

Valuation of Variable Annuities with Guaranteed Minimum Withdrawal Benefits under Continuous Withdrawal Strategies

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Abstract:

The life insurance industry has experienced substantial growth in recent years, accompanied by the development of a wide range of insurance and retirement products. Among these products, variable annuities (VAs) with Guaranteed Minimum Withdrawal Benefits (GMWBs) have attracted considerable attention because they combine the investment features of variable annuities with guarantees that protect policyholders against adverse market performance. In particular, a GMWB rider allows the policyholder to withdraw a predetermined amount periodically while providing a guarantee designed to recover the initial investment, subject to the contractual conditions. In this paper, we study the mathematical modeling and valuation of variable annuities with GMWB riders. A pricing framework for the associated guarantee is formulated, and numerical approximations are obtained using finite difference methods. Furthermore, the effects of various model and contract parameters on the value of the GMWB rider are investigated through numerical experiments.

Keywords: Variable Annuities, Guaranteed Minimum Withdrawal Benefit (GMWB), Life Insurance, Insurance Fees, Finite Difference Method.

MSC2010 Classification: 65M70; 35K15; 91G80.

1 Introduction

Increasing life expectancy, together with rising living and health-care costs, has made retirement periods both longer and more financially demanding. Consequently, financial products that provide long-term investment opportunities while offering protection against market and longevity risks have received increasing attention. Variable annuities (VAs) constitute an important class of retirement and insurance products whose benefits depend, at least in part, on the performance of investment options selected by the policyholder [3].

Variable annuities may incorporate various types of guarantees designed to protect policyholders against unfavorable market conditions. One of the most widely studied

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guarantees is the Guaranteed Minimum Withdrawal Benefit (GMWB). A variable annuity with a GMWB rider allows the policyholder to withdraw funds periodically, typically on an annual or semiannual basis, while providing a contractual guarantee on the amount that may be withdrawn over the lifetime of the contract. Under specified contractual conditions, the guarantee enables the policyholder to recover the initial premium through a sequence of withdrawals, even when the underlying investment account performs poorly.

The valuation and hedging of guarantees embedded in variable annuity contracts have been extensively investigated in the literature. Coleman et al. [7] considered risk-minimization hedging strategies for variable annuities in the presence of both equity and interest-rate risks. Risk-neutral option-pricing techniques for the valuation of Guaranteed Minimum Death Benefits (GMDBs) embedded in variable annuities were introduced in [4]. Horneff et al, develop the optimal withdrawal behavior and sensitivity to factors such as mortality, fees, portfolio choice, and guaranteed withdrawal rates[2]. Milevsky and Salisbury [5] developed a pricing framework for variable annuities with GMWBs under both static and dynamic withdrawal strategies. Under a static withdrawal strategy, the policyholder follows a predetermined withdrawal schedule, with withdrawals typically made at the contractual rate. In contrast, under a dynamic strategy, the policyholder is allowed to adjust the withdrawal decisions in response to prevailing financial conditions. The resulting optimal withdrawal problem seeks to determine a strategy that maximizes the value of the contract to the policyholder. Such flexibility introduces additional complexity into the valuation problem and generally leads to an optimal-control formulation. Kwok et al. [6] employed finite difference techniques for the numerical valuation of variable annuities under dynamic withdrawal strategies.

In this paper, we develop a mathematical formulation for the valuation of variable annuities with GMWB riders. The corresponding pricing equations are derived within an appropriate valuation framework and subsequently approximated using finite difference techniques. Particular attention is devoted to the numerical behavior of the resulting model and to the sensitivity of the contract value to changes in relevant financial and contractual parameters. Numerical experiments are presented to illustrate the performance of the proposed approach and to quantify the influence of these parameters on the value of the GMWB guarantee.

2 Model Formulation for Continuous Withdrawals

In this section, we formulate the valuation problem for a variable annuity (VA) contract equipped with a Guaranteed Minimum Withdrawal Benefit (GMWB) rider under the assumption that withdrawals can be made continuously over time. The GMWB rider provides the policyholder with the right to withdraw funds from the contract according to a prescribed set of contractual rules, while guaranteeing a minimum level of withdrawals even when the underlying investment account

performs poorly. Therefore, the contract combines participation in the financial market through the variable annuity account with downside protection provided by the insurer. To describe the contract, we distinguish between two state variables: the personal investment account and the guarantee account. Let W_t denote the market value of the policyholder's personal variable annuity account at time t , and let A_t denote the remaining balance of the GMWB guarantee account. At the inception of the contract, $t = 0$, the policyholder pays an initial premium w_0 , which is invested in the underlying fund. We assume that both accounts are initially equal to the premium, i.e.,

