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Efficient calculation of all steady states in large-scale overlapping generations models

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Abstract:
In this paper, we address the problem of analyzing and computing all steady states of an overlapping generation (OLG) model with production and many generations. The characterization of steady states coincides with a geometrical representation of the algebraic variety of a polynomial ideal, and, in principle, one can apply computational algebraic geometry methods to solve the problem. However, it is infeasible for standard methods to solve problems with a large number of variables and parameters. Instead, we use the specific structure of the economic problem to develop a new algorithm that does not employ the usual steps for the computation of Gröbner basis such as the computation of successive S-polynomial and expensive division.
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1 Introduction

In this paper, we show that methods from computational algebraic geometry can be applied to examine the multiplicity of equilibria in realistically calibrated computable equilibrium models. In particular, we develop a method to compute all steady states of dynamic models with overlapping generations and production. The method can be applied to models with around 60 generations, making it potentially relevant for applied work.

[14] argue that in many economic models, the equilibrium can be characterized by a system of polynomial equations and that, therefore, the computation of Gröbner bases can become an important tool for computational economics. However, as [16] point out, the applicability of Gröbner bases to applied economic problems

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is severely limited by the computational complexity of standard algorithms that are used to compute the basis. In particular, the well-known Buchberger algorithm's (see [5]) computational complexity is doubly exponential in the number of variables. In this paper, we show how to overcome this problem for a specific economic problem, namely the computation of all steady states in OLG models with Cobb-Douglas production. We show that methods from computational algebraic geometry can be used for large-scale problems if one adapts the algorithm to the specific economic problem under consideration.

Applied general equilibrium models with overlapping generations are used in many areas of modern economics, in particular in macroeconomics and public finance. Early work on such models include [19]'s, [3], [3], and [1]. These models have been extended over time to include multiple regions, multiple goods, and demographic change (see, e.g., [13]). Moreover, there is now literature in climate-change economics that utilizes OLG models to model the intergenerational externality that arises from the burning of fossil fuels (see, e.g. [6], [9], or [14]).

Unfortunately, the usefulness of the predictions of these equilibrium models and the ability to perform sensitivity analysis are seriously challenged in the presence of multiple equilibria. In particular, [11] show that the overlapping generations model might exhibit robust indeterminacy and that there might be infinitely many competitive equilibria. A simpler but related question concerns the number of steady states of the overlapping generations economy. If there is a unique steady state, the analysis of competitive equilibria becomes much easier and indeterminacy can often be ruled out ([11]). Unfortunately, it is now well understood that even when one focuses on steady states, sufficient assumptions for the global uniqueness of competitive equilibria are too restrictive to be applicable to models used in practice. However, it remains an open question whether the multiplicity of steady states is a problem that is likely to occur in so-called realistically calibrated overlapping generations models. Given specifications for endowments and preferences, the fact that the known sufficient conditions for uniqueness do not hold obviously does not imply that there must be several competitive equilibria. While [11]) construct robust examples of realistically calibrated OLG models with three-period lived agents where there are three real steady states, [15] argue that at least for pure exchange economies, multiplicity is not too common. They consider relatively small examples but search through a wide range of endowments to demonstrate that in many specifications, there is a unique steady state.

In this paper, we extend the analysis in [15] to OLG models with production and long-lived agents. We provide computational methods to find all steady-state equilibria in the model and apply the algorithm to a realistically calibrated model with agents that live for 60 periods. Clearly, standard algorithms for the computation of a Gröbner basis cannot be applied for the large system that arises from a 60-period model. Our strategy is instead to identify a single univariate polynomial that determines one of the unknown variables and then use the "Shape Lemma"([4]) to build up a Gröbner basis from this univariate polynomial. The crucial step, finding the right univariate polynomial to start with, is conducted by finding the trace of S-polynomials and performing the division algorithm theoretically. This is obviously a special feature of this model, and it is subject to further research to investigate if the strategy can be extended to other equilibrium models in economics.

One crucial element of the model we consider in this paper is that once steady state capital is determined, all other variables can be obtained from a convex optimization problem (or even have an analytic solution) Therefore, one can search for all steady states by simply plotting aggregate excess demand as a function of capital. Since this is one-dimensional, the number of solutions can often be determined without reasonable doubt. However, the algebraic approach taken in this paper still has several advantages. First, there are (perhaps non-generic but still relevant) cases where there are two equilibria, and one cannot determine numerically whether there are one or two equilibria. Second, as explained in detail in [14], using Gröbner basis, one can often make general statements about the number of equilibria (i.e., statements that hold for all, or at least an interesting set of endowments). We illustrate this point with examples below, but a general examination of the issue is left to future work.

The rest of the paper is organized as follows. In Section 2, we briefly review some basic concepts from algebraic geometry. In Section 3, we describe the economic model and define a steady state equilibrium. Sections 4 and 5 develop algorithms to find all real solutions. Section 6 provides examples that illustrate the usefulness of our approach. All of the proofs of theorems and propositions are collected in Appendix 1. Appendix 2 reports the full equilibrium of an example t of an OLG model with 60 generations.

2 Gröbner bases

In this section, we state the required mathematical facts of our analysis and introduce the necessary notation – see [5] for a thorough introduction. Let $\mathbb{K}$ be a field and $x_1, \dots, x_n$ be n (algebraically independent) variables. Each power product $x_1^{a_1} \cdots x_n^{a_n}$ is called a monomial where $a_1, \dots, a_n \in \mathbb{Z}_{\geq 0}$. Because of simplicity, we abbreviate such monomials by $\mathbf{x}^a$ where $\mathbf{x}$ is used for the sequence $x_1, \dots, x_n$ and $a = (a_1, \dots, a_n)$. We can sort the set of all monomials over $\mathbb{K}$ by so-called monomial orderings, which are defined as follows.

Definition 2.1. The total ordering $\prec$ on the set of monomials is called a monomial ordering whenever for each monomials $\mathbf{x}^a, \mathbf{x}^b$ and $\mathbf{x}^c$ we have:

- $\mathbf{x}^a \prec \mathbf{x}^b \Rightarrow \mathbf{x}^c\mathbf{x}^a \prec \mathbf{x}^c\mathbf{x}^b$, and
- $\prec$ is well-ordering.

We thank the anonymous referee for pointing this out to us. This is an important point that we overlooked in an earlier version of this paper.

There are infinitely many monomial orderings; each one is convenient for a different type of problem. Among them, we want to point out the pure and graded reverse lexicographic orderings denoted by $\prec_{lex}$ and $\prec_{grevlex}$. We say that

- $\mathbf{x}^a \prec_{lex} \mathbf{x}^b$ whenever $a_1 = b_1, \ldots, a_i = b_i$ and $a_{i+1} < b_{i+1}$ for an integer $1 \leq i < n$.
- $\mathbf{x}^a \prec_{grevlex} \mathbf{x}^b$ if
$$\sum_{i=1}^{n} a_i < \sum_{i=1}^{n} b_i$$
breaking ties when there exists an integer $1 \leq i < n$ such that
$$a_n = b_n, \ldots, a_{n-i} = b_{n-i} \text{ and } a_{n-i-1} > b_{n-i-1}.$$

It is worth noting that the former has many nice theoretical properties while the latter is more useful for computational purposes.

Each $\mathbb{K}-$linear combination of monomials is called a polynomial on $x_1, \ldots, x_n$ over $\mathbb{K}$. The set of all polynomials has the ring structure with usual polynomial addition and multiplication and is called the polynomial ring on $x_1, \ldots, x_n$ over $\mathbb{K}$ and denoted by $\mathbb{K}[x_1, \ldots, x_n]$ or just by $\mathbb{K}[\mathbf{x}]$. For given polynomials $f_1, \ldots, f_k$, the set
$$\mathcal{I} = \{\sum_{i=1}^{k} h_i f_i : h_i \in \mathbb{K}[\mathbf{x}]\} = \langle f_1, \ldots, f_k \rangle,$$
is called the *ideal* generated by $f_1, \ldots, f_k$.

Let f be a polynomial, and let $\prec$ be a monomial ordering. The greatest monomial w.r.t. $\prec$ contained in f is called the leading monomial of f, denoted by $\mathrm{LM}(f)$ and the coefficient of $\mathrm{LM}(f)$ is called the leading coefficient of f which is denoted by $\mathrm{LC}(f)$. Further, if F is a set of polynomials, $\mathrm{LM}(F)$ is defined to be $\{\mathrm{LM}(f) | f \in F\}$ and if $\mathcal{I}$ is an ideal, $in(\mathcal{I})$ is the ideal generated by $\mathrm{LM}(\mathcal{I})$ and is called the initial ideal of $\mathcal{I}$. We remind the reader of the concept of Gröbner basis of a polynomial ideal.

Definition 2.2. Let $\mathcal{I}$ be a polynomial ideal of $K[\mathbf{x}]$ and $\prec$ be a monomial ordering. The finite set $G \subset \mathcal{I}$ is called a Gröbner basis of $\mathcal{I}$ if for each non zero polynomial $f \in \mathcal{I}$, $\mathrm{LM}(f)$ is divisible by $\mathrm{LM}(g)$ for some $g \in G$.

Using the well-known Hilbert basis theorem (see, e.g., [5]), it is shown that each polynomial ideal possesses a Gröbner basis with respect to each monomial ordering. There are several algorithms also to compute Gröbner basis. The first and simplest one is Buchberger's algorithm, while the most efficient known algorithms are Faugère's F_5 algorithm ([8]) and other signature-based algorithms such as $\mathrm{G}^2\mathrm{V}$ ([10]). It is worth noting that Gröbner basis of an ideal is not necessarily unique. To obtain a unique representation, we need to define the concept of a reduced Gröbner basis. The reduced Gröbner basis of an ideal is unique up to the monomial ordering.

Definition 2.3. Let G be a Gröbner basis for the ideal $\mathcal{I}$ w.r.t. $\prec$. Then G is called a reduced Gröbner basis of $\mathcal{I}$ whenever each $g \in G$ is monic, i.e. $\mathrm{LC}(g) = 1$ and none of the monomials appearing in g is divisible by $\mathrm{LM}(h)$ for each $h \in G \setminus \{g\}$.

One of the most important applications of Gröbner basis is its help to solve a polynomial system. Let

$$\begin{cases} f_1 &= 0 \\ & \vdots \\ f_k &= 0 \end{cases}$$

be a polynomial system and $\mathcal{I} = \langle f_1, \ldots, f_k \rangle$ be the ideal generated by $f_1, \ldots, f_k$. We define the affine variety associated with the above system (or equivalently with the ideal $\mathcal{I}$) to be

$$\mathbf{V}(\mathcal{I}) = \mathbf{V}(f_1, \ldots, f_k) = \{\tau \in \overline{\mathbb{K}}^n | f_1(\tau) = \cdots = f_k(\tau) = 0\}$$

where $\overline{\mathbb{K}}$ is used to denote the algebraic closure of $\mathbb{K}$. Now let G be a Gröbner basis for $\mathcal{I}$ with respect to an arbitrary monomial ordering. As an interesting fact, $\mathcal{I} = \langle G \rangle$ which implies that $\mathbf{V}(\mathcal{I}) = \mathbf{V}(G)$. This is the key computational trick to solving a polynomial system. It is useful to illustrate this with a numerical example.

Example 2.1. We are going to solve the following polynomial system:

$$\begin{cases} x^2 - xyz + 1 &= 0 \\ y^3 + z^2 - 1 &= 0 \\ xy^2 + z^2 &= 0 \end{cases}$$

By the nice properties of pure lexicographical ordering, the reduced Gröbner basis of the ideal $\mathcal{I} = \langle x^2 - xyz + 1, y^3 + z^2 - 1, xy^2 + z^2 \rangle \subset \mathbb{Q}[x, y, z]$ has the form

$$G = \{g_1(z), x - g_2(z), y - g_3(z)\},$$

with respect to $z \prec_{lex} y \prec_{lex} x$, where

$$\begin{cases} g_1(z) &= z^{15} - 3z^{14} + 5z^{12} - 3z^{10} - z^9 - z^8 + 4z^6 - 6z^4 + 4z^2 - 1 \\ g_2(z) &= 2z^{14} - 9z^{13} + 11z^{12} + 2z^{11} - 7z^{10} - 3z^9 + 2z^8 - z^7 + 4z^6 + \\ & +7z^5 - 10z^4 - 6z^3 + 11z^2 + 2z - 4 \\ g_3(z) &= z^{13} - 3z^{12} + z^{11} + 2z^{10} + z^9 - z^8 - 2z^6 + 2z^4 - z^3 - 3z^2 + 1. \end{cases}$$

This special form of Gröbner basis for this system allows us to find $\mathbf{V}(G)$ by solving only one univariate polynomial $g_1(z)$ and putting the roots into the two last polynomials in G.

