Economic Models Involving Time Fractal

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Abstract:
In this article, the price adjustment equation has been proposed and studied in the frame of fractal calculus which plays an important role in market equilibrium. Fractal time has been recently suggested by researchers in physics due to the self-similar properties and fractional dimension. We investigate the economic models from the viewpoint of local and non-local fractal Caputo derivatives. We derive some novel analytical solutions via the fractal Laplace transform. In fractal calculus, a useful local fractal derivative is a generalized local derivative in the standard computational sense, and the non-local fractal Caputo fractal derivative is a generalization of the non-local fractional Caputo derivative. The economic models involving fractal time provide a new framework that depends on the dimension of fractal time. The suggested fractal models are considered as a generalization of standard models that present new models to economists for fitting the economic data. In addition, we carry out a comparative analysis to understand the advantages of the fractal calculus operator on the basis of the additional fractal dimension of time parameter, denoted by α, which is related to the local derivative, and we also indicate that when this dimension is equal to 1, we obtain the same results in the standard fractional calculus as well as when α and

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the nonlocal memory effect parameter, denoted by γ, of the nonlocal fractal derivative are both equal to 1, we obtain the same results in the standard calculus.

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1 Introduction

Fractal geometry is an extension of classical geometry. The word "fractal" was invented in the 1960s by B. Mandelbrot. Fractal geometry is used to illustrate real objects like trees, lightning, meandering rivers, and coastlines [1–9]. Fractals are geometric structures where their fractal dimension exceeds their topological dimension. Fractals have many properties such as complicated structure, self-similarity of structures, discrete scale symmetry, and fractal dimension which can be a real number. It is well known that many processes in nature can be represented by fractals; such types can be found anywhere in nature [1–15].

The analysis on fractals was constructed using various techniques, such as harmonic analysis, probabilistic methods, measure theory, fractional calculus, fractional spaces, and time-scale calculus [16–33]. A Riemann-like calculus, named fractal calculus or F^{α}-calculus, was formed on fractal sets, which is algorithmic and can be applied easily [34–38]. Fractal calculus has been applied in optics and solid-state physics, and it has been used to define a fractal logistic equation, fractal Shannon entropy, and fractal mean-square calculus [39–43]. The Fourier and Laplace transforms have been introduced for F^{α}-calculus in [39]. The analog of the Sumudu transform in F^{α}-calculus has been introduced in [44]. The appearance of various types of anomalous diffusion on the same completely disconnected set is presented in [40]. Fractal time was considered in many mathematical models for fractal-time processes [45–50]. Therefore, mathematical economics which is a interesting field of research in theoretical and applied sciences has attracted the interests of researchers due to its important role in proposing new mathematical models arising from economic phenomena and processes [62]. As a result, with the help of fractional calculus, many economic models can be formulated in the sense of fractional-order derivatives; hence, the dynamics of economic processes with memory effect and nonlocality can be well-described (for more background information about a survey of some recent proposed mathematical models with continuous time for interpreting the dynamics of economic models with long memory, we refer to [62], and see also [63, 64]). Due to geometrical fractal properties, fractals have been widely seen and applied in economics such as money flow, sales data, transactions' intervals, network structures of economic agents (i.e. money transfer in banks' network), market price fluctuation, income distributions, and market properties [65]. Therefore, simple mathematical models of fractals are needed to be proposed and analyzed to construct more realistic economic models for a possible comparative analysis with real data in economy in order to provide a good prediction for the complex real world economy as mentioned in [65]. Motivated by those previous research studies and the experimental results of the demand and supply balance where fractal properties can be seen in many economic data due to the fact that economic models are generally around the critical point of the phase transition and market price fluctuations, macroscopic financial data analysis of companies, and open markets' price fluctuations in [66], we involve the fractal time in our proposed economic models. For more detailed analysis of macroeconomic models in steadystate fractal, we refer to [67]. This paper is organized as follows: In Section 2, we provide a review of basic tools. We present a fractal economic model with its solutions in Section 3. In Section 4, we give a conclusion about our research work.

2 Some fundamental tools

In this section, we summarize and review some basic tools.

2.1 Economic models

Mathematical economic models, symbolizing economic procedure by certain variables, allowing economists to make expressive and favorable demands regarding controversial circumstances. As expected, economists have benefited from mathematical economic models which help in making strong predictions about maximum gain. Investigating the relationship between demand and supply, specifically those correlated with a price adjustment, has dealt with a primary state for equilibrium progressively.