$$W_0 = A_0 = w_0. \quad (1)$$

It is important to emphasize that W_t and A_t have different financial interpretations. The personal account W_t represents the actual market value of the investment and is therefore affected by the performance of the underlying risky asset, the insurance fees charged by the insurer, and withdrawals made by the policyholder. In contrast, the guarantee account A_t is a bookkeeping variable used to keep track of the remaining amount available under the GMWB guarantee. Hence, A_t is not directly invested in the financial market and is not affected by fluctuations in the underlying asset price. Instead, its value decreases as withdrawals are made. Let γ_t denote the instantaneous withdrawal rate selected by the policyholder at time t . Under the continuous-withdrawal formulation, the guarantee account evolves according to

$$A_t = A_0 - \int_0^t \gamma_s \, ds, \quad 0 \leq \gamma_s \leq \lambda, \quad (2)$$

where γ_t is control variable representing the continuous withdrawal rate and λ is the upper bound of γ_t . Let fund follows a Geometric Brownian Motion and W_t denote the value of the personal variable annuity account, by using the assumptions of the Black-Scholes model, we have

$$dW_t = (r - \alpha)W_t dt + \sigma W_t dB_t + dA_t, \quad W_t > 0, \quad (3)$$

where $\lambda > 0$ denotes the maximum admissible withdrawal rate specified by the contract. Thus, γ_t acts as a control variable and determines the rate at which the remaining guarantee balance is consumed. The constraint

$$0 \leq \gamma_t \leq \lambda$$

restricts the withdrawal strategy to the set of admissible controls. In particular, $\gamma_t = 0$ corresponds to making no withdrawal at time t , whereas $\gamma_t = \lambda$ corresponds to withdrawing at the maximum contractual rate.

Basically, the policyholder chooses the optimal withdrawal policy dynamically to maximize the present value of cash flows generated from holding the variable annuity policy. By this assumption the no-arbitrage value V of the variable annuity

with GMWB is given by

$$V(W, A, t) = \max_{\gamma} E_t \left(e^{-r(T-t)} \max(W_T, 0) + \int_t^T e^{-r(u-t)} f(\gamma_u) du \right). \quad (4)$$

From the piecewise linear properties of the function $f(\gamma_t)$, as shown in [6], the Hamilton-Jacobi-Bellman (HJB) equation in stochastic control problems leads to the following equations

$$V_t = -\frac{1}{2}\sigma^2 W^2 \frac{\partial^2 V}{\partial W^2} - (r - \alpha)W \frac{\partial V}{\partial W} + rV - \max_{\gamma_t} \left(f(\gamma_t) - \gamma \left(\frac{\partial V}{\partial W} + \frac{\partial V}{\partial A} \right) \right). \quad (5)$$

We notice that the account balance of the guarantee becomes insignificant when $A = 0$ or $W \rightarrow \infty$. For these cases, the value of the variable annuity becomes $We^{-\alpha(T-t)}$. In summary, final and boundary condition for the PDE problem (5) take the form

$$V(W, A, T) = \max(W, (1 - k)A), \quad (6)$$

$$V(W, 0, t) = We^{-\alpha(T-t)}, \quad \lim_{W \rightarrow \infty} V(W, A, t) = We^{-\alpha(T-t)}, \quad (7)$$

$$V(0, A, t) = (1 - k) \max(A - Gt', 0) + \frac{G}{r} \left(1 - e^{-r \min(A/G, t')} \right), \quad (8)$$

where $t' = \min \left(-\frac{\ln(1-k)}{r}, T - t \right)$.

$$V(W, A, t) = \max_{\gamma} E_t \left(e^{-r(T-t)} \max(W_T, 0) + \int_t^T e^{-r(u-t)} f(\gamma_u) du \right). \quad (9)$$

In equation (9), the function $f(\gamma_t)$ denote the rate of cash flow received by the policyholder due to continuous rate. According to [6], invoking the penalty function

$$f(\gamma_t) = \begin{cases} \gamma_t & 0 \leq \gamma_t \leq G \\ \gamma_t - k(\gamma_t - G) & \gamma_t > G \end{cases} \quad (10)$$

leads to a penalty charge k if the withdrawal rate exceeds the fixed contract withdrawal rate G .

3 Implementation and Numerical Results

To obtain a numerical approximation of PDE (5), the spatial domain is discretized by introducing a set of uniformly distributed grid points. The spatial derivatives appearing in the governing equation are then approximated using standard finite difference formulas of appropriate order. In particular, the first and second-order spatial derivatives are replaced by their corresponding finite difference approximations at the interior grid points, while the prescribed boundary conditions are

incorporated at the boundary nodes. This spatial discretization reduces the original continuous problem to a finite-dimensional semi-discrete system and provides a convenient and computationally efficient framework for approximating the value of the variable annuity contract over the computational domain.

