Journal of Mathematics and Modeling in Finance (JMMF)
Vol. 4, No. 1, Winter & Spring 2024
Research paper

A comparison of the linear model and the efficient frontier for the evaluation of portfolio performance

Fatemeh Fattahi[1], **Farhad Hosseinzadeh Lotfi**[2], **Andrew C. Worthington**[3]

[1] Department of Mathematics and Computer Sciences, Kharazmi University, Tehran, Iran
f.fattahi3000@gmail.com

[2] Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran
farhad@hosseinzadeh.ir

[3] Department of Accounting, Finance and Economics, Griffith University, Brisbane, QLD, Australia
a.worthington@griffith.edu.au

Abstract:
Data envelopment analysis (DEA) is a methodology widely used for evaluating the relative performance of portfolios under a meanvariance framework. However, there has been little discussion of whether nonlinear models best suit this purpose. Moreover, when using DEA linear models, the portfolio efficiency obtained is not comparable to those on the efficient portfolio frontier. This is because a separable piecewise linear boundary usually below the efficient frontier is considered the efficient frontier, so the model does not fully explore the possibility of portfolio benchmarks. In this paper, and with use of the dual-Lagrangian function, we propose a linear model under a meanvariance framework to evaluate better the performance of portfolios relative to those on the efficient frontier.

Keywords: Data Envelopment Analysis, Efficiency, Portfolio, Dual-lagrangine.
Classification: MSC2010 or JEL Classifications: 99G10, 90C08, 49N05, 49N10.

1 Introduction

One of the most important principles in portfolio management is the evaluation of portfolio performance, including selecting the best portfolio in terms of returns. A central purpose of creating a portfolio is the risk reduction of investment, such that the return from one asset compensates for the loss from another. To address this, Markowitz (1952) [15] introduced the frontier method under a meanvariance framework. This is a nonlinear model and its operation calculates the distance between the portfolio and its image on the efficient frontier. Tobin (1958) [21] and Hanoch and Levy (1969) [4] later took steps to improve the Markowitz model, while Sharpe (1966) [18], Treynor (1965) [22] and Jensen (1968) [6] provide suitable

[1]Corresponding author
Received: 13/01/2024 Accepted: 16/05/2024
https://doi.org/10.22054/JMMF.2024.76182.1115

benchmarks for portfolio assessment. Of these, the so-called Sharpe index is the risk premium per unit of total risk, the Treynor index is the risk premium per unit of the systematic risk, and the Jensen index is the difference between the actual portfolio return and the estimated benchmark return.

Subsequently a number of other studies further explored aspects of the model, including Turner and Weigel (1992) [23] and Sharpe, Gordon and Jeffry (1995) [19]. Yoshimoto (1996) [25] considered multi-period portfolio selection with transaction costs using Markowitzs basic model. In contrast, Morey and Morey (1999) [17] developed a model for evaluating portfolio performance with significant change to the Markowitz model based on data envelopment analysis (DEA), by simultaneously considering the variance as an input and the mean return as an output over the same time horizon. Importantly, these models are nonlinear in the sense that both the Markowitz model and the Morey and Morey model depend particularly on the assumption that the asset return distribution is normal. This is problematic in that the return distributions of most financial assets exhibit strong asymmetry. As a result, the meanvariance model is unsuitable for the performance evaluation of these types of assets.

As variance as a risk measure in the presence of asymmetrical returns creates inappropriate results because the positive and negative changes in asset returns are considered alike, semi-variance is a good alternative. Markowitz (1992) [16], Choobineh and Branting (1986) [3] and Kaplan and Alldredge (1997) [8] have all examined the characteristics and computational problems of the semi-variance model, showing for the most part that the semi-variance model can well measure risk. Subsequently, Joro and Na (2006) [7] presented a nonlinear meanvarianceskewness model that considers the mean and skewness of each portfolio as its output and its variance as the input, while Lozano and Gutiérrez (2008) [12] described several linear diversification models with a single input and output. While these are completely different to DEA models, they are consistent with the second-order stochastic dominance (SSD) in the sense that being efficient according to these proposed models is a necessary condition for being SSD efficient.

Later, Branda and Kopa (2012) [1] considered the empirical influence of various risk measures on DEA efficiency and its relation to SSD and Briec et al. (2006) [2] introduced a nonparametric efficiency measurement approach for static portfolio selection in the meanvarianceskewness space. Subsequently, Kerstens et al. (2011) [9] provided the geometric display of the meanvarianceskewness portfolio frontier based upon the shortage function, Lozza et al. (2011) [13] proposed several models for portfolio selection and Soleimani et al. (2009) [20] added market sectors to the Markowitz model as an additional constraint and considered the cardinality constraints and minimization of transaction costs. Lastly, Liu et al. (2015) [11] examined the convergence of DEA models under the meanvariance framework, which suggested that appropriate DEA models with sufficient data could be used to effectively approximate portfolio efficiency. They also examined some basic portfolios using DEA models to estimate their portfolio efficiency via their efficiency scores and verified the validity of the models using simulations.