The existence of univariate polynomials depends on the dimension of the ideal (see [4]), which is defined as follows.

Definition 2.4. Let $\mathcal{I} \subset \mathbb{K}[\mathbf{x}]$ be an ideal and $\mathbf{u}$ be a set of variables. We call $\mathbf{u}$ an independent set with respect to $\mathcal{I}$, whenever $\mathcal{I} \cap \mathbb{K}[\mathbf{u}] = \{0\}$. The cardinality of a maximal independent set with respect to $\mathcal{I}$ is called the dimension of $\mathcal{I}$. Furthermore, we say that $\mathcal{I}$ is a zero-dimensional ideal when the dimension of $\mathcal{I}$ is zero, and positive dimensional otherwise.

3 The overlapping generations model

We now introduce the economic model and define a steady state equilibrium. In the next section, we will show that this naturally gives rise to a zero-dimensional ideal whose Gröbner basis we will compute.

We consider a discrete-time, infinite-horizon model with a single (representative) firm and overlapping generations of consumers. Time is indexed by $t = 0, \ldots, \infty$; there are three commodities: a single consumption good, capital, and labor.

3.1 Firms

The consumption good is produced by the following Cobb-Douglas production function

$$Y_t = K_t^{\alpha} L_t^{1-\alpha} + (1-\delta)K_t \tag{1}$$

where Y_t is the final output, whose price is normalized to 1, K_t, L_t, reference the two inputs used to produce this output, namely capital, and labor. The capital share is α, depreciation is denoted by δ.

Profit maximization requires

$$\alpha K_t^{\alpha-1} L_t^{1-\alpha} = r_t + \delta \tag{2}$$

and

$$(1-\alpha) K_t^{\alpha} L_t^{1-\alpha} = w_t \tag{3}$$

where r_t, and w_t reference the real interest rate and the real wage rate.

3.2 Households

Each household lives for A periods. Households born at time t maximize utility defined by

$$U_t(c) = U(c) = \sum_{j=1}^{A} \beta^j v(c_{t+j-1,j}) \tag{4}$$

subject to a budget constraints of the form

$$k_{t+1,j+1} = (1+r_t)k_{t,j} + w_t l_j - c_{t,j},\ c_{t,j} \geq 0 \tag{5}$$

As is standard, we assume $\alpha \in (0,1)$ and $\delta \in [0,1]$.

where $c_{t,j}$, $k_{t,j}$, l_j correspond to consumption, assets and labor supply of generation j at time t.

We assume that households do not value leisure and normalize

$$\sum_{j=1}^{A} l_j = 1.$$

For our analysis, we need to characterize an optimal solution to an agent's optimization problem by polynomial equations. It is therefore convenient to focus on the case where agents have CRRA utility of the form

$$v(c) = \begin{cases} -c^{-\gamma} & \gamma > 0 \\ \ln(c) & \gamma = 0. \end{cases}$$

For simplicity, we ignore the case of a coefficient of risk-aversion less than one.

3.3 Competitive equilibrium and steady states

A competitive equilibrium consists of a sequence of prices $(r_t, w_t)_{t=0}^{\infty}$ as well as choices for all agents $(k_{t+j-1,j}, c_{t+j-1,j})_{j=1,\ldots,A,t=0,\ldots,\infty}$ such that each agent t maximizes (4) subject to the budget constraint (5) holding for all $j = 1, \ldots, A$, the firm maximizes profits, and such that all markets clear, i.e.

$$L_t = 1, K_t = \sum_{j=1}^{A-1} k_{t+1,j} \text{ for all } t = 0, \ldots, \infty.$$

As we explained in the introduction, in order to analyze competitive equilibria in this model, it is useful first to analyze the steady states (or steady-state equilibria). These are competitive equilibria where prices and choices are constant over time. Formally we have the following definition.

Definition 3.1. A steady state equilibrium consists of asset holdings and consumptions $(c_a, k_a)_{a=1}^{A}$, as well as prices r and w such that

$$(c_a, k_a)_{a=1}^{A} \in \arg \max_{c \in \mathcal{R}_+^A, k \in \mathcal{R}^A} U(c) \text{ s.t. } k_{j+1} = (1+r)k_j + wl_j - c_j,\ j = 1, \ldots, A$$

and such that

$$\alpha K^{\alpha-1} L^{1-\alpha} = r + \delta$$

and

$$(1-\alpha) K^{\alpha} L^{1-\alpha} = w,$$

with $L = 1, K = \sum_{j=1}^{A} k_j$.

We now develop an algorithm to compute all steady states. In the following, we will refer to them simply as "equilibria". For this, we will assume that all parameters, $\alpha, \beta, \gamma, \delta, l_1, \ldots, l_A$ are rational numbers – since the rationals are dense in the reals, this is certainly without loss of generality.

4 The Main Idea

In this section, we construct a system of polynomial equations which has the property that the equilibria of the OLG model are solutions to the equations. Then, we explain the general shape of a Gröbner basis for this system and develop an algorithm to compute it.

From Definition 3.1 we obtain the following system of equations with $2(A+1)$ variables (and the same number of equations).

$$\begin{cases} c_a^{-\gamma-1} &= \beta(1+r)c_{a+1}^{-\gamma-1} \ (a=1,\dots,A-1) \\ 1+r &= \alpha K^{\alpha-1}+1-\delta \\ c_a &= k_{a-1}(1+r)+wl_a-k_a, \ (a=1,\dots,A) \\ w &= (1-\alpha)K^{\alpha} \\ K &= \sum_{a=0}^{A} k_a \\ k_0 &= k_A = 0 \end{cases} \tag{6}$$

Recall that A is a positive integer, $\gamma \in [0,+\infty)$, $\beta \in (0,+\infty)$, $\alpha \in (0,1)$, $\delta \in [0,1]$ and for each a, $l_a \geq 0$. Throughout we normalize $\sum_{a=1}^{A} l_a = 1$.

We start with the trivial case of $\gamma = 0$ and then consider the more interesting and difficult case of positive γ. In each case, we perform the following steps.

- We find a suitable change of variables so that System (6) can be converted to a system of polynomial equations.
- We construct a univariate polynomial in one of the variables whose solutions in that variable coincide with the associated solutions of the whole system of equations.
- We develop an efficient method to construct the elements of Gröbner basis by a complete formulation, building up from the univariate polynomial we constructed above.

To explain the strategy in detail, we first consider the simplest case of log-utility. For the case without production [13] show that there must exist a unique competitive equilibrium (and therefore a unique steady state) when $\gamma \leq 0$. To the best of our knowledge, no such results are available for the case with production and only partial depreciation, but it is likely that there is a simple uniqueness proof also for this case. Our main reason for examining this case is that it allows us to introduce the basic idea in the simplest setting – the first equation in (6) becomes linear, which makes the argument much simpler to follow.

4.1 The case of $\gamma = 0$

Throughout this section, assume that $\gamma = 0$ and $\alpha = m/n$ where m, n are natural numbers and $m < n$. In order to write the equations as polynomials, we introduce a new auxiliary variable S such that $S^n = K$ and so $K^{\alpha-1} = S^{m-n}$. As $m-n<0$, we multiply the equation $r = \alpha K^{\alpha-1} - \delta$ by S^{n-m} to obtain $(r+\delta)S^{n-m} = \alpha$. We have the following polynomial system, which is obviously equivalent to System (6).

$$\left\{\begin{array}{rcl} c_{a+1} & = & \beta(1+r)c_a \quad (a=1,\ldots,A-1) \\ c_a & = & k_{a-1}(1+r)+wl_a-k_a \quad (a=1,\ldots,A) \\ (r+\delta)S^{n-m} & = & \alpha \\ w & = & (1-\alpha)S^m \\ K & = & \sum_{a=0}^{A} k_a \\ K & = & S^n \\ k_0 & = & k_A = 0 \end{array}\right. \tag{7}$$

In the sequel, we study the polynomial ideal associated with the system (7). Suppose that

$$\begin{array}{rcl} \mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}} & = & \langle f_1,\ldots,f_{A-1},g_1,\ldots,g_A,h_r,h_w,h_K,q_k\rangle \\ & \subset & \mathbb{R}[c_1,\ldots,c_A,k_0,\ldots,k_A,K,S,w,r] \end{array}$$

is the polynomial ideal associated to System (7), where $\vec{l}$ (in the index of $\mathcal{I}$) denotes the vector of labor endowments $l_1,\ldots,l_A$ and

$$\begin{array}{rcl} f_a & = & c_{a+1}-\beta(1+r)c_a \quad (a=1,\ldots,A-1), \\ g_a & = & c_a-k_{a-1}(1+r)-wl_a+k_a \quad (a=1,\ldots,A,g_0=k_0,g_A=k_A), \\ h_r & = & (r+\delta)S^{n-m}-\alpha, \\ h_w & = & w-(1-\alpha)S^m, \\ h_K & = & K-S^n, \\ q_k & = & K-\displaystyle\sum_{a=0}^{A} k_a. \end{array}$$

In what follows, we try to formulate a Gröbner basis for $\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}$ w.r.t. a lexicographical monomial ordering in which S is the smallest variable. To make the results more obvious, we state an illustrative example and continue it during the following propositions, theorems, and corollaries.

Example 4.1. Let $A = 3$, $\alpha = 1/2$ $(m=1, n=2)$, $\beta = 2$ and $\delta = 1$. So, the polynomial ideal associated with System (7) is

$$\begin{aligned}
\mathcal{I}_{3,1/2,2,1,(1/3,1/3,1/3)} = \langle f_1 &= c_2 - 2(1+r)c_1, \\
f_2 &= c_3 - 2(1+r)c_2, \\
g_1 &= c_1 - wl_1 + k_1, \\
g_2 &= c_2 - k_1(1+r) - wl_2 + k_2, \\
g_3 &= c_3 - k_2(1+r) - wl_3, \\
h_r &= (r+1)S - 1/2, \\
h_w &= w - 1/2S, \\
h_K &= K - S^2, \\
q_k &= K - (k_1+k_2)\rangle
\end{aligned}$$

In the following proposition, we provide two useful polynomials in $\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}$ which tends to observe the univariate polynomial, if there exists, in the desired Gröbner basis.

Proposition 4.1. Using the above notation we have

(i) $\Psi := \sum_{i=0}^{A-1} \beta^i (1+r)^{A-1} c_1 - \sum_{i=0}^{A-1} (1+r)^i l_{A-i} w \in \mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}$

(ii) $\Xi := (1-(\beta(1+r))^A)c_1 - (1-\beta(1+r))(rS^n + w) \in \mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}$

Example 4.2. For Example 4.1, one can easily observe that

$$\Psi = 7(1+r)^2 c_1 - (l_3 + (1+r)l_2 + (1+r)^2 l_1)w$$

and

$$\Xi = (1-(2(1+r))^3)c_1 - (1-2(1+r))(rS^2 + w)$$

and one can verify that both are elements of $\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}$.

The crucial step of the argument is now to derive a polynomial only in S that is an element of the ideal $\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}$. We have the following result.

Theorem 4.2. For each A, α, β, δ and each vector of labor-endowments, $\vec{l}$, there exist scalars $\lambda_0, \ldots, \lambda_{2(A-1)} \in \mathbb{Q}$ such that

$$\mathfrak{f}_{A,\alpha,\beta,\delta,\vec{l}} := S^m(n\delta S^{n-m} - m)((n\beta(\delta-1)+n)S^{n-m} - m\beta) \sum_{i=0}^{2(A-1)} \lambda_i S^{(n-m)i} \in \mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}.$$

The proof in the appendix shows the construction step by step.

