|x-c_i|^2}{2\sigma^2}\right)],$$

where (c_i) denotes the center of the $(i)-th$ neuron and (σ) is the Gaussian spread parameter. In this study, the spread parameter was fixed at $[\sigma = 0.1,]$ which provided an accurate approximation while maintaining numerical stability. The predictive performance of the proposed hybrid method was evaluated using the Mean Squared Error (MSE),

$$[\text{MSE} = \frac{1}{N}\sum_{i=1}^{N}(y_i - \hat{y}_i)^2] \ ,$$

where (y_i) denotes the option price and $(\hat{y}_i)$ is the corresponding RBF prediction. In the next section, we provide The numerical findings show that the RBF neural network offers a smooth alternative approximation for pricing American options under the uncertain volatility model with stochastic bounds while effectively modeling the option prices by the finite difference approach.

6 Numerical Illustration

In this section, we present some numerical tests to demonstrate the effectiveness of the novel representations proposed in this work on the one hand, and the reliability of our neural network-based algorithm on the other.

The model proposed is compared to market prices of an American put option, obtained from https://finance.yahoo.com/quote/%5EXSP/options/, (accessed on 01 September 2022). Its underlying asset is the SP&500 index, and we will take different values of strike price as shown in Tables1,2. We consider the uncertain model with the following parameters: $T = 0.25,, S0 = 375.00, r = 0, \kappa = 15, \theta = 0.04, \rho = -0.9, \nu_0 = 0.04, d = 0.75$ and $u = 1.25$. Therefore, the two deterministic bounds for V_0 are given by $\sigma_{min} = d\sqrt{\nu_0} = 0.15$ and $\sigma_{max} = u\sqrt{\nu_0} = 0.25$ which are the bounds of the uncertain volatility model. As anticipated, when V_0 is convex (resp., concave), the Black-Scholes prices with constant volatility 0.25 (resp. 0.15), are a good approximation.

In this study we supposed that the market condition has been verified, so the Table 1 and 2 described the different values of the put option while changing the value of the payoff, and the value of the price of volatility risk, such that in Table 1 we take the volatility of volatility $\xi = 0.5$, and in Table 2 we have $\xi = 0.01$ in order to prove the role of the difference that produce when changing those two parameters (λ, ξ).

Table 1: *Comparison of worst-case scenario option prices using* $V_0 + \sqrt{\varepsilon}V_1$*, and* V^ε*,the Black Scholes model* V_{bs}*, and the market prices* ($\xi = 0.5$)*.*

	$K = 415$		$K = 395$		$K = 370$	
	$\lambda = 0.1$	$\lambda = 0.05$	$\lambda = 0.1$	$\lambda = 0.05$	$\lambda = 0.1$	$\lambda = 0.05$
$V_0 + \sqrt{\xi}V_1$	79.893	75.08	21.18	23.5	0.23	0.115
V^ε	79.77	75.604	21.17	23.49	0.18	0.147
V_{bs}	74.82		25.52		0.47	
V_{market}	77.25		23.4		0.13	

Table 2: *Comparison of worst-case scenario option prices using* $V_0 + \sqrt{\varepsilon}V_1$*, and* V^ε*,the Black Scholes model* V_{bs}*, and the market prices* ($\xi = 0.01$)*.*

	$K = 415$		$K = 395$		$K = 370$	
	$\lambda = 0.1$	$\lambda = 0.05$	$\lambda = 0.1$	$\lambda = 0.05$	$\lambda = 0.1$	$\lambda = 0.05$
$V_0 + \sqrt{\xi}V_1$	75.55	78.15	21.94	24.6	0.19	0.141
V^ε	75.62	77.00	22.63	23.48	0.187	0.134
V_{bs}	74.82		25.52		0.47	
V_{market}	77.25		23.4		0.13	

From the numerical results, we see that the approximation procedure works really well even when ξ and λ are not so small such as $\xi = 0.5, \lambda = 0.1$. Fixing the value of ξ, we notice that whenever the value of the price of volatility risk λ takes a small value, the price obtained by our representation converges to the market value price. On the other hand, if we use small values of ξ, the proposed method's error compared to the traditional Black Scholes model is very small. Additionally, the outcomes demonstrate the convergence mentioned in the Theorem 4.1.

As expected, the suggested optimized artificial neural network approach provides a better perspective to describe the behaviour of the presented option pricing model, and may be chosen owing to its reliability and accuracy while requiring a minimal computational effort, since the CPU time in each manipulation was very small (cpu=0.0509s).

7 Conclusion

This paper explores the behavior of American-style option prices in the worst case scenario under an uncertain volatility model with stochastic bounds represented by the Heston model. The market price of volatility risk is taken into consideration. The findings of this study contribute to a greater understanding of the advantages of uncertain volatility models with stochastic bounds over models with constant bounds for option pricing and risk management. The impact of uncertain volatility with stochastic bounds on option price is analyzed and compared to the results obtained from the Black-Scholes model and real market value using our proposed neural network approach, which combines radial basis function neural network and finite difference method. The results were found to be closer to market values than traditional methods and the convergence time was short, leading to improved efficiency and accuracy of the RBF-NN method.