Both of the market equilibrium and economic development play a major role in addressing real-world problems. In addition, market competition plays an important role for buyers and sellers of merchandise. Buyers and sellers represent the two economic actors for goods whose costs may rise or fall quickly. For more details, we refer to [51–60]. It is really hard to identify whether buyers select cheap products or expensive products, since snob impact may cause buyers to select expensive products. Therefore, there is a considerable difficulty concerning the buyer's options. A totally competitive market is formed by several buyers and sellers of an economic item where each item has no control over the price of the market. In this model, the total claimed amount by the product's purchasers is represented as a function of the product price known as the demand function. Similarly, the total amount provided by the sellers of product is expressed as a function of the price of the product known as the supply function. We consider two methods and their solutions as follows:
First model: Let us consider the **demand and supply functions** of the form [51–59]

$$q_d(t) = d_0 - d_1 p(t) \quad \text{and} \quad q_s(t) = s_0 - s_1 p(t), \tag{1}$$

respectively where p is the market price of the product, q_d is the quantity demanded, q_s is the quantity supplied, and $d_0,\ d_1,\ s_0,\ s_1$ are positive constants. It is easy to see that the **equilibrium price** can be achieved by considering $q_d = q_s$, and it is given by

$$p^* = \frac{(d_0 + s_0)}{(d_1 + s_1)}. \tag{2}$$

Economists believe that markets are in equilibrium, meaning that the supply of a product is precisely equivalent to the demand of the product. For example, consider the basic **price adjustment equation**:

$$\frac{dp}{dt} = \lambda \left(q_d - q_s \right), \tag{3}$$

where $\lambda > 0$ is the **speed of adjustment** constant. This shows that price increases when demand exceeds supply and decreases when supply exceeds demand. Inserting Eq. (1) in Eq. (3), we get

$$p'(t) + \lambda \left(d_1 + s_1 \right) p(t) = \lambda \left(d_0 + s_0 \right). \tag{4}$$

The solution of Eq.(4) is

$$p\left(t\right)=p^{*}+\left[p(0)-p^{*}\right]e^{-\lambda\left(d_{1}+s_{1}\right)}. \tag{5}$$

If $t\rightarrow\infty$, then we have $p(t)=p^{*}$ which is called **globally stable**.
Second model: By adding the **expectations of agents** factor in Eq.(1), we have

$$q_{d}\left(t\right)=d_{0}-d_{1}p\left(t\right)+d_{2}p'\left(t\right) \quad \text{and} \quad q_{s}\left(t\right)=s_{0}-d_{1}p\left(t\right)+s_{2}p'\left(t\right), \tag{6}$$

where d_2 and s_2 are **supplemental factors**. By equating $q_d(t)$ and $q_s(t)$ as we did above, we obtain

$$p'\left(t\right)+\lambda\frac{\left(d_{1}+s_{1}\right)}{\left(d_{2}+s_{2}\right)}p\left(t\right)=\lambda\frac{\left(d_{0}+s_{0}\right)}{\left(d_{2}+s_{2}\right)}. \tag{7}$$

Therefore, the solution of Eq.(7) is

$$p\left(t\right)=p*+\left(b-p*\right)e^{\mu t}, \tag{8}$$

where $\mu=\frac{(d_{1}+s_{1})}{(d_{2}+s_{2})}$ and $p\left(t\right)$ is called **market clearing time paths**. In addition, when $p'(t)=0$ for each $t\geq 0$, then the market is in **dynamic equilibrium** which means equilibrium in a changing economy. We can easily show that the dynamic equilibrium in this model is $p(t)=p*,\ \forall\ t$.

2.2 Fractal calculus

In this subsection, we give a brief review of local and non-local fractal calculus.

2.3 Staircase functions

In this subsection, we present some basic tools of fractal calculus on thin Cantor-like set (κ) which is shown in Fig. 1a [34–36, 61].
Definition 2.1.1. Let $p[a_1,a_2]$ be a subdivision of an interval $I=[a_1,a_2]$ which is a collection of points $\{a_1=t_0,t_1,...,t_n=a_2\}$, such that $t_i<t_{i+1}$. For more details, we refer to [34–36, 61].
Definition 2.1.2. Assume that $\kappa\subset\mathbf{R}$, is a thin Cantor-like set and $p[a_1,a_2]$ is a subdivision. The mass function is given by [34–36, 61]

$$\Psi^{\alpha}\left(\kappa,a_{1},a_{2}\right)=\lim_{\varsigma\rightarrow0}\Psi_{\varsigma}^{\alpha}, \tag{9}$$

where

$$\Psi_{\varsigma}^{\alpha} = \inf_{\{p[a_1,a_2]:|p|\leq\varsigma\}} \sum_{j=0}^{m-1} \Gamma(\alpha+1)(t_{j+1}-t_j)^{\alpha}\phi(\kappa,[t_{j+1}-t_j]) \tag{10}$$

and

$$\phi(\kappa,[t_{j+1}-t_j]) = \begin{cases} 1, & \kappa\cap[t_{j+1}-t_j]\neq\emptyset; \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

$$|p| = \max_{0\leq j\leq m}(t_{j+1}-t_j).$$

Definition 2.1.3. Assume that $c_0 \in \mathbf{R}$. The staircase function of order α is given by [34–36]