Remark 4.3. We show in the proof of Theorem 4.2 that the general form of $\mathfrak{f}_{A,\alpha,\beta,\delta,\vec{l}}$ can be written as

$$(\alpha + (1-\delta)S^{n-m})^{A-1}((1-\beta+\beta\delta)S^{n-m} - \alpha\beta)(S^m - \delta S^n)S^{(n-m)(A-1)}$$
$$-\frac{(1-\alpha)}{\sum_{i=0}^{A-1}\beta^i}S^m(S^{(n-m)A}$$
$$-(\beta(1-\delta)S^{n-m} + \alpha\beta)^A)\sum_{i=0}^{A-1} l_{A-i}(S^{n-m})^{A-1-i}(\alpha + (1-\delta)S^{n-m})^i.$$

This allows us to obtain $\lambda_0, \ldots, \lambda_{2(A-1)}$ constructively. In the sequel we use $\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}$ to denote the factor $\sum_{i=0}^{2(A-1)} \lambda_i S^{(n-m)i}$ obtained in Theorem 4.2.

It is easy to derive the degree of $\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}$.

Corollary 4.3. For each A, α, β, δ and $\vec{l}$, the degree of $\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}$ equals to $(n-m)(2A-3)$ if $\delta = 1$ and $(n-m)(2A-2)$, otherwise.

Example 4.4. In this example, we verify the existence of a univariate equation on S, obtained by the equations of System (7). As we saw in the previous example, $\Psi = \Xi = 0$ and so $S^5((1-(2(1+r))^3)\Psi - 7(1+r)^2\Xi) = 0$ which can be written as

$$-14S^7r^4 + \frac{8}{3}S^5r^5w - 35S^7r^3 + 16S^5r^4w - 28S^7r^2 + 26S^5r^3w - 7S^7r$$
$$+\frac{46}{3}S^5r^2w + 3S^5rw = 0.$$

As $h_r = h_w = 0$, it follows that $Sr = 1/2 - S$ and $w = 1/2S$. Consequently,

$$-\frac{1}{48}S(2S-1)(2S-2)(2S^3 + 25S^2 - 5S - 1) = 0.$$

It can be checked that the only economically sensible solutions to this equation are also solutions to

$$(2S^3 + 25S^2 - 5S - 1) = 0.$$

Concerning the general structure of the factors of $\mathfrak{f}_{A,\alpha,\beta,\delta,\vec{l}}$, it is possible to verify their compatibility with an economic equilibrium by examining the other equations of (7) from the algebraic and economic point of view. By doing so, we obtain the following corollary.

Corollary 4.4. For each A, α, β, δ and each sequence of labor-endowments, $\vec{l}$, the solutions of System (7) are contained in $\mathbf{V}(\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}} + \langle\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}\rangle)$.

In the sequel we state an algorithm which computes $\mathcal{G}_{A,\alpha,\beta,\delta,\vec{l}}$, a Gröbner basis consisting $\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}$ together with polynomials $u - \rho_u$ for each variable u, where $\rho_u \in \mathbb{Q}[S]$. In addition $\mathcal{G}_{A,\alpha,\beta,\delta,\vec{l}}$ has this property that

$$\mathbf{V}(\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}} + \langle\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}\rangle) \subseteq \mathbf{V}(\mathcal{G}_{A,\alpha,\beta,\delta,\vec{l}}),$$

which allows us to determine all equilibria by calculating $\mathbf{V}(\mathcal{G}_{A,\alpha,\beta,\delta,\vec{l}})$.

Note that for this simple case of log-utility, we can derive a simple expression for demand, and given a positive solution for S we can directly compute the steady-state interest rate, wages, and all individuals' consumption. The rest of this section explains how to do this abstractly. For the economic problem at hand, this is irrelevant, but it is important to understand the mathematical technique.

The following theorem states how the variable r can be written as a polynomial with a determined degree on S.

Theorem 4.5. For each A, α, β, δ and $\vec{l}$, there exists a polynomial $\rho_r \in \mathbb{Q}[S]$ such that $r - \rho_r \in \mathcal{G}_{A,\alpha,\beta,\delta,\vec{l}}$. Furthermore, degree of ρ_r equals to $(n-m)(2A-3)$, if $\delta \neq 1$ and $(n-m)(2A-4)$ otherwise.

The following theorem states that the variable c_1 can be represented as a polynomial on S. It is worth noting that the proof of this theorem is existential.

Theorem 4.6. For each A, α, β, δ and $\vec{l}$, there is a polynomial $\rho_c \in \mathbb{R}[S]$ such that $c_1 - \rho_c \in \mathcal{G}_{A,\alpha,\beta,\delta,\vec{l}}$.

Regarding Theorem 4.6 and its proof, there are $\rho_1, \rho_2 \in \mathbb{Q}[S]$ such that $\rho_1 c_1 - \rho_2 = 0$ modulo the assumed polynomial ideal, and so there exists $\rho_c \in \mathbb{Q}[S]$ in the way that $c_1 = \rho_c$. However, it concludes that $\frac{\rho_2}{\rho_1} = \rho_c$ modulo $\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}$ and so, we need to declare a sub-algorithm to find the polynomial representation of $\frac{\rho_2}{\rho_1}$. In the following lemma we state a simple algorithm based on linear algebraic techniques to compute the quotient of two polynomials modulo a given polynomial.

Lemma 4.7. Let $f_1, f_2, f_3 \in \mathbb{Q}[x]$ with $\mathbf{V}(f_1) \cap \mathbf{V}(f_2) = \emptyset$ and $\mathcal{I} = \langle f_1, uf_2 - f_3 \rangle \subset \mathbb{Q}[x, u]$. Then, the reduced Gröbner basis of $\mathcal{I}$ with respect to the lexicographical monomial ordering $x \prec u$ equals to $\{f_1, u - g\}$, where $g \in \mathbb{Q}[x]$ and $gf_2 - f_3$ coincides with zero, modulo f_1.

By the notions of the above lemma, we call g, the polynomial representation of the quotient f_3/f_2 modulo f_1. One can design the following algorithm to compute the polynomial representation of a quotient of univariate polynomials with respect to an univariate polynomial.

Algorithm 1 POL-QUO

Require: $f_1, f_2, f_3 \in \mathbb{Q}[x]$.
Ensure: $g \in \mathbb{Q}[x]$ such that $gf_2 - f_3 = 0$ modulo f_1.
  $d := \deg(f_1)$;
  $g := \sum_{i=0}^{d-1} \epsilon_i x^i$;
  Compute the remainder of $gf_2 - f_3$ on division by f_1;
  Represent the remainder as $\sum_{i=0}^{d-1} \bar{\epsilon}_i x^i$;
  Solve the system $\bar{\epsilon}_0 = 0, \ldots, \bar{\epsilon}_{d-1} = 0$ to obtain $\epsilon_0, \ldots, \epsilon_{d-1}$;
  Return (g);

Now, we are able to present an algorithm to compute $\mathcal{G}_{A,\alpha,\beta,\delta,\vec{l}}$.

Theorem 4.8. For each A, α, β, δ and the sequence of labors $\vec{l}$, the following algorithm computes $\mathcal{G}_{A,\alpha,\beta,\delta,\vec{l}}$ as a Gröbner basis w.r.t. a lexicographical monomial ordering in which S is the smallest variable and the equilibria of the corresponding OLG model are contained in $\mathbf{V}(\mathcal{G}_{A,\alpha,\beta,\delta,\vec{l}})$.

Algorithm 2 OLG-Gröbner

Require: A; the number of generations, and the parameters $\alpha = \frac{m}{n}, \beta, \delta$ and $\vec{l}$.
Ensure: The reduced Gröbner basis $\mathcal{G}_{A,\alpha,\beta,\delta,\vec{l}}$.

$G := \{\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}, k_0, k_A\}$;
$G := G \cup \{K - \mathrm{NF}_{\{\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}\}}(S^n)\}$;
$G := G \cup \{w - \mathrm{NF}_{\{\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}\}}((1-\alpha)S^m)\}$;
$G := G \cup \{r + \delta + \frac{\alpha}{\lambda_0} \sum_{i=1}^{2(A-1)+\epsilon} \lambda_i S^{(n-m)(i-1)}\}$; ($\epsilon = -1$ if $\delta = 1$ and $\epsilon = 0$, otherwise.)
$P := \mathrm{NF}_G(\sum_{i=1}^{A} \beta^{i-1}(1+r)^{A-1})$;
$Q := \mathrm{NF}_G(\frac{1-\alpha}{A} \sum_{i=1}^{A}(1+r)^{A-i} S^m)$;
$G := G \cup \{c_1 - \textsc{Pol-Quo}(\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}, P, Q)\}$;
$G := G \cup \{c_a - \mathrm{NF}_G((\beta(1+r))^{a-1} c_1) \mid a = 2, \ldots, A\}$;
for $a = 1, \ldots, A-1$ **do**
  $G := G \cup \{k_a - \mathrm{NF}_G(k_{a-1}(1+r) + w l_a - c_a)\}$;
end for
Return (G);

5 The case of $\gamma > 0$

In this section, we analyze the solutions of System (6) for the case where $\gamma > 0$. It is obvious that in this case, the equations are not in polynomial form. To convert System (6) to a polynomial system, we introduce an auxiliary variable p and let $p^{\gamma+1} = \beta(1+r)$. As in Section 3, assume that $\alpha = m/n$ where m, n are natural numbers and $m < n$. As above, we have another auxiliary variable, S, with $S^n = K$ and so $K^{\alpha-1} = S^{m-n}$. Again, we multiply the equation $r = \alpha K^{\alpha-1} - \delta$ by S^{n-m}. After these changes, we obtain the following polynomial system, which is equivalent to System (6).

$$\begin{cases} c_{a+1} & = & pc_a \quad (a = 1, \dots, A-1) \\ c_a & = & k_{a-1}(1+r) + wl_a - k_a \quad (a = 1, \dots, A) \\ (r+\delta)S^{n-m} & = & \alpha \\ w & = & (1-\alpha)S^m \\ K & = & \sum_{a=0}^{A} k_a \\ K & = & S^n \\ p^{\gamma+1} & = & \beta(1+r) \\ k_0 & = & k_A = 0 \end{cases} \tag{8}$$

In the sequel, we begin to present the properties of the polynomial ideal associated to System (8). Suppose that

$$\begin{array}{rcl} \mathcal{I}^{+}_{A,\gamma,\alpha,\beta,\delta,\vec{l}} & = & \langle f_1, \dots, f_{A-1}, g_1, \dots, g_A, h_r, h_w, h_K, h_p, q_k \rangle \\ & \subset & \mathbb{R}[c_1, \dots, c_A, k_0, \dots, k_A, K, S, w, r, p] \end{array}$$

is the polynomial ideal associated to System (8), where $\vec{l}$ (in the index of $\mathcal{I}^+$) denotes the sequence of labor-endowments $l_1, \dots, l_A$, $\sum_a l_a = 1$, and

$$\begin{array}{rcl} f_a & = & c_{a+1} - pc_a \quad (a = 1, \dots, A-1), \\ g_a & = & c_a - k_{a-1}(1+r) - wl_a + k_a \quad (a = 1, \dots, A, g_0 = k_0, g_A = k_A), \\ h_r & = & (r+\delta)S^{n-m} - \alpha, \\ h_w & = & w - (1-\alpha)S^m, \\ h_K & = & K - S^n, \\ h_p & = & p^{\gamma+1} - \beta(1+r), \\ q_k & = & K - \displaystyle\sum_{a=0}^{A} k_a. \end{array}$$

In the following discussion, we try to formulate a Gröbner basis for $\mathcal{I}^{+}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$ w.r.t. a lexicographical monomial ordering in which p is the smallest variable.

In the following proposition, we provide two useful polynomials in $\mathcal{I}^{+}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$.

Proposition 5.1. Regarding the above notations, we have

(i) $\Psi^+ := \sum_{i=0}^{A-1}(1+r)^i p^{A-1-i} c_1 - \sum_{i=0}^{A-1}(1+r)^i l_{A-i} w \in \mathcal{I}^{+}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$,

(ii) $\Xi^+ := (1-p^A)c_1 - (1-p)(rS^n + w) \in \mathcal{I}^{+}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$.

Proof. The proof is similar to the proof of Proposition 4.1. □

In the following theorem, we derive a polynomial only in p which is an element of the ideal $\mathcal{I}^{+}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$.