Appendix

1

Proposition 2. The process S has finite moments of any order uniformly in $0 \leqslant \varepsilon^*$ for $t \in [t_0, T]$. Therefore we have

$$\mathbb{E}_{t,s,\nu}[\int_{t_0}^{T} \mid S_t \mid^k dt] \leqslant \mathbb{E}_{0,s,\nu}[\int_{0}^{T} \mid S_t \mid^k dt] \leqslant M_k(T, s, \nu), \tag{45}$$

where $M_k(T, s, \nu)$ may depend on (k, T, s, ν) but not on ε^*.

Proof. The proof is analogous to the proof of Proposition A.3 in Appendix A of [9]. □

Proposition 3. The process ν has finite moments of any order uniformly in $0 \leqslant \varepsilon^* \leqslant 1$ for $t \in [t_0, T]$. Thus:

$$\mathbb{E}_{t,s,\nu}[\int_{t_0}^{T} \mid \nu_t \mid^k dt] \leqslant \mathbb{E}_{0,\nu}[\int_{0}^{T} \mid \nu_t \mid^k dt] \leqslant N_k(T, \nu), \tag{46}$$

where $N_k(T, \nu)$ may depend on (k, T, ν) but not on ε^*.

Proof. The following moment estimate is obtained by adapting the standard moment estimates for asset price processes in stochastic volatility models. The proof follows arguments similar to those presented in Fouque et al. [7]. □

2

Proposition 4. Let S^ε satisfies the stochastic differential equation 9 and S^0 satisfies the equation 24, then:

$$\mathbb{E}(S_T^\varepsilon - S_T^0)^2 \leqslant K_0 \xi, \tag{47}$$

where K_0 is a positive constant independent of ε.

Proof. We integrate the Equation 9 and Equation 24, then we get:

$$S_T^\varepsilon = \int_{t_0}^{T} r S_t^\varepsilon dt + \int_{t_0}^{T} \beta_t \sqrt{\nu} S_t^\varepsilon dW_t, \tag{48}$$

and

$$S_T^0 = \int_{t_0}^{T} rS_t^0 dt + \int_{t_0}^{T} \beta\sqrt{\nu_0} S_t^0 dW_t. \tag{49}$$

The difference of Equation 48 and Equation 49 is given by

$$S_T^\varepsilon - S_T^0 = \int_{t_0}^{T} r(S_t^\varepsilon - S_t^0)dt + \int_{t_0}^{T} \beta\sqrt{\nu}(S_t^\varepsilon - S_t^0)dW_t. \tag{50}$$

Let $X_t = S_t^\varepsilon - S_t^0$, so we have $X_{t_0} = 0$ and

$$X_T = \int_{t_0}^{T} rX_t dt + \int_{t_0}^{T} \beta(\sqrt{\nu} - \sqrt{\nu_0})X_t dW_t. \tag{51}$$

Then we calculate the expectation X_t,

$$\begin{aligned}
\mathbb{E}_{(t,s,\nu)}[X_T^2] &\leqslant 2\mathbb{E}_{(t,s,\nu)}[(\int_{t_0}^{T} rX_t dt)^2 + (\int_{t_0}^{T} \beta(\sqrt{\nu} - \sqrt{\nu_0})X_t dW_t)^2] \\
&\leqslant \int_{t_0}^{T} (2r^2T)\mathbb{E}_{(t,s,\nu)}[X_t^2]dt + 2\beta^2 \int_{t_0}^{T} \mathbb{E}_{(t,s,\nu)}[(\sqrt{\nu} - \sqrt{\nu_0})^2 (S_t^\varepsilon)^2]dt.
\end{aligned}$$

Note that using the result that S_t and ν_t have finite moments uniformly in ε^*, we can show that $\int_{t_0}^{T} \mathbb{E}_{(t,s,\nu)}[(\sqrt{\nu} - \sqrt{\nu_0})^2 (S_t^\varepsilon)^2]dt \leqslant K\varepsilon^*$, where K is a function independent of ε^*, then we obtain the desired result

$$\mathbb{E}(S_T^\varepsilon - S_T^0)^2 = \mathbb{E}_{(t,s,\nu)}[X_T^2] \leqslant K_0\varepsilon^*.$$

□

3

Theorem 3.1. *(The Arzela-Ascoli Theorem) If a sequence $\{f_n\}_1^\infty$ in $C(X)$ is bounded and equicontinuous then it has a uniformly convergent subsequence.*

Theorem 3.2. *(The Arzela-Ascoli Theorem in High Derivative) Let f_l, $l = 1, 2, \ldots$, be $\mathbb{C}^\infty$ functions defined on an open set $D \subset \mathbb{R}^n$. Assume that $sup_{l \geqslant 1} sup_{x \in K} \mid \partial_x^\alpha f_l \mid$ is finite for each multi-index α and for each compact subset K in D. Then we have: There exist a subsequence $\{f_{l(i)}\}_{i=1}^\infty$ of $\{f_l\}_{l=1}^\infty$ and an $f \in \mathcal{C}^\infty(D)$ such that for any multi-index α, $\partial_x^\alpha f_{l(i)}$ converges uniformly to $\partial_x^\alpha f$ on any compact subset in D as i tend to infinity.*

Proof. The proof is given in [10]. □

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How to Cite: Zaineb El Kharrazi[1], Sahar Saoud[2], Zouhir Mahani[3], *Implementing an Optimized Neural Network to Price American Option under Uncertain Volatility Model with Stochastic Bounds*, Journal of Mathematics and Modeling in Finance (JMMF), Vol. 7, No. 1, Pages:201–224, (2027).