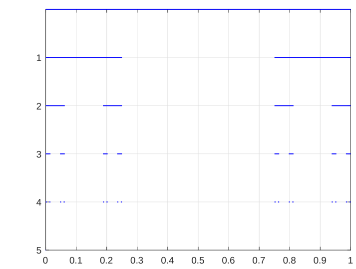
$$S_{\kappa}^{\alpha}(t) = \begin{cases} \Psi^{\alpha}(\kappa,c_0,t), & \text{if t} \geq c_0, \\ -\Psi^{\alpha}(\kappa,c_0,t), & \text{otherwise.} \end{cases} \tag{12}$$

The integral staircase function is illustrated in Fig. 1b.
Definition 2.1.4. The $\Psi-$dimension is defined using the mass function as [34–36]

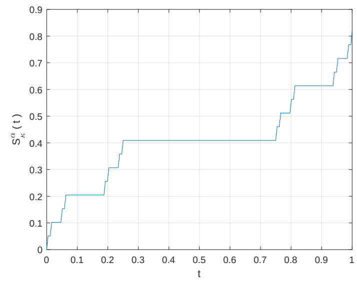
$$\begin{aligned} \dim_{\Psi}(\kappa\cap[a_1,a_2]) &= \inf\{\alpha : \Psi^{\alpha}(\kappa,a_1,a_2)=0\}, \\ &= \sup\{\alpha : \Psi^{\alpha}(\kappa,a_1,a_2)=\infty\}. \end{aligned} \tag{13}$$

Fig. 1c presents $\Psi-$ dimension which is an intersection point of the red line with the blue line.
Definition 2.1.5. The characteristic function $\chi\kappa(\alpha,t)$ for a given thin Cantor set is defined by [34–36]

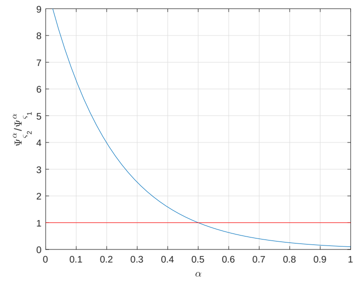
$$\chi\kappa(\alpha,t) = \begin{cases} \dfrac{1}{\Gamma(\alpha+1)}, & t\in\kappa, \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$



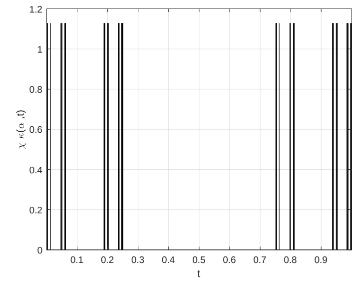
(a) The thin Cantor-like set ($\kappa = 1/2$) by iteration.



(b) The integral staircase function for the thin Cantor set κ for the case of $\kappa = 1/2$.



(c) Ψ − dimension of the thin Cantor set $\kappa = 1/2$.



(d) Characteristic function thin Cantor set with $\kappa = 1/2$.

Figure 1: Graphs corresponding to thin Cantor set with $\kappa = 1/2$.

In Fig. 1d, we have plotted the characteristic function of thin Cantor set.

2.4 Local Fractal Calculus

Definition 2.2.1. If κ is α-perfect set, then the κ-derivative of $f(t)$ at t is defined by [34–36]

$$D_{\kappa}^{\alpha} f(t_0) = \begin{cases} \kappa\text{-}\lim\limits_{t \to t_0} \dfrac{f(t) - f(t_0)}{S_{\kappa}^{\alpha}(t) - S_{\kappa}^{\alpha}(t_0)}, & \text{if } t_0 \in \kappa, \\ 0 & \text{otherwise}, \end{cases} \tag{15}$$

if the limit exists.
Definition 2.2.2. The κ-integral of $f(t)$ on $[a_1, a_2]$ is defined by [34–36]

$$\int_{a_1}^{a_2} f(t)\, d_\kappa^\alpha t \approx \sum_{j=1}^{n} f_j(t)\left(S_\kappa^\alpha(t_j) - S_\kappa^\alpha(t_{j-1})\right). \tag{16}$$

2.5 Non-Local Fractal Calculus

Definition 2.3.1 For a function $f(t),\ t \in \kappa$, the **fractal left-Riemann-Liouville integral** is defined by [34–37]

$${}_a\mathbb{J}_t^\gamma f(t) = \frac{1}{\Gamma_\kappa^\alpha(\gamma)} \int_a^t \frac{f(x)}{\left(S_\kappa^\alpha(t) - S_\kappa^\alpha(x)\right)^{\alpha-\gamma}} d_\kappa^\alpha x. \tag{17}$$

where $t > a$.
Definition 2.3.2 The **fractal left-sided Caputo derivative** is given by [37]

$${}_a^C\mathbb{D}_t^\gamma f(t) = \frac{1}{\Gamma_\kappa^\alpha(n-\gamma)} \int_a^t \frac{(D_\kappa^\alpha)^n f(x)}{(S_\kappa^\alpha(t) - S_\kappa^\alpha(x))^{-n\alpha+\gamma+\alpha}} d_\kappa^\alpha x. \tag{18}$$