Theorem 5.2. For each $\gamma > 0, A, \alpha, \beta, \delta$ and $\vec{l}$, there exists $\tilde{\mathfrak{f}}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}} \in \mathbb{Q}[p]$ such that

$$\mathfrak{f}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}} := S^m(p-1)(p^{\gamma+1}-\beta)\tilde{\mathfrak{f}}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}} \in \mathcal{I}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}.$$

Furthermore, $\tilde{\mathfrak{f}}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$ is not divisible by $p-1$.

The proof in the appendix shows step by step how to construct the polynomial, the basic idea is the same as for the log-case, but there are several additional steps.

Remark 5.1. It follows from the the proof of Theorem 5.2 that $\mathfrak{f}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$ can be written as

$$\begin{aligned} &S^m(\sum_{i=0}^{A-1} \beta^{A-1-i} p^{A-1+\gamma i}(1-p)(p^{\gamma+1}-\beta(1-\delta)-\alpha\beta\delta) \qquad (9)\\ &-\frac{n-m}{n}(1-p^A)(p^{\gamma+1}-\beta(1-\delta))\sum_{i=0}^{A-1}\beta^{A-1-i}p^{(\gamma+1)i}l_{A-i}), \end{aligned}$$

which demonstrates the general form of $\tilde{\mathfrak{f}}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$ by its construction.

In addition, it is worth noting that $p^A \neq 1$. To see the reason, let us sum up the equations $c_a = p^{a-1}c_1$ for all $a = 1, \ldots, A$ which implies that

$$\sum_{a=1}^{A} c_a = \frac{p^A - 1}{p-1} c_1.$$

Thus, if $p^A = 1$ then the summation of all amounts of consumptions equals to zero which contradicts the economical hypothesis that all consumptions are positive. Therefore, in the rest of our argument, we deal with

$$\frac{\tilde{\mathfrak{f}}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}}{\gcd(\tilde{\mathfrak{f}}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}, p^A-1)}.$$

Nevertheless, we denote it again by $\tilde{\mathfrak{f}}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$ to avoid complication in notions.

Corollary 5.3. The equilibria of the main problem (8) are contained in $\mathbf{V}(\mathcal{I}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}} + \langle \tilde{\mathfrak{f}}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}} \rangle)$.

Proof. From Theorem 5.2 and its proof, it follows that

$$\begin{aligned} \mathcal{I}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}} &= \mathcal{I}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}} + \langle S^m\rangle \cap \mathcal{I}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}} + \langle p^{\gamma+1}-\beta\rangle \cap \mathcal{I}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}} \\ &+ \langle p-1\rangle \cap \mathcal{I}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}} + \langle \tilde{\mathfrak{f}}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}\rangle \end{aligned}$$

which is equivalent to

$$\begin{aligned} \mathbf{V}(\mathcal{I}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}) &= \mathbf{V}(\mathcal{I}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}} + \langle S^m\rangle) \cup \mathbf{V}(\mathcal{I}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}} + \langle p^{\gamma+1}-\beta\rangle) \\ &\cup \mathbf{V}(\mathcal{I}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}} + \langle p-1\rangle) \cup \mathbf{V}(\mathcal{I}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}} + \langle \mathfrak{f}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}\rangle), \end{aligned}$$

It is easy to see that, by our economic assumptions, none of the terms S^m, $p^{\gamma+1} - \beta$, and $p - 1$ can be zero. Therefore, the equilibria of the main problem are contained in $\mathbf{V}(\mathcal{I}^{+}_{A,\gamma,\alpha,\beta,\delta,\vec{l}} + \langle \mathfrak{f}^{+}_{A,\gamma,\alpha,\beta,\delta,\vec{l}} \rangle)$. $\square$

This is the main result of the paper. We can find all steady-states of our OLG model by finding all zeros of the univariate polynomial in (9). The following example illustrates this point.

Example 5.2. Let $A = 3, \gamma = 1, \beta = 2, \delta = \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $\vec{l} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. As mentioned in the proof of Theorem 5.2,

$$\tilde{\mathfrak{f}}^{+}_{3,1,\frac{1}{2},2,\frac{1}{2},(\frac{1}{3},\frac{1}{3},\frac{1}{3})} = p^6 + p^5 - 2p^4 - 9p^3 - 16p^2 + 2p + 2.$$

In the sequel we state an algorithm which computes $\mathcal{G}^{+}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$, a Gröbner basis consisting $\mathfrak{f}^{+}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$ together with polynomials $u - \rho_u$ for each variable u, where $\rho_u \in \mathbb{Q}[S, p]$. In addition, $\mathcal{G}^{+}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$ has this property that

$$\mathbf{V}(\mathcal{I}^{+}_{A,\gamma,\alpha,\beta,\delta,\vec{l}} + \langle \mathfrak{f}^{+}_{A,\gamma,\alpha,\beta,\delta,\vec{l}} \rangle) \subset \mathbf{V}(\mathcal{G}^{+}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}),$$

which allows us to find all equilibria by calculating $\mathbf{V}(\mathcal{G}^{+}_{A,\gamma,\alpha,\beta,\delta,\vec{l}})$.

As in the log-case above, analytic expressions are known for individual demand functions, and one can approach this step differently. However, again, it is useful to show the general method.

Theorem 5.4. For each $\gamma > 0, A, \alpha, \beta, \delta$ and vector of labor endowments $\vec{l}$, the following algorithm computes the reduced Gröbner basis $\mathcal{G}^{+}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$ with respect to a lexicographical monomial ordering in which S, p are the smaller than other variables and $p \prec S$, such that $\mathbf{V}(\mathcal{G}^{+}_{A,\gamma,\alpha,\beta,\delta,\vec{l}})$ contains the equilibria of the corresponding OLG model.

Algorithm 3 OLG-GRÖBNER(+)

Require: A; the number of generations, and the parameters $\alpha = \frac{m}{n}, \beta, \delta, \gamma > 0$ and $\vec{l}$.

Ensure: The reduced Gröbner basis $\mathcal{G}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$, whose variety contains equilibria of the corresponding OLG model.

$G := \{\tilde{\mathfrak{f}}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}, k_0, k_A\}$;

$G := G \cup \{r-$the remainder of $\frac{p^{\gamma+1}}{\beta} - 1$ on division by $\tilde{\mathfrak{f}}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$;

$R := S^{n-m} - \alpha\beta$POL-QUO$(\mathfrak{f}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}, p^{\gamma+1} - \beta + \beta\delta, 1)$

$G := G \cup \{R\}$;

$G := G \cup \{w-$ the remainder of $(1-\alpha)S^m$ on division by $R\}$;

$G := G \cup \{K-$ the remainder of S^n on division by R and $\tilde{\mathfrak{f}}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}\}$;

$c_{temp} :=$ POL-QUO$(\tilde{\mathfrak{f}}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}, 1-p^A, 1-p)(rS^n + w)$;

$G := G \cup \{c_1 - \mathrm{NF}_G(c_{temp})\}$;

$G := G \cup \{c_a - \mathrm{NF}_G(p^{a-1}c_1) \mid a = 2, \ldots, A\}$;

for $a = 2, \ldots, A-1$ **do**

  $G := G \cup \{k_a - \mathrm{NF}_G(k_{a-1}(1+r) + wl_a - c_a)\}$;

end for

Return (G);

Remark 5.3. It is worth noting that for each equilibrium, all consumptions must be positive, and so p, as the ratio of two successive consumptions gives a positive real value. Therefore, to find the equilibria we search for positive real roots of $\tilde{\mathfrak{f}}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$, such as $p_1, \ldots, p_\ell$. For each $i = 1, \ldots, \ell$ we substitute $p = p_i$ in R (the polynomial on S and p) to find the positive values of S. Now, it is enough to substitute these values in

$$\{g = 0 \mid g \in \mathcal{G}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}\},$$

to find the value of all variables. Obviously, the solution with all positive components exhibits an equilibrium of the corresponding OLG model.

Example 5.4. Continuing Example 5.2 we have

$$G = \{\tilde{\mathfrak{f}}^+_{3,1,\frac{1}{2},2,\frac{1}{2},(\frac{1}{3},\frac{1}{3},\frac{1}{3})}, k_0, k_3\}.$$

In the next step, the algorithm calculates the remainder of $\frac{p^2}{2} - 1$ on division by $\tilde{\mathfrak{f}}^+_{3,1,\frac{1}{2},2,\frac{1}{2},(\frac{1}{3},\frac{1}{3},\frac{1}{3})}$ which equals to $\frac{p^2}{2} - 1$, again. Thus,

$$G = \{\tilde{\mathfrak{f}}^+_{3,1,\frac{1}{2},2,\frac{1}{2},(\frac{1}{3},\frac{1}{3},\frac{1}{3})}, k_0, k_3, r - (\frac{p^2}{2} - 1)\}.$$

In the sequel, the algorithm calls the POL-QUO sub-algorithm to compute

$$\frac{1}{p^2-1} \; modulo \; \tilde{\mathfrak{f}}^+_{3,1,\frac{1}{2},2,\frac{1}{2},(\frac{1}{3},\frac{1}{3},\frac{1}{3})}.$$

In doing so, it considers a the polynomial

$$g = \epsilon_0 + \epsilon_1 p + \epsilon_2 p^2 + \epsilon_3 p^3 + \epsilon_4 p^4 + \epsilon_5 p^5$$

which unknown coefficients such that $g(p^2-1)-1=0$ modulo $\tilde{\mathfrak{f}}^+_{3,1,\frac{1}{2},2,\frac{1}{2},(\frac{1}{3},\frac{1}{3},\frac{1}{3})}$. Following this condition, we achieve the following linear system of equations

$$\begin{cases} -\epsilon_0 - 2\epsilon_4 + 2\epsilon_5 - 1 = 0, \\ -\epsilon_1 - 2\epsilon_4 = 0, \\ \epsilon_0 - \epsilon_2 + 16\epsilon_4 - 18\epsilon_5 = 0, \\ \epsilon_1 - \epsilon_3 + 9\epsilon_4 + 7\epsilon_5 = 0, \\ \epsilon_2 + \epsilon_4 + 7\epsilon_5 = 0, \\ \epsilon_3 - \epsilon_4 + 2\epsilon_5 = 0 \end{cases}$$

whose solution is

$$\epsilon_0 = -73/63, \epsilon_1 = -2/21, \epsilon_2 = 11/63, \epsilon_3 = 1/9, \epsilon_4 = 1/21, \epsilon_5 = -2/63,$$

and so

$$g = -73/63 - 2/21p + 11/63p^2 + 1/9p^3 + 1/21p^4 - 2/63p^5,$$

which implies that

$$G = \{\tilde{\mathfrak{f}}^+_{3,1,\frac{1}{2},2,\frac{1}{2},(\frac{1}{3},\frac{1}{3},\frac{1}{3})}, k_0, k_3, r - \frac{p^2}{2} - 1, S - g\}.$$

Now, the algorithm calculates the remainder of $\frac{1}{2}S$ (resp. S^2) on division by $S-g$ which equals to $\frac{1}{2}g$ (resp. $\bar{g} := 113/3969p^5 - 20/441p^4 - 76/567p^3 - 422/3969p^2 + 82/441p + 5185/3969$). It follows that

$$G = \{\tilde{\mathfrak{f}}^+_{3,1,\frac{1}{2},2,\frac{1}{2},(\frac{1}{3},\frac{1}{3},\frac{1}{3})}, k_0, k_3, r - \frac{p^2}{2} - 1, S - g, w - \frac{1}{2}g, K - \bar{g}\}.$$

In this step, the algorithm recalls again the POL-QUO sub-algorithm to compute $(1-p)/(1-p^3)$ modulo $\tilde{\mathfrak{f}}^+_{3,1,\frac{1}{2},2,\frac{1}{2},(\frac{1}{3},\frac{1}{3},\frac{1}{3})}$ which equals to

$$h := 5/57p^5 + 2/57p^4 - 5/19p^3 - 12/19p^2 - 47/57p + 61/57,$$

and computes

$$\bar{h} := \mathrm{NF}_G(h(rS^n + w)) = -\frac{1139}{7938}p^5 - \frac{11}{294}p^4 + \frac{218}{567}p^3 + \frac{4324}{3969}p^2 + \frac{131}{98}p - \frac{14977}{7938}.$$