In evaluating portfolio performance, it is increasingly evident that nonlinear models capture the complexities inherent in financial markets more effectively than traditional linear models. However, the usefulness of Data Envelopment Analysis (DEA) in linear settings often falls short when juxtaposed with the nonlinear efficiency portfolio frontier, particularly in volatile market conditions. As a result, Markowitz (1952) [15] introduced the frontier method under a mean-variance framework, which has undergone various enhancements by researchers like Tobin (1958) [21] and Hanoch and Levy (1969) [4]. Sharpe (1966) [18], Treynor (1965) [22], and Jensen (1968) [6] further developed benchmarks for a more nuanced portfolio assessment.

Recent advancements in machine learning have significantly enhanced asset return predictions and portfolio optimization. Ma, Han, and Wang (2021) [14] demonstrated how deep learning could be leveraged for portfolio optimization, offering more accurate forecasts and adaptive strategies compared to traditional models. Similarly, Lim, Cao, and Quek (2022) [10] have shown how dynamic portfolio rebalancing can be effectively managed through reinforcement learning, adapting to market conditions in real-time and thus potentially outperforming static models in turbulent environments.

Moreover, the increasing relevance of ESG (Environmental, Social, and Governance) factors in investment decisions reflects a broader shift towards sustainability. Iazzolino et al. (2023) [5] examined the impact of ESG factors on the financial efficiency of portfolios, emphasizing the importance of incorporating these dimensions into performance evaluation models to meet new investor expectations and regulatory standards.

The overall finding of the existing literature is that evaluating the performance of portfolios requires nonlinear models or, if using DEA linear models, the efficiency is not comparable to the efficiency portfolio frontier. In this paper, we propose a linear model with the assistance of the DEA to evaluate the performance of any portfolio desired For this purpose, we draw on the concept of the dual-lagrangine and the earlier Markowitz and Morey and Morey models. Motivation for this linear model stems from the recognition of the computational complexities associated with nonlinear models, particularly in time-sensitive decision-making environments. Nonlinear models often entail intricate computational procedures and may exhibit challenges in real-time applications due to their computational intensity and potential convergence issues. In contrast, the proposed linear model offers a simpler and more computationally efficient alternative, making it well-suited for practical portfolio management tasks. By leveraging the power of linear programming techniques within the framework of Data Envelopment Analysis (DEA), our model aims to provide a more tractable and scalable approach to portfolio performance evaluation, facilitating timely decision-making in dynamic market conditions. Moreover, the transparency and interpretability inherent in linear models enhance their appeal, allowing for easier validation and implementation by practitioners. Overall, the motivation behind this linear model lies in its potential to bridge the gap between theoretical sophistication and practical utility in portfolio management. The remainder of the paper is as follows. Section 2 reviews the Markowitz and Morey and Morey models and the concept of the efficient portfolio frontier. Section 3 develops the linear model and Section 4 provides an empirical application to Chinese stocks. Section 5 concludes.

2 Preliminaries

2.1 Lagrange duality

In mathematical optimization theory, duality is the principle that allows for the consideration of an optimization problem from two distinct perspectives: the primal and the dual problem. The solution to the dual problem provides a lower bound for the solution of the primal problem, which typically involves minimization. However, the optimal values of the primal and dual problems are not necessarily equal in general. This discrepancy is known as the duality gap. For convex optimization problems, the duality gap is zero under certain constraint qualification conditions.

Let n , m, and p be positive integers. Let X be a subset of R^n (usually a box-constrained one), let f, $g_i(x)$, and $h_j(x)$ be real-valued functions on X for each i in $\{1, ..., m\}$ and each j in $\{1, ..., p\}$, with at least one of f, $g_i(x)$, and $h_j(x)$ being nonlinear.

A nonlinear programming problem is an optimization problem of the form (1):

$$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & g_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_j(x) = 0, \quad j = 1, \ldots, p \\ & x \in X \end{aligned} \tag{1}$$

With the domain $D \subseteq R^n$, the Lagrangian function $\Lambda : R^n \times R^m \times R^p \to R$ is defined as:

$$\Lambda(x, u, \nu) = f(x) + \sum_{i=1}^{m} u_i g(x) + \sum_{j=1}^{p} \nu_j h_j(x) \tag{2}$$

$$\theta(x, u, \nu) = \inf_x \{f(x) + \sum_{i=1}^{m} u_i g(x) + \sum_{j=1}^{p} \nu_j h_j(x)\} \tag{3}$$

and the dual-Lagrangian problem is:

$$\begin{array}{lll} \underset{u,\nu}{max} & \underset{x}{inf} & f(x)+\sum_{i=1}^{m}u_i g_i(x)+\sum_{j=1}^{p}\nu_j h_j(x) \\ s.t. & & u\geq 0 \end{array} \tag{4}$$

where the objective function is the Lagrangian dual function. Provided the functions $f, g_1, \cdots, g_m$ and $h_1, \cdots, h_p$ are continuously differentiable, the infimum occurs where the gradient is equal to zero. The problem (5) is then the Wolfe dual problem [24]. In mathematical optimization, Wolfe duality, named after Philip Wolfe, is type of dual problem in which the objective function and constraints are all differentiable functions. Using this concept a lower bound for a minimization problem can be found because of the weak duality principle.