where $n\alpha - \alpha < \gamma \leq n\alpha,\ n \in \mathbf{N}$.
Definition 2.3.3 The **fractal left-sided RiemannLiouville derivative** is given by [37]

$${}_a\mathbb{D}_t^\gamma f(t) = \frac{1}{\Gamma_\kappa^\alpha(n-\gamma)} \left(D_\kappa^\alpha\right)^n \int_a^t \frac{f(x)}{\left(S_\kappa^\alpha(t) - S_\kappa^\alpha(x)\right)^{-n\alpha+\gamma+\alpha}} d_\kappa^\alpha x. \tag{19}$$

where $n\alpha - \alpha \leq \gamma < n\alpha,\ n \in \mathbf{N}$.
We define here the fractal Laplace transforms in order to apply them to the economic model.
The **fractal Laplace transform** (FLT) of a function $f(t)$, where t is in the thin Cantor-like sets, is presented by [37]

$$B(w) = L_\kappa^\alpha(f) = \int_0^\infty \exp\left(-S_\kappa^\alpha(t)\, S_\kappa^\alpha(w)\right) f(t)\, d_\kappa^\alpha t, \tag{20}$$

where $B(w)$ and $L(f) : f(t) \to B(w)$ are called the fractal Laplace transforms of $f(t)$.
If $\exists\ M, T > 0$, and $at^\alpha < S_\kappa^\alpha(t) < bt^\alpha;\ a,\ b \in \mathbf{R}$ [50] such that

$$e^{-\varepsilon S_\kappa^\alpha(t)}|f(t)| < e^{-\varepsilon t^\alpha}|f(t)| \leq M \qquad \forall\ t > T, \tag{21}$$

then the fractal integral in Eq.(22) exists.
The inverse fractal Laplace transform is defined as follows:

$$f(t) = \left(L_{\kappa}^{\alpha}\right)^{-1}\left(B\left(w\right)\right). \tag{22}$$

The **fractal convolution** is defined as

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d_{\kappa}^{\alpha}\tau = \int_0^t f(\tau)g(t-\tau)d_{\kappa}^{\alpha}\tau, \tag{23}$$

and its fractal Laplace transform is given by

$$L_{\kappa}^{\alpha}(f(t) * g(t)) = B(w)G(w), \tag{24}$$

where $G\left(w\right) = L_{\kappa}^{\alpha}\left(g\right)$.
The fractal **Laplace transform of the local fractal derivative** is defined by

$$L_{\kappa}^{\alpha}\left(\left(D_t^{\alpha}\right)^n f\left(t\right)\right) = S_{\kappa}^{\alpha}\left(w\right)^n L_{\kappa}^{\alpha}\left(f\left(t\right)\right) - \sum_{k=0}^{n-1} S_{\kappa}^{\alpha}\left(w\right)^{n-k-1}\left(D_{\kappa,t}^{\alpha}\right)^n f|_{t=0}. \tag{25}$$

The **fractal Laplace transform of the fractal left-sided RiemannLiouville derivative** is defined by

$$L_{\kappa}^{\alpha}(({}_0\mathbb{D}_t^{\gamma})f\left(t\right)) = S_{\kappa}^{\alpha}\left(w\right)^{\gamma} B(w) - \sum_{k=0}^{n-1} S_{\kappa}^{\alpha}\left(w\right)^k \left[{}_0\mathbb{D}_{\kappa,t}^{\gamma-k-1} f\left(t\right)\right]_{t=0}. \tag{26}$$

The fractal Laplace transform of **the fractal Caputo derivative** is defined by [37]

$$L_{\kappa}^{\alpha}\left({}_0^C\mathbb{D}_t^{\gamma} f\left(t\right)\right) = S_{\kappa}^{\alpha}(w)^{\gamma} L_{\kappa}^{\alpha}\left(f\left(t\right)\right) - \sum_{j=0}^{n-1} S_{\kappa}^{\alpha}(w)^{\gamma-j-1}(D_{\kappa}^{\alpha})^j(f\left(t\right)|_{t=0}. \tag{27}$$

The **fractal Mittag-Lefler function** of one parameter is given by [37]

$$E_{\kappa,\gamma}^{\alpha}\left(t\right) = \sum_{j=0}^{\infty} \frac{S_{\kappa}^{\alpha}(t)^j}{\Gamma^{\alpha}\left(\gamma j + 1\right)}, \qquad \gamma > 0. \tag{28}$$