Therefore,

$$G = \{\tilde{\mathfrak{f}}^+_{3,1,\frac{1}{2},2,\frac{1}{2},(\frac{1}{3},\frac{1}{3},\frac{1}{3})}, k_0, k_3, r - \frac{p^2}{2} - 1, S - g, w - \frac{1}{2}g, K - \bar{g}, c_1 - \bar{h}\}.$$

In the sequel, after performing some simple normal forms, the set G will be completed as follows,

$$\begin{aligned}
&\{p^6 + p^5 - 2p^4 - 9p^3 - 16p^2 + 2p + 2,\\
&\quad S - (-\tfrac{73}{63} - \tfrac{2}{21}p + \tfrac{11}{63}p^2 + \tfrac{1}{9}p^3 + \tfrac{1}{21}p^4 - \tfrac{2}{63}p^5),\\
&\quad r - (\tfrac{p^2}{2} - 1),\\
&\quad w - \tfrac{1}{2}(-\tfrac{2}{63}p^5 + \tfrac{1}{21}p^4 + \tfrac{1}{9}p^3 + \tfrac{11}{63}p^2 - \tfrac{2}{21}p - \tfrac{73}{63}),\\
&\quad K - (\tfrac{113}{3969}p^5 - \tfrac{20}{441}p^4 - \tfrac{76}{567}p^3 - \tfrac{422}{3969}p^2 + \tfrac{82}{441}p + \tfrac{5185}{3969}),\\
&\quad c_1 - (-\tfrac{1139}{7938}p^5 - \tfrac{11}{294}p^4 + \tfrac{218}{567}p^3 + \tfrac{4324}{3969}p^2 + \tfrac{131}{98}p - \tfrac{14977}{7938}),\\
&\quad c_2 - (\tfrac{421}{3969}p^5 + \tfrac{43}{441}p^4 - \tfrac{229}{1134}p^3 - \tfrac{7613}{7938}p^2 - \tfrac{1411}{882}p + \tfrac{1139}{3969}),\\
&\quad c_3 - (-\tfrac{34}{3969}p^5 + \tfrac{1}{98}p^4 - \tfrac{5}{1134}p^3 + \tfrac{773}{7938}p^2 + \tfrac{11}{147}p - \tfrac{842}{3969}),\\
&\quad k_1 - (\tfrac{1097}{7938}p^5 + \tfrac{20}{441}p^4 - \tfrac{415}{1134}p^3 - \tfrac{8417}{7938}p^2 - \tfrac{1193}{882}p + \tfrac{6722}{3969}),\\
&\quad k_2 - (-\tfrac{871}{7938}p^5 - \tfrac{40}{441}p^4 + \tfrac{263}{1134}p^3 + \tfrac{7573}{7938}p^2 + \tfrac{1357}{882}p - \tfrac{1537}{3969}),\\
&\quad k_0, k_3\}
\end{aligned}$$

Now, we search for equilibria. Standard numerical techniques (see [14]) can be employed to see that $p^6+p^5-2p^4-9p^3-16p^2+2p+2$ has two positive real roots in $[95/256, 3/8]$ and $[309/128, 155/64]$ which can be approximated to 0.3747 and 2.4174 respectively. For the first, S gives a negative solution, and so we focus on the second. Substituting this value for p, we find the solution

$$\begin{aligned}
r = 1.9218, w = 0.1032,\\
c_1 = 0.0200, c_2 = 0.0483, c_3 = 0.1168,\\
k_0 = 0, k_1 = 0.0144, k_2 = 0.0282, k_3 = 0, K = 0.0426.
\end{aligned}$$

As $\mathbf{V}(\mathcal{G}^+_{A,\gamma,\alpha,\beta,\delta,\bar{l}})$ contains just one positive solution, this solution is the unique steady-state equilibrium of the corresponding OLG model.

6 Some examples

In this section, we provide several examples that illustrate our approach.

6.1 Bounding the number of steady states

As mentioned in the introduction, one can also use numerical methods to find the number of approximate equilibria. In many cases, that also provides the number of exact equilibria without reasonable doubt, and this is, of course, much simpler and more efficient than our approach. However, in this section, we provide some examples that our approach allows us to make statements about the number of equilibria for an interesting range of parameters and not for only one example.

For this, we recall some well-known upper bounds on the number of solutions for univariate polynomials. Given any univariate polynomial, $f = \sum_{i=0}^d a_i x^i$, with

$a_i \in \mathcal{R}$ for all i, the number of its complex zeros is obviously bounded by its degree d. Define the number of sign changes of f to be the number of elements of $\{a_i \neq 0, i = 0, \ldots, d-1 : \operatorname{sign}(a_i) = -\operatorname{sign}(a_{i+1})\}$. The classical Descartes's Rule of Signs states that the number of real positive zeros of f does not exceed the number of sign changes.

As in example 5.2 we assume $A = 3, \gamma = 1$, and $\alpha = 1/2$. We also assume $\delta = 1$ and $\beta = 1$ and we leave labor-endowments $\vec{l} = (1_1, l_2, l_3)$ as free parameters. It can be easily verified from (9) that

$$\mathfrak{f}^{+}_{3,1,\frac{1}{2},1,1,(l_1,l_2,l_3)} = S/2(p-1)p^2(p^2+p+1)(l_1p^4 + (l_2-2)p^2 + l_3 + 1).$$

The only economically relevant zeros must satisfy

$$l_1p^4 + (l_2-2)p^2 + l_3 + 1 = 0.$$

By Descartes' rule of signs there can be at most two positive real solutions (note that $l_2 - 2 < 0$). The index theorem (see [7]) implies that there must be a unique steady state for all endowments.

It turns out that this result generalizes to arbitrary A. For example, for $A = 10$, we obtain that all relevant solutions must satisfy

$$l_{10} + l_9p^2 + l_8p^4 + l_7p^6 + p^7 + l_6p^8 - 2p^9 + l_5p^{10} + l_4p^{12} + l_3p^{14} + l_2p^{16} + l_1p^{18} = 0.$$

Again, there can be at most 2 positive real solutions. One can try different values for α, β, A, and δ and find similar patterns. In particular, one can also treat the capital share as a parameter.

The situation becomes more complicated for larger values of γ. For example, for $\gamma = 2$, keeping α, β, A and δ as above, the number of sign changes in the relevant polynomial is already 3, allowing for the possibility of three equilibria. We obtain

$$\mathfrak{f}^{+}_{3,2,\frac{1}{2},1,1,(l_1,l_2,l_3)} = S/2(-1+p)(1+p+p^2)(1+(l_3-1)p+p^2-2p^3+(l_2+2)p^4-2p^5+l_1p^7).$$

Nevertheless, if we take $l_3 = 0$ and use $l_1 = 1 - l_2$ we obtain that all economically relevant solutions must satisfy

$$1 - p + p^2 - 2p^3 + (l_2+2)p^4 - 2p^5 + (1-l_2)p^7 = 0$$

where can be at most six positive real solutions.

Example 6.1. (A numerical example) As a numerical example to show the performance of the introduced algorithm, we have executed it for the OLG model with $A = 60, \gamma = 10, \beta = 4, \delta = 3/4, \alpha = 7/10$ and $l_a = 1/60$ for each $a = 1, \ldots, 60$. Table 1 in Appendix 6.2 demonstrates the equilibrium of this model.

6.2 Experimental Results

We have implemented the algorithms due to the `OLGGrob` package in `Maple 2018`. To evaluate the performance of the algorithms, we have executed them for some values of parameters. The computations were performed on a personal computer with Intel(R) core(TM) 2 Duo CPU E7300 @ 2.66, 2.67 GHz processor, 4 GB RAM and 64 bits running under the MS Windows operating system. In the following tables the Time (resp. Memory) column shows the the consumed CPU time in seconds (resp. the amount of megabytes of memory used) by the algorithm. In addition, the S-time column shows the time of finding the positive real roots of the univariate polynomial together with substitution and solving the rest of polynomial equations in $\mathcal{G}_{A,\alpha,\beta,\delta,\vec{l}}$ or $\mathcal{G}^{+}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$. It is worth noting that in the following timing tables $\gamma = 10$ (for the case of $\gamma > 0$), $\beta = 4, \delta = 3/4, \alpha = 7/10$ and $l_a = 1/A$ for each $a = 1, \ldots, A$.

A	Time	Memory	S-time
5	0.05	7	0.03
10	0.40	29	0.08
15	1.58	156	0.23
20	4.43	636	0.31
25	11.90	1922	0.44
30	26.33	4963	0.51
35	64.33	11563	0.66
40	128.06	24081	1.01
45	231.12	46271	1.42
50	443.20	84537	2.11
55	723.21	145561	2.66
60	1105.52	236480	3.23
65	1768.47	379125	3.12
70	2622.03	578096	3.76

Table 1: Algorithm behaviour for $\gamma = 0$

A	Time	Memory	S-time
5	0.19	21	0.16
10	1.79	250	1.29
15	6.33	1119	3.49
20	17.22	3294	5.91
25	37.60	7688	10.58
30	65.57	14817	16.11
35	117.17	27103	24.20
40	192.93	44971	33.73
45	294.76	70098	52.45
50	468.38	106270	66.85
55	280.33	31765	88.36
60	1077.00	213982	57.03
65	1409.98	296077	75.75
70	1699.21	394247	90.01

Table 2: Algorithm behaviour for $\gamma > 0$

The following diagrams exhibit the growth rate of the Time (resp. Memory) amount with respect to A. Regarding the general form of the univariate polynomials (see Theorems 4.2 and 5.2 and their proofs), the coefficients depend on the values of parameters. Therefore, some specific values may cause the algorithm to need less time and memory than the similar case for smaller A. This observation can be seen in the case of $\gamma = 10$ for $A = 55$ where the diagram descends for both Time and Memory.

See `http://faculty.du.ac.ir/basiri/software/` to download the package instructions.

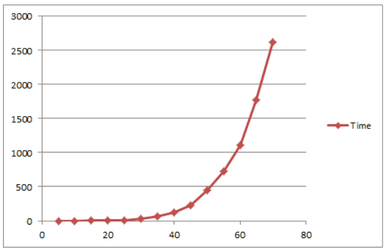

Figure 1: Time behaviour when A increases.

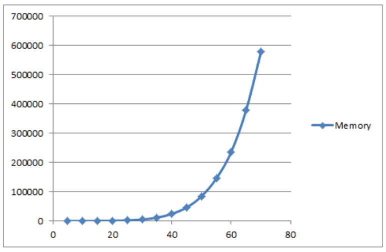

Figure 2: Memory behaviour when A increases.

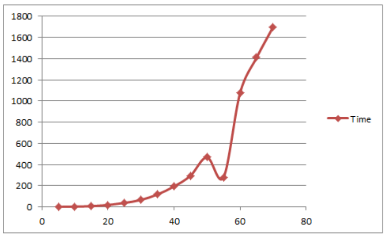

Figure 3: Time behaviour when A increases.

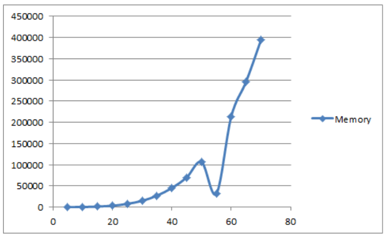

Figure 4: Memory behaviour when A increases.

Appendix 1

Proof of Proposition 4.1

Proof. (i) First, we claim that for each $a = 1, \dots, A$,

$$\psi_a := k_a + \sum_{i=0}^{a-1} \beta^i (1+r)^{a-1} c_1 - \sum_{i=0}^{a-1} (1+r)^i l_{a-i} w \in \mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}.$$

We prove this claim by induction on a. It is obvious that the hypothesis holds for $a = 1$. So, we prove it for $a+1$. Regarding the existence of $f_1, \dots, f_{A-1}$ in $\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}$, it implies that for each $a = 2, \dots, A$,

$$\xi_a := \sum_{i=1}^{a-1} (\beta(1+r))^{a-1-i} f_i = c_a - (\beta(1+r))^{a-1} c_1 \in \mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}.$$

Now, by a simple calculation

$$\psi_{a+1} = (1+r)\psi_a + g_{a+1} - \xi_{a+1},$$

which completes the induction. It is obvious that $\Psi = \psi_A$ which terminates the proof.