After discretization of the spatial derivatives, the original PDE is transformed into a system of ordinary differential equations with respect to the time variable, following the method-of-lines approach. The resulting semi-discrete system can be written in the general form $dV/dt = F(t, V)$, where V contains the numerical approximations of the contract value at the spatial grid points. This system can subsequently be integrated in time using standard numerical ODE techniques. In the present implementation, the computations can be carried out in MATLAB using the built-in solver `ode45`, which employs an adaptive explicit Runge–Kutta scheme. Alternatively, conventional time-stepping methods, such as explicit or implicit Euler and Crank–Nicolson schemes, may be employed depending on the desired accuracy, stability properties, and computational efficiency. The resulting fully discrete solution provides an approximation of the GMWB contract value at each spatial node and time level.

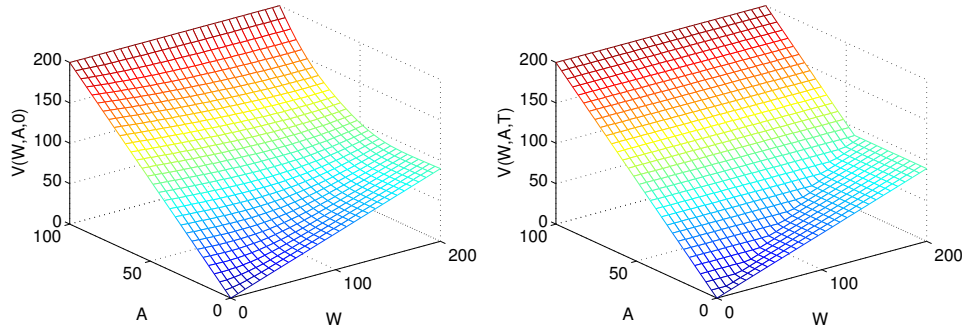


Figure 1: From left to right, the value of variable annuity with GMWB as a function of W and A at the maturity T and $t = 0$

For the numerical illustrations, we use the parameter values from [6] given by $r = 0.05$, $\alpha = 0.03$, $\sigma = 0.3$, $k = 0.1$, $\lambda = A_0 = 100$, $G = 10$, and $T = 10$. The computational domain is defined by $[0, W_{\max}] \times [0, A_0]$ and $W_{\max} = 200$. The value of variable annuity with GMWB as a function of W and A is shown in figure 1 at the maturity T and $t = 0$. For example, the value of the variable annuity rider is 92.23 for the balance guarantee level $A = 85$ and $W = 90$.

The insurance company charges a proportional insurance fee, denoted by α , which is incorporated into the dynamics of the policyholder's investment account. The fair, or no-arbitrage, fee is defined as the value of α for which the initial value of the variable annuity contract coincides with the initial premium w_0 paid by the policyholder. In other words, the fair fee is determined by solving the condition

$$V(w_0, w_0, 0; \alpha) = w_0,$$

where the dependence of the contract value on the insurance fee α is made explicit. This condition ensures that, under the assumed model and risk-neutral valuation framework, the present value of the benefits provided by the GMWB rider is consistent with the initial premium charged for the contract. Figure 2 illustrates the value of the variable annuity contract as a function of the initial account value W at $t = 0$ for different values of the proportional insurance fee. As expected, the contract value depends on the level of α , since an increase in the insurance fee reduces the effective growth rate of the investment account and consequently affects the value of the guarantee. From the numerical results presented in Figure 2, the fair insurance fee is approximately $\alpha = 0.04$, for which the contract value at inception is approximately equal to the initial premium w_0 . Thus, $\alpha \approx 0.04$ represents the estimated break-even fee under the model assumptions and parameter values considered in this study.

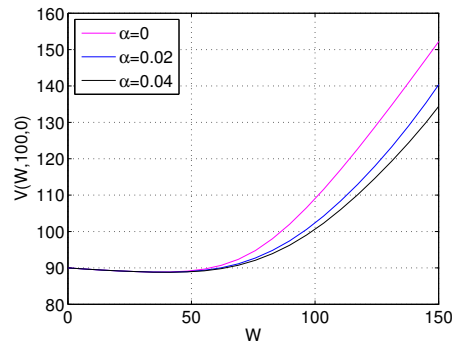


Figure 2: The value of the variable annuity with GMWB as a function of W at $t = 0$, $A = 100$, for different values of the insurance fee α .

4 Conclusion

In this paper, we have developed a mathematical and numerical framework for the valuation of variable annuities with Guaranteed Minimum Withdrawal Benefits (GMWBs) under a continuous withdrawal strategy. The contract was formulated as a stochastic optimal control problem in which the policyholder dynamically determines the withdrawal rate subject to the contractual constraints. The evolution of the investment account was modeled under the risk-neutral measure using a geometric Brownian motion with a proportional insurance fee, while a separate guarantee account was introduced to track the remaining withdrawal benefit. The resulting equation was discretized in the spatial variables using a finite difference scheme,

and the resulting system of ordinary differential equations was solved numerically using standard time-integration techniques. Numerical experiments were conducted to investigate the dependence of the contract value on the principal financial and contractual parameters. In particular, the results demonstrate the significant influence of the proportional insurance fee on the value of the variable annuity and provide an estimate of the fair insurance fee under the model assumptions.

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