Some important formulas of local and non-local fractal calculus can be stated as follows:

$$
\begin{aligned}
D_{\kappa}^{\alpha} c\chi_{\kappa} &= 0, \quad c \text{ is constant}, &(29)\\
{}_0\mathbb{D}_t^{\gamma} c\chi_{\kappa} &= \frac{cS_{\kappa}^{\alpha}(t)^{-\gamma}}{\Gamma_{\kappa}^{\alpha}(1-\gamma)} &(30)\\
D_{\kappa}^{\alpha} S_{\kappa}^{\alpha}(t) &= \chi\kappa(\alpha, t) &(31)\\
D_{\kappa}^{\alpha} S_{\kappa}^{\alpha}(t)^{m} &= m\chi_{\kappa} S_{\kappa}^{\alpha}(t)^{m-1} &(32)\\
D_{\kappa}^{\alpha} \sin(S_{\kappa}^{\alpha}(t)) &= \chi\kappa(\alpha, t)\cos(S_{\kappa}^{\alpha}(t)) &(33)\\
D_{\kappa}^{\alpha}(f(t)g(t)) &= D_{\kappa}^{\alpha}(f(t))g(t) + f(t)D_{\kappa}^{\alpha}(g(t)), &(34)\\
{}_0\mathbb{D}_t^{\gamma} f(t)g(t) &= \sum_{n=0}^{\infty} \binom{\gamma}{n}\, {}_0\mathbb{D}_t^{n} f(t)\, {}_0\mathbb{D}_t^{\gamma-n} g(t), &(35)\\
\int S_{\kappa}^{\alpha}(t)^{n} d_{\kappa}^{\alpha} t &= \frac{S_{\kappa}^{\alpha}(t)^{n+1}}{n+1} + c &(36)\\
\sin(S_{\kappa}^{\alpha}(t)) &= \sum_{i=1}^{\infty} (-1)^{i-1} \frac{S_{\kappa}^{\alpha}(t)^{2i-1}}{(2i-1)!}, &(37)\\
L_{\kappa}^{\alpha}\left(E_{\kappa,\gamma}^{\alpha}\left(-at^{\gamma}\right)\right) &= \frac{S_{\kappa}^{\alpha}(w)^{\gamma-1}}{\left(S_{\kappa}^{\alpha}(w)^{\gamma} + a\right)} &(38)\\
L_{\kappa}^{\alpha}\left(1 - E_{\kappa,\gamma}^{\alpha}\left(-at^{\gamma}\right)\right) &= \frac{a}{S_{\kappa}^{\alpha}(w)\left(S_{\kappa}^{\alpha}(w)^{\gamma} + a\right)}. &(39)
\end{aligned}
$$

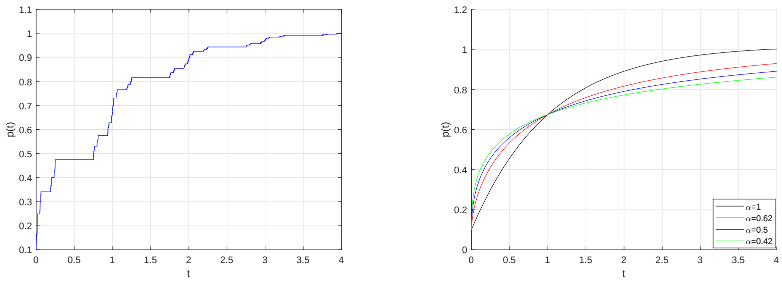
3 Application

In this section, we apply our main results for economic models which are investigated in details via local and nonlocal fractal Laplace transforms. The crucial claim is to obtain much better results via underlying derivatives and integrals for differentiable functions.

3.1 Economic model in the frame of local fractal derivative

Local fractal first model: The price adjustment equation in local fractal calculus is presented without taking into account the expectations of the agent (for standard version see [51–53]) as follows:

$$
D_{\kappa,t}^{\alpha} p\left(t\right) + \lambda\left(d_1 + s_1\right) p\left(t\right) = \lambda\left(d_0 + s_0\right), \tag{40}
$$



(a) The graph of Eq.(42) in the case of $\kappa = 1/2, d_0 = 30,\ s_0 = 20,\ d_1 = 40,\ s_1 = 9,\ \lambda = 0.02,\ b = 0.1,$ and $\alpha = 0.5$.

(b) The sketch of Eq. (43) using $d_0 = 30,\ s_0 = 20,\ d_1 = 40,\ s_1 = 9,\ \lambda = 0.02,\ b = 0.1$.

Figure 2: The graph of the local fractal first model

using the initial condition $p\left(0\right) = b$.
By applying the local fractal Laplace transform to both sides of Eq. (40), we obtain

$$p^{\alpha}_{\kappa}\left(w\right) = \frac{\lambda\left(d_0 + s_0\right)}{S^{\alpha}_{\kappa}\left(w\right)\left(S^{\alpha}_{\kappa}\left(w\right) + \lambda\left(d_1 + s_1\right)\right)} + \frac{b}{\left(S^{\alpha}_{\kappa}\left(w\right) + \lambda\left(d_1 + s_1\right)\right)}. \tag{41}$$

By taking the inverse Laplace transform of Eq. (41), we get the following solution:

$$p\left(t\right) = p* + \left(b - p*\right) e^{-\lambda\left(d_0 + s_0\right) S^{\alpha}_{\kappa}\left(t\right)}. \tag{42}$$

By using $S^{\alpha}_{\kappa}\left(t\right) \leq t^{\alpha}$, we have

$$p\left(t\right) \approx p* + \left(b - p*\right) e^{-\lambda\left(d_0 + s_0\right) t^{\alpha}}, \tag{43}$$

where $p* = \frac{d_0 + s_0}{d_1 + s_1}$. In Figs. 2a and 2b, we have plotted Eqs.(42) and (43) for the case when $p* = 1.02$.