(ii) As $g_1, \ldots, g_A$ belong to $\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}$, it concludes that

$$\sum_{i=0}^{A} g_i = \sum_{a=1}^{A} c_a - (1+r)\sum_{a=1}^{A} k_{a-1} - w\sum_{a=1}^{A} l_a + \sum_{a=1}^{A} k_a \in \mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}$$

which approves by applying $q_k \in \mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}$ and $\sum_a l_a = 1$ that

$$\sum_{a=1}^{A} c_a - rK - w \in \mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}} \tag{10}$$

On the other hand, by the notions of the proof of (i),

$$\sum_{a=1}^{A} \xi_a = \sum_{a=1}^{A} c_a - c_1 \sum_{a=1}^{A} (\beta(1+r))^{a-1} \in \mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}. \tag{11}$$

Now, subtracting the obtained polynomials in (10) and (11) and summation with rh_K, we observe that

$$\varphi := c_1 \sum_{a=1}^{A} (\beta(1+r))^{a-1} - rS^n - w \in \mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}} \tag{12}$$

Now, it is easy to see that $\Xi = (1-\beta(1+r))\varphi$ which certainly belongs to $\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}$. □

Proof of Theorem 4.2

Proof. To obtain $\mathfrak{f}_{A,\alpha,\beta,\delta,\vec{l}}$, we perform four steps of calculations as follows,
Step 1: Eliminating c_1 from Ξ and Ψ by computing

$$(1-(\beta(1+r))^A)\Psi - \sum_{i=0}^{A-1} \beta^i (1+r)^{A-1}\Xi$$

Step 2: Multiplying the result by $S^{(2A-1)(n-m)}$ to obtain the following structure which allows to rewrite r by an expression on S in Step 3.

$$\sum_{i=0}^{A-1} \beta^i (S^{n-m} + rS^{n-m})^{A-1}(S^{n-m} - \beta(S^{n-m} + rS^{n-m}))(rS^n + w)S^{(n-m)(A-1)}$$
$$-(S^{(n-m)A} - (\beta(S^{n-m} + rS^{n-m}))^A) \sum_{i=0}^{A-1} l_{A-i}(S^{n-m})^{A-1-i}(S^{n-m} + rS^{n-m})^i w$$

Step 3: Summation with suitable multiple of h_r to eliminate all rS^{n-m} terms and dividing by $\sum_{i=0}^{A-1} \beta^i$.

$$(\alpha + (1-\delta)S^{n-m})^{A-1}((1-\beta+\beta\delta)S^{n-m} - \alpha\beta)((\alpha - \delta S^{n-m})S^m + w)S^{(n-m)(A-1)}$$
$$-(S^{(n-m)A} - (\beta(1-\delta)S^{n-m} + \alpha\beta)^A) \sum_{i=0}^{A-1} l_{A-i}(S^{n-m})^{A-1-i}(\alpha + (1-\delta)S^{n-m})^i \frac{w}{\sum_{i=0}^{A-1} \beta^i}$$

Step 4: Summation with suitable multiple of h_w to substitute w with $(1-\alpha)S^m$.

$$\begin{aligned}
&(\alpha+(1-\delta)S^{n-m})^{A-1}((1-\beta+\beta\delta)S^{n-m}-\alpha\beta)(S^m-\delta S^n)S^{(n-m)(A-1)}\\
&-\frac{(1-\alpha)}{\sum_{i=0}^{A-1}\beta^i}S^m(S^{(n-m)A}\\
&-(\beta(1-\delta)S^{n-m}+\alpha\beta)^A)\sum_{i=0}^{A-1}l_{A-i}(S^{n-m})^{A-1-i}(\alpha+(1-\delta)S^{n-m})^i
\end{aligned}$$

Now we show that $\mathfrak{f}_{A,\alpha,\beta,\delta,\vec{l}}$ is divisible by $\mathfrak{g}_1 := n\delta S^{n-m}-m$ and $\mathfrak{g}_2 := (n\beta(\delta-1)+n)S^{n-m}-m\beta$, provided that $\delta \neq 0$ and $\beta(\delta-1)\neq -1$. In doing so, suppose that η is a root of $\mathfrak{g}_1$ and consequently $\eta^{n-m}=\frac{m}{n\delta}$. By evaluating $\mathfrak{f}_{A,\alpha,\beta,\delta,\vec{l}}$ on η we observe that

$$\mathfrak{f}_{A,\alpha,\beta,\delta,\vec{l}}(\eta)=(\frac{m}{n\delta})^{2A+m-1}((1-\beta)(1-\alpha)-\frac{1-\alpha}{\sum_{i=0}^{A-1}\beta^i}(1-\beta^A)\sum_{i=0}^{A-1}l_{A-i})=0$$

Similarly, let ξ be a root of $\mathfrak{g}_2$ which implies that $\xi^{n-m}=\frac{m\beta}{n\beta(\delta-1)+n}$. By evaluating $\mathfrak{f}_{A,\alpha,\beta,\delta,\vec{l}}$ on ξ we conclude that $\mathfrak{f}_{A,\alpha,\beta,\delta,\vec{l}}(\xi)=0$. Now it is easy to see that the roots of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ have the multiplicity 1 and therefore, $\mathfrak{f}_{A,\alpha,\beta,\delta,\vec{l}}$ is divisible by $\mathfrak{g}_1\mathfrak{g}_2$. □

Proof of Corollary 4.3

To prove this corollary, it is enough to analyse the degree of $\mathfrak{f}_{A,\alpha,\beta,\delta,\vec{l}}$, obtained in the proof of Theorem 4.2. In doing so, it is sufficient to analyse those cases in which $\delta=0$, $\delta=1$ or $\beta(1-\delta)=1$. This analysis shows that

$$\deg(\mathfrak{f}_{A,\alpha,\beta,\delta,\vec{l}})=\begin{cases}(n-m)(2A-1)+m & \delta=1 \text{ or } \delta=0\\ (n-m)(2A-2)+n & \beta(1-\delta)=1 \text{ and } \delta\neq 0\\ (n-m)(2A-1)+n & \text{Otherwise.}\end{cases}$$

Now, it is enough to subtract $\deg(S^m)+\deg(\mathfrak{g}_1)+\deg(\mathfrak{g}_2)$ from $\deg(\mathfrak{f}_{A,\alpha,\beta,\delta,\vec{l}})$ in each one of the above cases. Note that if $\delta=0$ (resp. $\beta(1-\delta)=1$) then, $\deg(\mathfrak{g}_1)=0$ (resp. $\deg(\mathfrak{g}_2)=0$). Finally, it follows that

$$\deg(\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}})=\begin{cases}(n-m)(2A-3) & \delta=1\\ (n-m)(2A-2) & \text{Otherwise.}\end{cases}$$

Proof of Corollary 4.4

Proof. Regarding the result of Theorem 4.2, one can decompose the ideal $\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}$ as

$$\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}=(\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}+\langle S^m\rangle)\cap(\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}+\langle\mathfrak{g}_1\rangle)\cap(\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}+\langle\mathfrak{g}_2\rangle)\cap(\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}+\langle\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}\rangle),$$

in which, $g_1 = (n\delta S^{n-m} - m)$ and $g_2 = (n\beta(\delta-1)+n)S^{n-m} - m\beta$. This decomposition tends to the following separation on the variety of the ideal $\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}$.

$$\begin{aligned}\mathbf{V}(\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}) &= \mathbf{V}(\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}} + \langle S^m\rangle) \cup \mathbf{V}(\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}} + \langle \mathfrak{g}_1\rangle) \cup \mathbf{V}(\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}} + \langle \mathfrak{g}_2\rangle)\\ &\quad \cup \mathbf{V}(\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}} + \langle \tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}\rangle.\end{aligned}$$

Now, we study three first components to verify their emptiness.

- If there exists a solution in $\mathbf{V}(\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}} + \langle S^m\rangle)$ then, it implies that under the evaluation of this solution $S = 0$. However, as $h_r \in \mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}$, it concludes that $\alpha = 0$ which is a contradiction with the selection of $\alpha \in (0,1)$.

- If there exists a solution in $\mathbf{V}(\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}} + \langle \mathfrak{g}_1\rangle)$ then, under the evaluation of this solution, $\delta S^{n-m} - \alpha = 0$. On the other hand, as $h_r \in \mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}}$ we deduce that $rS^{n-m} = 0$. If $S = 0$, as $\mathfrak{g}_1 = 0$ it concludes that $m = 0$ which is impossible and thus $r = 0$. In this situation we can observe that for each $a = 1,\ldots,A$, $c_a = \beta^{a-1}c_1$ and $c_a = k_{a-1} + w/A - k_a$. So we can write

$$\begin{aligned}\sum_{a=1}^{A} a\beta^{a-1}c_1 &= \sum_{a=1}^{A} a(k_a + \frac{w}{A} - k_{a+1})\\ &= K + \frac{A+1}{2}w\\ &= S^n + \frac{A+1}{2}(1-\alpha)S^m \end{aligned}\tag{13}$$

On the other hand,

$$\begin{aligned}\sum_{a=0}^{A-1} \beta^a c_1 &= \sum_{a=0}^{A-1} k_a + \frac{w}{A} - k_{a+1}\\ &= w\\ &= (1-\alpha)S^m \end{aligned}\tag{14}$$

Also, from h_r we infer that $S^{n-m} = \frac{\alpha}{\delta}$. Now from (13) and (14) we can conclude that

$$\begin{aligned}\frac{\sum_{a=1}^{A} a\beta^{a-1}}{\sum_{a=0}^{A-1}\beta^a}(1-\alpha)S^m &= S^n + \frac{A+1}{2}(1-\alpha)S^m\\ &= \frac{\alpha}{\delta}S^m + \frac{A+1}{2}(1-\alpha)S^m \end{aligned}\tag{15}$$

Therefore,

$$\frac{\sum_{a=1}^{A} a\beta^{a-1}}{\sum_{a=0}^{A-1}\beta^a} = \frac{\alpha}{\delta(1-\alpha)} + \frac{A+1}{2}. \tag{16}$$

It deduces that $r = 0$ if and only if the given A, β, α, δ satisfy (16) and so, from algebraic point of view, System (7) has some solutions even if $r = 0$. AS an example, for $A = 4$, $\beta = 2$, $\alpha = 3/7$, and $\delta = 45/46$ System (7) has the following solution:

$$r = 0, K = 0.236, S = 0.814, w = 0.308, c_1 = 0.021, c_2 = 0.041, c_3 = 0.082,$$
$$c_4 = 0.164, k_1 = 0.056, k_2 = 0.092, k_3 = 0.087$$

However, from economical point of view, r can not be zero which implies that $\mathbf{V}(\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}} + \langle \mathfrak{g}_1 \rangle)$ contains no conceptual equilibrium.

- If there exists a solution in $\mathbf{V}(\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}} + \langle \mathfrak{g}_2 \rangle)$ then, $(1 + r) = 1/\beta$. It follows that $c_1 = \cdots = c_A$ and so for each $a = 1, \ldots, A$,

$$c_1 = \frac{1}{\beta} k_{a-1} + \frac{w}{A} - k_a.$$

Applying summation on the both sides of this equality, we observe that

$$\begin{aligned} Ac_1 &= (\frac{1}{\beta} - 1)K + w && (17) \\ &= (\frac{1}{\beta} - 1)S^n + (1 - \alpha)S^m. && (18) \end{aligned}$$

On the other hand, from Proposition 4.1 we observed that $\Psi = 0$. Regarding the assumed condition, this equation can be rewritten as

$$\begin{aligned} Ac_1 &= w \\ &= (1 - \alpha)S^m && (19) \end{aligned}$$

Now, from (17) and (19) we conclude that

$$(\frac{1}{\beta} - 1)S^n = 0.$$

As $S \neq 0$ (see the first case), $\beta = 1$ which implies that $r = 0$ which is contained in the previous case.