Local fractal second model: The price adjustment equation in local fractal calculus is presented by taking into account the expectations of the agents (for standard version see [51–53]) as follows:

$$D^{\alpha}_{\kappa,t} p\left(t\right) - \frac{\left(d_1 + s_1\right)}{\left(d_2 + s_2\right)} p\left(t\right) = -\frac{\left(d_0 + s_0\right)}{\left(d_2 + s_2\right)}, \tag{44}$$

using the initial conditions $p\left(0\right) = b$.
By applying the local fractal Laplace transform to both sides of Eq. (44),

we get

$$p_{\kappa}^{\alpha}(w) = -\frac{(d_0+s_0)}{(d_1+s_1)} \frac{1}{S_{\kappa}^{\alpha}(w)\left(S_{\kappa}^{\alpha}(w) - \frac{(d_1+s_1)}{(d_2+s_2)}\right)} + \frac{b}{\left(S_{\kappa}^{\alpha}(w) - \frac{(d_1+s_1)}{(d_2+s_2)}\right)}. \tag{45}$$

By taking the inverse Laplace of Eq. (45), the following solution can be obtained as follows:

$$p(t) = p* + (b - p*)\, e^{\frac{(d_1+s_1)}{(d_2+s_2)} S_{\kappa}^{\alpha}(t)}. \tag{46}$$

By $S_{\kappa}^{\alpha}(t) \leq t^{\alpha}$, we get

$$p(t) \approx p* + (b - p*)\, e^{\frac{(d_1+s_1)}{(d_2+s_2)} t^{\alpha}}, \tag{47}$$

where $p* = \frac{d_0+s_0}{d_1+s_1}$.

3.2 Economic model in the frame of non-local fractal Caputo derivative

Non-local fractal first model: The price adjustment equation in non-local fractal calculus is presented without allowing the expectations of agents (for the fractional standard calculus, we refer to [54–59]) as follows:

$${}_{0}^{C}\mathbb{D}_{\kappa,t}^{\gamma} p(t) + \lambda (d_1+s_1)\, p(t) = \lambda (d_0+s_0), \tag{48}$$

using $p(0) = b$.

By applying the non-local fractal Laplace transform to (48), we get

$$\begin{aligned} p_{\kappa}^{\alpha}(w) = &\frac{\lambda (d_0+s_0)}{S_{\kappa}^{\alpha}(w)\left(S_{\kappa}^{\alpha}(w)^{\gamma} + \lambda (d_1+s_1)\right)} \\ &+ \frac{b S_{\kappa}^{\alpha}(w)^{\gamma-1}}{\left(S_{\kappa}^{\alpha}(w)^{\gamma} + \lambda (d_1+s_1)\right)}, \end{aligned}$$

which implies the following:

$$\begin{aligned} p_{\kappa}^{\alpha}(w) = &\frac{(d_0+s_0)}{(d_1+s_1)} \frac{\lambda (d_1+s_1)}{S_{\kappa}^{\alpha}(w)\left(S_{\kappa}^{\alpha}(w)^{\gamma} + \lambda (d_1+s_1)\right)} \\ &+ \frac{b S_{\kappa}^{\alpha}(w)^{\gamma}}{S_{\kappa}^{\alpha}(w)\left(S_{\kappa}^{\alpha}(w)^{\gamma} + (\lambda d_1+s_1)\right)}. \end{aligned} \tag{49}$$

By taking the inverse Laplace of both sides of Eq. (49), the following solution can be achieved:

$$p(t) = \frac{(d_0+s_0)}{(d_1+s_1)}\left(1 - E^{\alpha}_{\kappa,\gamma}\left(-\lambda\left(d_1+s_1\right)S^{\alpha}_{\kappa}(t)^{\gamma}\right)\right) + bE^{\alpha}_{\kappa,\gamma}\left(-\lambda\left(d_1+s_1\right)S^{\alpha}_{\kappa}(t)^{\gamma}\right). \tag{50}$$

In view of Eq.(28), Eq. (50) takes the following form:

$$p(t) = \frac{(d_0+s_0)}{(d_1+s_1)}\left(1 - \sum_{j=0}^{\infty}\left(\frac{\left(-\lambda\left(d_1+s_1\right)S^{\alpha}_{\kappa}(t)^{\gamma}\right)^{j}}{\Gamma(\gamma j+1)}\right)\right) + b\sum_{j=0}^{\infty}\left(\frac{\left(-\lambda\left(d_1+s_1\right)S^{\alpha}_{\kappa}(t)^{\gamma}\right)^{j}}{\Gamma(\gamma j+1)}\right). \tag{51}$$