Therefore, the solutions of (7) or equivalently, the equilibria of the main problem are contained in $\mathbf{V}(\mathcal{I}_{A,\alpha,\beta,\delta,\vec{l}} + \langle \tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}} \rangle)$. □

Proof of Theorem 4.5

Proof. We proceed with two polynomials $\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}$ and h_r. Suppose that $\delta \neq 1$ and so, $\deg(\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}) = (n - m)(2A - 2)$. Let also $\prec$ be a lexicographical monomial ordering under which $S \prec r$. By this assumption, $\mathrm{LM}(\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}) = S^{2(A-1)(n-m)}$

and $\mathrm{LM}(h_r) = rS^{n-m}$. At the first step, we eliminate the leading monomials by

$$\begin{aligned}
& r\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}} - \lambda_{2(A-1)} S^{(n-m)(2A-3)} h_r \\
&= \sum_{i=0}^{2A-3} \lambda_i r S^{(n-m)i} - \lambda_{2(A-1)}(\delta S^{2(A-1)(n-m)} - \alpha S^{(n-m)(2A-3)}) \\
&= \lambda_0 r + rS^{n-m} \sum_{i=1}^{2A-3} \lambda_i S^{(n-m)(i-1)} - \lambda_{2(A-1)}(\delta S^{2(A-1)(n-m)} - \alpha S^{(n-m)(2A-3)})
\end{aligned}$$

Now we can substitute rS^{n-m} by $\alpha - \delta S^{n-m}$ due to some reduction steps on h_r. Furthermore, we can reduce $S^{2(A-1)(n-m)}$ by $\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}$ to observe,

$$\begin{aligned}
& \lambda_0 r + (\alpha - \delta S^{n-m}) \sum_{i=1}^{2A-3} \lambda_i S^{(n-m)(i-1)} + \lambda_{2(A-1)}\Big(\frac{\delta}{\lambda_{2(A-1)}} \sum_{i=0}^{2A-3} \lambda_i S^{(n-m)i} \\
& + \alpha S^{(n-m)(2A-3)}\Big).
\end{aligned}$$

Finally, it is enough to consider

$$\begin{aligned}
\rho_r \quad := \quad & -\frac{1}{\lambda_0}\Big[(\alpha - \delta S^{n-m}) \sum_{i=1}^{2A-3} \lambda_i S^{(n-m)(i-1)} + \lambda_{2(A-1)}\Big(\frac{\delta}{\lambda_{2(A-1)}} \sum_{i=0}^{2A-3} \lambda_i S^{(n-m)i} \\
& + \alpha S^{(n-m)(2A-3)}\Big)\Big].
\end{aligned}$$

Suppose now that $\delta = 1$. In this situation, $\deg(\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}) = (n-m)(2A-3)$, however similar calculations with the same trace can be done to observe ρ_r, but in this case $\deg(\rho_r) = (n-m)(2A-4)$. ∎

Proof of Theorem 4.6

Proof. We proceed by polynomials $\mathfrak{f}_{A,\alpha,\beta,\delta,\vec{l}}, g_1, \ldots, g_A$ and $\tilde{h}_r$. at first, we can apply $c_i - (\beta(1+r))^{i-1} c_1$ and $w - (1-\alpha)S^m$ to simplify g_i as follows,

$$\tilde{g_i} := (\beta(1+r))^{i-1} c_1 - k_{i-1}(1+r) - \frac{1-\alpha}{A} S^m + k_i$$

It is easy to see now that we can eliminate k_a by performing $(1+r)\tilde{g_a} + \tilde{g}_{a+1}$ for each $a = 1, \ldots, A$. It concludes that in the expression

$$\sum_{i=1}^{A} (1+r)^{A-i} \tilde{g_i}$$

all k_a for $a = 1, \dots, A$ will be eliminated. More precisely,

$$\begin{aligned}
&\sum_{i=1}^{A}(1+r)^{A-i}\tilde{g}_i = \sum_{i=1}^{A}(1+r)^{A-i}\{(\beta(1+r))^{i-1}c_1 - k_{i-1}(1+r) - \frac{1-\alpha}{A}S^m + k_i\} \\
&= \sum_{i=1}^{A}(1+r)^{A-i}(\beta(1+r))^{i-1}c_1 - \sum_{i=1}^{A}(1+r)^{A-i+1}k_{i-1} - \sum_{i=1}^{A}\frac{1-\alpha}{A}S^m(1+r)^{A-i} \\
&\quad + \sum_{i=1}^{A}(1+r)^{A-i}k_i \\
&= \sum_{i=1}^{A}\beta^{i-1}(1+r)^{A-1}c_1 - \frac{1-\alpha}{A}\sum_{i=1}^{A}(1+r)^{A-i}S^m
\end{aligned}$$

Let us consider this polynomial as $\rho_1 c_1 - \rho_2$ where, $\rho_1 = \sum_{i=1}^{A}\beta^{i-1}(1+r)^{A-1}$ and $\rho_2 = \frac{1-\alpha}{A}\sum_{i=1}^{A}(1+r)^{A-i}S^m$. We are going to prove that for each zero of $\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}$, c_1 gives just finitely many values, and so it can be represented by a polynomial on S. In doing so, it is enough to say that ρ_1 and ρ_2 can not vanish simultaneously. To prove this, assume in contradiction that $\rho_1 = \rho_2 = 0$ which implies that $r = -1$. Thus, $\Xi\ |_{r=-1} = 0$ and consequently, $c_1 - (w - S^n)$ or equivalently $c_1 - (1-\alpha)S^m + S^n$ belongs to the ideal. However, as c_1 gives infinitely many values, S must also give infinitely many values which is a contradiction since a univariate on S belongs to the ideal. □

Proof of Theorem 4.8

Proof. First of all, it is easy to see that polynomials of $\mathcal{G}_{A,\alpha,\beta,\delta,\vec{l}}$ have distinct leading monomials with respect to a lexicographical monomial ordering in which S is the smallest variable. It remains to show that how the elements of $\mathcal{G}_{A,\alpha,\beta,\delta,\vec{l}}$ are obtained from the equations of System (7) which proves that the equilibria of the corresponding OLG model are contained in $\mathbf{V}(\mathcal{G}_{A,\alpha,\beta,\delta,\vec{l}})$. After this, it can also be verified easily that $\mathcal{G}_{A,\alpha,\beta,\delta,\vec{l}}$ generates the equations of System (7) which concludes that $\mathcal{G}_{A,\alpha,\beta,\delta,\vec{l}}$ is the required Gröbner basis. To do so, we show that how each variable can be rewritten as a polynomial on S by combining the equations of System (7).

From Theorem 4.2 and the equations of System (7), it concludes that $\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}$, $K - \mathrm{NF}_{\{\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}\}}(S^n)$, $w - \mathrm{NF}_{\{\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}\}}((1-\alpha)S^m)$, k_0 and k_A are (obvious) algebraic combinations of (7). In the sequel, we search for the polynomial with r as its leading monomial (note that $r = \alpha S^{m-n} - \delta$ but $m - n < 0$). Considering Theorem 4.2, Corollary 4.3 and the notations of the algorithm, $\deg(\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}) = (2(A-1)+\epsilon)(n-m)$, where ϵ equals to 1 (if $\delta = 1$) or 0 (if $\delta \neq 1$). We continue with the case $\delta \neq 1$ as the other case will be observed in a similar construction. Applying Theorem 4.2, there are scalars $\lambda_0, \dots, \lambda_{2(A-1)}$ such that

$$\sum_{i=0}^{2(A-1)} \lambda_i S^{(n-m)i} = 0.$$

Multiplying by S^{m-n}, it concludes that

$$\lambda_0 S^{m-n} + \sum_{i=1}^{2(A-1)} \lambda_i S^{(n-m)(i-1)} = 0,$$

and so,

$$\alpha S^{m-n} - \delta = -\frac{\alpha}{\lambda_0} \sum_{i=1}^{2(A-1)} \lambda_i S^{(n-m)(i-1)} - \delta,$$

whose degree is smaller than the degree of $\tilde{\mathfrak{f}}_{A,\alpha,\beta,\delta,\vec{l}}$, and so no simplification is needed. Thus, we attend

$$r + \delta + \frac{\alpha}{\lambda_0} \sum_{i=1}^{2(A-1)} \lambda_i S^{(n-m)(i-1)}.$$

To rewrite c_1 as a polynomial on S, we apply the result of Theorem 4.6 which implies that modulo the ideal,

$$(\sum_{i=1}^{A} \beta^{i-1}(1+r)^{A-1})c_1 - \frac{1-\alpha}{A} \sum_{i=1}^{A} (1+r)^{A-i} S^m) = 0.$$

However as observed above, r can be rewritten as a polynomial on S. So, performing normal form on the above equation we receive to

$$c_1 = \frac{Q}{P},$$

where

$$P = \mathrm{NF}_G(\sum_{i=1}^{A} \beta^{i-1}(1+r)^{A-1}), Q = \mathrm{NF}_G(\frac{1-\alpha}{A} \sum_{i=1}^{A} (1+r)^{A-i} S^m).$$

Now, applying Pol-Quo algorithm causes to observe the polynomial representation of c_1 on S, whose existence was proven in Theorem 4.6. It remains to deal with $c_2, \ldots, c_A$ and $k_1, \ldots, k_{A-1}$. It is worth noting that from System (7) for each $a = 2, \ldots, A$, $c_a - (\beta(1+r))^{a-1} c_1 = 0$. As shown above, r and c_1 have a polynomial representation on S and so, performing normal form leads to rewrite each c_a on S. Having a similar conclusion, for each $a = 1, \ldots, A-1$,

$$k_a = k_{a-1}(1+r) + w l_a - c_a,$$

and consequently k_a can be represented on S by a polynomial, which terminates the proof. □

Proof of Theorem 5.2

Proof. To obtain $\mathfrak{f}^{+}_{A,\alpha,\beta,\delta,\vec{l}}$, we perform four steps of calculations as follows,
Step 1: Eliminate c_1 from Ξ^+ and Ψ^+ by computing

$$(1-p^A)\Psi^+ - \sum_{i=0}^{A-1}(1+r)^i p^{A-1-i}\Xi^+$$

Step 2: Multiply the result by $S^{(A-1)(n-m)}$ to obtain the following structure which allows to rewrite r by an expression on S in Step 3.

$$\sum_{i=0}^{A-1} S^{(n-m)(A-1-i)}(S^{n-m}+rS^{n-m})^i p^{A-1-i}(1-p)(rS^n+w)$$
$$-(1-p^A)\sum_{i=0}^{A-1} S^{(n-m)(A-1-i)}(S^{n-m}+rS^{n-m})^i l_{A-i}w$$

Step 3: Summation with suitable multiple of h_r to eliminate all rS^{n-m} terms.

$$\sum_{i=0}^{A-1} S^{(n-m)(A-1-i)}(\alpha+(1-\delta)S^{n-m})^i p^{A-1-i}(1-p)(S^m(\alpha-\delta S^{n-m})+w) \tag{20}$$
$$-(1-p^A)\sum_{i=0}^{A-1} S^{(n-m)(A-1-i)}(\alpha+(1-\delta)S^{n-m})^i l_{A-i}w$$

Step 4: Summation with suitable multiple of h_w to substitute w with $(1-\alpha)S^m$ and factoring S^m.

$$S^m(\sum_{i=0}^{A-1} S^{(n-m)(A-1-i)}(\alpha+(1-\delta)S^{n-m})^i p^{A-1-i}(1-p)(\alpha-\delta S^{n-m}+\frac{n-m}{n})$$
$$-\frac{n-m}{n}(1-p^A)\sum_{i=0}^{A-1} S^{(n-m)(A-1-i)}(\alpha+(1-\delta)S^{n-m})^i l_{A-i})$$

Step 5: Finally, to substitute the exponents of S by expressions on p, we apply $\sigma := (p^{\gamma+1}-\beta(1-\delta))S^{n-m}-\alpha\beta$ which is obtained from $S^{n-m}h_p+\beta h_r \in \mathcal{I}^{+}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$. In doing so, we multiply the polynomial of the previous step by $(p^{\gamma+1}-\beta(1-\delta))^A$ and sum the result by suitable multiples of σ. Because of simplicity, we skip this process for S^m.

$$S^m(\sum_{i=0}^{A-1} \beta^{A-1-i}p^{A-1+\gamma i}(1-p)(p^{\gamma+1}-\beta(1-\delta)-\alpha\beta\delta)$$
$$-\frac{n-m}{n}(1-p^A)(p^{\gamma+1}-\beta(1-\delta))\sum_{i=0}^{A-1}\beta^{A-1-i}p^{(\gamma+1)i}l_{A-i})$$

Till now, we have observed a bivariate polynomial on S and p. Let us represent this polynomial as $S^m \mathfrak{h}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$. It remains to show that $\mathfrak{h}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$ is divisible by $p^{\gamma+1} - \beta$. To do so, we assume $p^{\gamma+1} - \beta = 0$ and prove that $\mathfrak{h}_{A,\gamma,\alpha,\beta,\delta,\vec{l}} = 0$. From the equality $p^{\gamma+1} = \beta(1+r)$ it follows that $r = 0$ and so, from h_r it implies that $\delta S^{n-m} = \alpha$. However, applying this equality implies that Polynomial (20) vanishes. Immediately, it proves that $\mathfrak{h}_{A,\gamma,\alpha,\beta,\delta,\vec{l}} = 0$ and this terminates the proof. □

Proof of Theorem 5.4

Proof. Let G be the reduced Gröbner basis of $\mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$ with respect to the lexicographical monomial ordering $\prec$, in which

$$p \prec S \prec w \prec r \prec K \prec c_1 \prec \cdots \prec c_A \prec k_0 \prec \cdots \prec k_A$$

At the first step, it is obvious that $\mathfrak{f}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$ is the generator of $\mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}} \cap \mathbb{Q}[p]$. Therefore, as p is the smallest variable under $\prec$, it follows that $\mathfrak{f}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}} \in G$. It is also easy to see that $k_0, k_A \in G$. As $p^{\gamma+1} - \beta(1+r) \in \mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$, it follows that $r - \frac{p^{\gamma+1}}{\beta} + 1 \in \mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$. Now, since the leading monomial equals to r, this implies that $r - \frac{p^{\gamma+1}}{\beta} + 1$ appears in the Gröbner basis. However, from the hypothesis, G is reduced and so $r - \bar{r} \in G$, where $\bar{r}$ is the remainder of $\frac{p^{\gamma+1}}{\beta} - 1$ on division by $\mathfrak{f}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$. As $h_r \in \mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$, it concludes that $(p^{\gamma+1} - \beta + \beta\delta)S^{n-m} - \alpha\beta \in \mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$. Now, as $p^{\gamma+1} - \beta + \beta\delta$ is an univariate polynomial on p, regarding Lemma 4.7, Algorithm Pol-Quo calculates a polynomial $g \in \mathbb{Q}[p]$ such that $S^{n-m} - g \in \mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$. To show that $S^{n-m} - g \in G$, it is enough to prove that each value of p leads to $n - m$ values of S. To see this, let $\tilde{p}$ (resp. $\tilde{S}$) be the value of p (resp. S) in a solution of System (8). So,

$$\tilde{S}^{n-m} = \frac{\alpha\beta}{\tilde{p}^{\gamma+1} - \beta + \beta\delta}.$$

Note that $\tilde{p}^{\gamma+1} - \beta + \beta\delta$ can never be zero as else, $\alpha = 0$ which contradicts $\alpha \in (0,1)$. Thus, there exist $n - m$ values for S per each value of p and so there exists no polynomial in G whose leading monomial is a pure power of S with a smaller exponent than $n - m$. It follows that $R = S^{n-m} - g \in G$. In the sequel, for each one of the variables $\{w, K, c_1, \ldots, c_A, k_1, \ldots, k_{A-1}\}$ we generate a polynomial whose leading monomial is the selected variable. For w and K, the leading monomials of h_w and h_K are w and K respectively. So, the algorithm computes the remainder of the tail of these polynomials on division by R (to keep G reduced) and then puts them into G. From Proposition 5.1 we know that $\Xi^+ \in \mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$. On the other hand, applying Lemma 4.7 and Algorithm Pol-Quo, there exists a polynomial $g \in \mathbb{Q}[p]$ such that $(1 - p^A)g - (1 - p)$ coincides with zero modulo $\mathfrak{f}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$ and so $(1 - p^A)g - (1 - p) \in \mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$. It follows that

$$g\Xi^+ - c_1((1 - p^A)g - (1 - p)) = (1 - p)(c_1 - g(rS^n + w)) \in \mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}.$$

Thus,

$$\mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}} = \mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}} + \langle 1-p\rangle \cap \mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}} + \langle c_1 - g(rS^n + w)\rangle.$$

However, regarding to Theorem 5.2, $\mathfrak{f}^+_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$ has no common root with $p-1$ and so $\mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}} + \langle 1-p\rangle = \langle 1\rangle$. This implies that

$$\mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}} = \mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}} + \langle c_1 - g(rS^n + w)\rangle,$$

and thus $c_1 - g(rS^n + w) \in \mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$. The leading monomial of this polynomial is c_1 and after computing the normal form of $g(rS^n + w)$ with respect to G, it must appear in G. Now it is easy to see that for each $a = 2,\ldots,A$, $c_a - p^{a-1}c_1$ must belong to the Gröbner basis as its leading monomial equals to c_a. So, after computing the normal form, the algorithm puts these polynomials in G. Similarly, $k_1 - \frac{w}{A} + c_1 \in \mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$ and its leading monomial equals to k_1. So, after computing the normal form, the algorithm puts this polynomial to G. Finally, for each $a = 2,\ldots,A-1$, $k_a + c_a - k_{a-1}(1+r) - \frac{w}{A} \in \mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$ with k_a as the leading monomial. However, a polynomial with k_1 as the leading monomial is already computed in G. Therefore, computing recursively, the algorithm computes a polynomial per each k_a, whose leading monomial equals to k_a. Till now, it is proved that $\mathrm{LM}(G) = \langle p^{(\gamma+2)(A-1)}, S^{m-n}, K, r, w, c_1, \ldots, c_A, k_0, \ldots, k_A\rangle$. It concludes by the first Buchberger's criterion and the process of the algorithm that G is the reduced Gröbner basis of $\mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$ with respect to $\prec$ since $G \subset \mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}$ and as is shown, $\mathrm{LM}(\mathcal{J}_{A,\gamma,\alpha,\beta,\delta,\vec{l}}) = \langle \mathrm{LM}(G)\rangle$. $\square$

Appendix 2

$K = 2.0409244293245756050$	$r = -0.18486791759699104768$	$w = 0.49431080261690741940$
$c_1 = 0.21080527207953000000e-4$	$c_2 = 0.23471671066456365742e-4$	$c_3 = 0.26134040191087705069e-4$
$c_4 = 0.29098400602641880000e-4$	$c_5 = 0.32399002598392350000e-4$	$c_6 = 0.36073990586778370000e-4$
$c_7 = 0.40165828616700368477e-4$	$c_8 = 0.44721799901326440764e-4$	$c_9 = 0.49794550633324491000e-4$
$c_{10} = 0.55442698621800398110e-4$	$c_{11} = 0.61731510573991087405e-4$	$c_{12} = 0.687336564570838346e-4$
$c_{13} = 0.76530049003456682572e-4$	$c_{14} = 0.85210778763012292319e-4$	$c_{15} = 0.948761565136893900e-4$
$c_{16} = 0.10563786650274360000e-3$	$c_{17} = 0.11762026826081099200e-3$	$c_{18} = 0.130961822141259078e-3$
$c_{19} = 0.14581669565479547076e-3$	$c_{20} = 0.16235654319204755023e-3$	$c_{21} = 0.180772489702336418e-3$
$c_{22} = 0.20127733930783384839e-3$	$c_{23} = 0.22410803428974351850e-3$	$c_{24} = 0.249528393033405892e-3$
$c_{25} = 0.27783215864423255709e-3$	$c_{26} = 0.30934639398617138000e-3$	$c_{27} = 0.344435256911831550e-3$
$c_{28} = 0.38350421687710063400e-3$	$c_{29} = 0.42700473129640346847e-3$	$c_{30} = 0.475439467220943050e-3$
$c_{31} = 0.52936810864205471943e-3$	$c_{32} = 0.58941382399908838991e-3$	$c_{33} = 0.656270467064720049e-3$
$c_{34} = 0.73071059485823423229e-3$	$c_{35} = 0.81359439474105328149e-3$	$c_{36} = 0.905879624311400331e-3$
$c_{37} = 0.10086326792615874100e-2$	$c_{38} = 0.11230409131820538000e-2$	$c_{39} = 0.125042636372800160e-2$
$c_{40} = 0.13922610233390371485e-2$	$c_{41} = 0.15501838519599945408e-2$	$c_{42} = 0.172601971509094157e-2$
$c_{43} = 0.19218004710552185743e-2$	$c_{44} = 0.21397884498345092004e-2$	$c_{45} = 0.238250259536279309e-2$
$c_{46} = 0.26527475729414137612e-2$	$c_{47} = 0.29536461783578346058e-2$	$c_{48} = 0.328867542337575485e-2$
$c_{49} = 0.36617067129177452400e-2$	$c_{50} = 0.40770505830502145670e-2$	$c_{51} = 0.453950650882590608e-2$
$c_{52} = 0.50544183654055072608e-2$	$c_{53} = 0.56277361785634599678e-2$	$c_{54} = 0.626608487977449163e-2$
$c_{55} = 0.69768408601141727714e-2$	$c_{56} = 0.77682172076010988438e-2$	$c_{57} = 0.864935861293036643e-2$
$c_{58} = 0.96304470402611016504e-2$	$c_{59} = 0.10722819384265801755e-1$	$c_{60} = 0.119390984722517174e-1$
$k_0 = k_{60} = 0$	$k_1 = 0.82174328497216710000e-2$	$k_2 = 0.14913334856395323240e-1$
$k_3 = 0.20368717033876494447e-1$	$k_4 = 0.24812609707824340510e-1$	$k_5 = 0.28431668596177993680e-1$
$k_6 = 0.31378004615236389054e-1$	$k_7 = 0.33775565792044630356e-1$	$k_8 = 0.35725338855376325467e-1$
$k_9 = 0.37309588682031969556e-1$	$k_{10} = 0.38595313394254801325e-1$	$k_{11} = 0.39637060044371333389e-1$
$k_{12} = 0.40479219014729689990e-1$	$k_{13} = 0.41157893417407663139e-1$	$k_{14} = 0.417024219666761047e-1$
$k_{15} = 0.42136619279252907370e-1$	$k_{16} = 0.42479785729603989060e-1$	$k_{17} = 0.427475293104362962e-1$
$k_{18} = 0.42952434139174670900e-1$	$k_{19} = 0.43104603765397919991e-1$	$k_{20} = 0.432121022621816280e-1$
$k_{21} = 0.43281311789195841367e-1$	$k_{22} = 0.43317221845477611169e-1$	$k_{23} = 0.433236625894234975e-1$
$k_{24} = 0.43303492287701580924e-1$	$k_{25} = 0.43258747062051861087e-1$	$k_{26} = 0.431907595579170382e-1$
$k_{27} = 0.43100251899427354640e-1$	$k_{28} = 0.42987407242914697159e-1$	$k_{29} = 0.428519234286492616e-1$
$k_{30} = 0.42693051489079422105e-1$	$k_{31} = 0.42509621232719172990e-1$	$k_{32} = 0.423000556305243953e-1$
$k_{33} = 0.42062375341726129328e-1$	$k_{34} = 0.41794194385188166064e-1$	$k_{35} = 0.414927076837453089e-1$
$k_{36} = 0.41154670971397371698e-1$	$k_{37} = 0.40776373347350818700e-1$	$k_{38} = 0.403536025834632314e-1$
$k_{39} = 0.39881603119541163004e-1$	$k_{40} = 0.39355026553998825837e-1$	$k_{41} = 0.387678742729666603e-1$
$k_{42} = 0.38113431748311598361e-1$	$k_{43} = 0.37384193894405603517e-1$	$k_{44} = 0.365717807452013024e-1$
$k_{45} = 0.35666842577598145899e-1$	$k_{46} = 0.34658953467009207458e-1$	$k_{47} = 0.335364921120302402e-1$
$k_{48} = 0.32286508605305268160e-1$	$k_{49} = 0.30894575657150494796e-1$	$k_{50} = 0.293446225842622927e-1$
$k_{51} = 0.27618750182555386287e-1$	$k_{52} = 0.25697024361212759967e-1$	$k_{53} = 0.235572461774944954e-1$
$k_{54} = 0.21174695629506616340e-1$	$k_{55} = 0.18521846259555937144e-1$	$k_{56} = 0.155680472808386360e-1$
$k_{57} = 0.12279169562987250218e-1$	$k_{58} = 0.86172113927335123920e-2$	$k_{59} = 0.453985945976353349e-2$

Table 1: The equilibrium of OLG model in Example 6.1

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