In view of $S^{\alpha}_{\kappa}(t) \leq t^{\alpha}$, we have

$$p(t) \approx \frac{(d_0+s_0)}{(d_1+s_1)}\left(1 - \sum_{j=0}^{\infty}\left(\frac{\left(-\lambda\left(d_1+s_1\right)t^{\alpha\gamma}\right)^{j}}{\Gamma(\gamma j+1)}\right)\right) + b\sum_{j=0}^{\infty}\left(\frac{\left(-\lambda\left(d_1+s_1\right)t^{\alpha\gamma}\right)^{j}}{\Gamma(\gamma j+1)}\right). \tag{52}$$

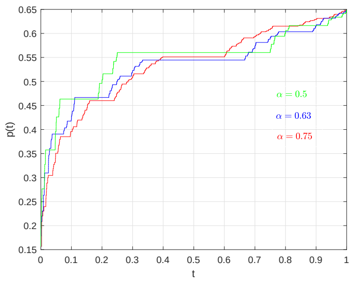
In Figs. 3a, 3c, and 3b, 3d, we have plotted Eqs. (51) and (52), respectively.

Non-local fractal second model: we consider a model that takes into account the expectations of agents. The price adjustment equation with non-local fractal Caputo derivative is given by

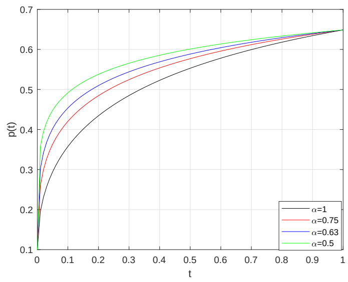
$${}^{C}_{0}\mathbb{D}^{\gamma}_{\kappa,t}p(t) - \frac{(d_1+s_1)}{(d_2+s_2)}p(t) = \frac{-(d_0+s_0)}{(d_2+s_2)}, \tag{53}$$

using $p(0) = b$.
By applying the non-local fractal Caputo Laplace transform to Eq. (53), we get

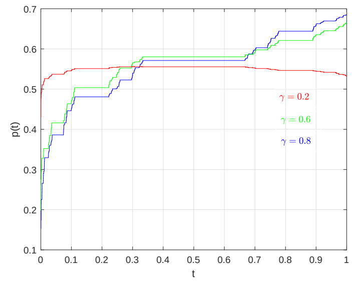
$$p^{\alpha}_{\kappa}(w) = -\frac{(d_0+s_0)}{(d_2+s_2)}\frac{1}{S^{\alpha}_{\kappa}(w)\left(S^{\alpha}_{\kappa}(w)^{\gamma} - \frac{(d_1+s_1)}{(d_2+s_2)}\right)} + \frac{bS^{\alpha}_{\kappa}(w)^{\gamma-1}}{\left(S^{\alpha}_{\kappa}(w)^{\gamma} - \frac{(d_1+s_1)}{(d_2+s_2)}\right)},$$



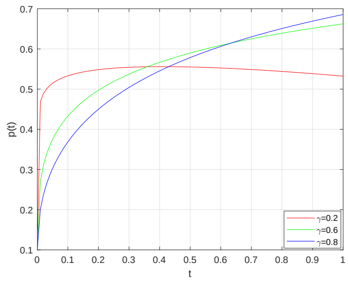
(a) The sketch of Eq. (51) using $d_0 = 30; s_0 = 20; d_1 = 40; s_1 = 5; b = 0.1; \lambda = 0.02; \gamma = 0.5$.



(b) The sketch of Eq. (52) using $d_0 = 30; s_0 = 20; d_1 = 40; s_1 = 5; b = 0.1; \lambda = 0.02; \gamma = 0.5$.



(c) The sketch of Eq. (51) using $d_0 = 30; s_0 = 20; d_1 = 40; s_1 = 5; b = 0.1; \lambda = 0.02; \alpha = 0.5$.



(d) The sketch of Eq. (52) using $d_0 = 30; s_0 = 20; d_1 = 40; s_1 = 5; \lambda = 0.02; b = 0.1; \alpha = 0.5$

Figure 3: The graph of non-local fractal first model

which implies the following:

$$p_\kappa^\alpha(w) = \frac{(d_0 + s_0)}{(d_1 + s_1)} \frac{-\frac{(d_1+s_1)}{(d_2+s_2)}}{S_\kappa^\alpha(w)\left(S_\kappa^\alpha(w)^\gamma - \frac{(d_1+s_1)}{(d_2+s_2)}\right)} + \frac{bS_\kappa^\alpha(w)^\gamma}{S_\kappa^\alpha(w)\left(S_\kappa^\alpha(w)^\gamma - \frac{(d_1+s_1)}{(d_2+s_2)}\right)}. \tag{54}$$

By taking the inverse Laplace of both sides of Eq. (54), we get the following solution:

$$p(t) = \frac{(d_0 + s_0)}{(d_1 + s_1)} \left(1 - E_{\kappa,\gamma}^\alpha\left(\frac{(d_1 + s_1)}{(d_2 + s_2)} S_\kappa^\alpha(t)^\gamma\right)\right) + bE_{\kappa,\gamma}^\alpha\left(\frac{(d_1 + s_1)}{(d_2 + s_2)} S_\kappa^\alpha(t)^\gamma\right). \tag{55}$$

With the help of Eq.(28), Eq.(55) takes the following form:

$$p\left(t\right)=\frac{\left(d_{0}+s_{0}\right)}{\left(d_{1}+s_{1}\right)}\left(1-\sum_{j=0}^{\infty}\frac{\left(\left(\frac{\left(d_{1}+s_{1}\right)}{\left(d_{2}+s_{2}\right)}\right)S_{\kappa}^{\alpha}(t)^{\gamma}\right)^{j}}{\Gamma\left(\gamma j+1\right)}\right)+b\sum_{j=0}^{\infty}\frac{\left(\left(\frac{\left(d_{1}+s_{1}\right)}{\left(d_{2}+s_{2}\right)}\right)S_{\kappa}^{\alpha}(t)^{\gamma}\right)^{j}}{\Gamma\left(\gamma j+1\right)}. \tag{56}$$

By applying $S_{\kappa}^{\alpha}\left(t\right)\leq t^{\alpha}$, we obtain

$$p\left(t\right)\approx\frac{\left(d_{0}+s_{0}\right)}{\left(d_{1}+s_{1}\right)}\left(1-\sum_{j=0}^{\infty}\frac{\left(\left(\frac{\left(d_{1}+s_{1}\right)}{\left(d_{2}+s_{2}\right)}\right)t^{\alpha\gamma}\right)^{j}}{\Gamma\left(\gamma j+1\right)}\right)+b\sum_{j=0}^{\infty}\frac{\left(\left(\frac{\left(d_{1}+s_{1}\right)}{\left(d_{2}+s_{2}\right)}\right)t^{\alpha\gamma}\right)^{j}}{\Gamma\left(\gamma j+1\right)}. \tag{57}$$

Remark: Note that one can obtain the standard results in our research study by choosing $\alpha=\gamma=1$. From financial and economic point of view, our obtained results in figures 2-3 can be interpreted as follows: On one hand, in figure 2(a), the geometrical property of fractal can been seen breaking down on a large time scale where the market price is fluctuating where it is discrete at some points and then it starts going on continuous fluctuations. In figure 2(b), the critical point can be seen at $p(t)\approx 0.65$ where $t=1$ where typical economic systems are around the critical point. On the other hand, in figure 3(a), the market price fluctuation is discrete and the price is tending like a zigzag with stabilization from $t=0.4$ to $t=0.6$. In figure 3(c), the geometrical property of fractal can been seen here breaking down on a short time scale where the market price tends to be more discrete from $t=0$ to $t=0.3$ and from $t=0.65$ to $t=1$ when $\gamma=0.6;0.8$, while it is more continuous from $t=0$ to $t=1$ when $\gamma=0.2$ with less price fluctuations. Then, the economic system stabilizes from $t=0.35$ to $t=0.64$. In figure 3(b), the critical point can be seen at $p(t)\approx 0.65$ where $t=1$ which is in agreement with figure 2(b). In figure 3(d), one critical point can be seen at $p(t)\approx 0.52$ where $t=0.45$ for $\gamma=0.2;0.8$, and another critical point can be seen at $p(t)\approx 0.52$ where $t=0.65$ for $\gamma=0.6;0.8$.

Future Research Work: This research study can be extended further to discuss market interaction such as companies interaction data, money flow data, material flow data, and sales data analysis [65].

4 Conclusion

In this study, we have investigated market equilibrium models from the viewpoint of the fractal local derivative and non-local fractal Caputo derivative. The price adjustment equation, which has a significant position in the marketplace in order to achieve equilibrium, is solved by the underlying fractional derivatives by both considering and not considering the perceptions of the agents. Thus, four different solutions have been obtained for each sense. The fractal Laplace transform has been used to obtain some new analytical solutions for the models under consideration. The advantage of these models over their corresponding classical versions is the presence of one arbitrary order of derivatives in the local fractal sense that has a physical meaning, and two arbitrary orders in the non-local fractal sense that allow taking the benefits of memory effect. In addition, we have analyzed the obtained solutions in-depth with illustrated descriptions and comparative analysis. We have also demonstrated two cases when $\alpha = 1$ and $\alpha = \gamma = 1$, we obtain the same results in the standard fractional version and standard Calculus version, respectively. To study the effect of the fractal and fractional-order derivative, we have changed the value of α, and fixed the value of γ in Figs. 3a and 3b, while we have also changed the value of γ and fixed the value of α in Figs 3c and 3d.

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