$$\begin{array}{ll} \underset{u,\nu,x}{max} & f(x)+\sum_{i=1}^{m}u_i g_i(x)+\sum_{j=1}^{p}\nu_j h_j(x) \\ s.t. & \nabla f(x)+\sum_{i=1}^{m}u_i\nabla g_i(x)+\sum_{j=1}^{p}\nu_j\nabla h_j(x)=0 \\ & u\geq 0 \end{array} \tag{5}$$

Theorem 2.1. ***(weak duality)***

Let be a feasible solution to the primal, and (u, ν) be a feasible solution to the dual. Then:

$$\theta(u,\nu)\leq f(x) \tag{6}$$

Theorem 2.2. ***(Complementary Slackness Theorem)***

If x is feasible for model (1) and (u, ν) is feasible for model (5) and if is optimal to model (1) and is optimal to model (5) then

$$u_i g_i(x)=0 \quad i=1,\cdots,m$$

2.2 Relationship between Morey and Morey and Markowitz models

In the realm of investment management, a profound understanding of efficient portfolios and their efficiency frontier is crucial. These concepts, emphasized in various investment models including the Morey and Markowitz models, enable us to tailor our investment strategies based on two fundamental criteria: maximizing expected returns for a specified level of risk, or minimizing risk for a given expected return.

An efficient portfolio is defined as one that either delivers the highest expected return for a specified level of risk, or offers the lowest level of risk for a predetermined expected return.

The efficient portfolio frontier represents the collection of optimal portfolios that provide either the highest expected return for a specified level of risk or the lowest risk for a given level of expected return.

Markowitz [15] introduced the nonlinear model for evaluating portfolio in 1952, as based on two indexes, one for mean and another for variance. Markowitz's essential work thus laid the basis for the frontier approach under the meanvariance framework.

$$\begin{aligned} \min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij} \\ s.t. \quad & \sum_{j=1}^{n}\lambda_j\mu_j \geq \mu_{Expected} \\ & \sum_{j=1}^{n}\lambda_j = 1 \\ & \lambda_j \geq 0 \end{aligned} \tag{7}$$

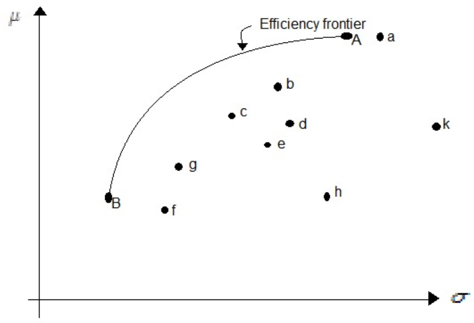

Figure 1: Portfolio Efficiency Frontier

Difference between the efficiency frontiers in the Markowitz model and DEA: First, the efficient frontier in Markowitz model does not a weakly efficient frontier. Second, in DEA, while all Decision Making Units (DMUs) are located below the efficient frontier, but as shown in Figure 1, portfolio k is not located below the efficient portfolio frontier. Third, no one individual stock may be on the efficient frontier in the Markowitz model by definition. In fact, the efficient portfolio frontier results from combining stocks with each other. In contrast, in DEA, at least one stock is certainly efficient.

Suppose units $\{a, b, c, d, e, f, g, h, k\}$ are stocks, given we know a portfolio is a grouping of financial assets such as stocks. To evaluate the efficiency of any portfolio, we have to calculate its distance to the efficient frontier, and many models then evaluate the efficiency of the portfolio, including the nonlinear Morey and Morey model. The input-orientated form of this model is as follows.

$$
\begin{aligned}
\min \quad & \theta \\
s.t. \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j \sigma_{ij} \leq \theta \sigma_o \\
& \sum_{j=1}^{n} \lambda_j \mu_j \geq \mu_{Expected} \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \geq 0
\end{aligned}
\tag{8}
$$

Models (7) and (8) have a global optimal given their convexity and the optimal solution λ is the same in both models. Consequently, according to the second constraint in model (8), the optimal value of the model (7) is equal to $\theta^* \sigma_o$.

$$
\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i^* \lambda_j^* \sigma_{ij} = \theta^* \sigma_o \tag{9}
$$

One of the challenges associated with nonlinear models is the considerable time they demand for solving. As an alternative, the assessment of these units often employs a linear DEA model known as BCC (Banker, Charnes, and Cooper). These models gauge portfolio performance by measuring the distance from the piecewise linear frontier. However, it's problematic that this frontier typically lies below the efficient portfolio frontier. Consequently, decision-makers might err in selecting portfolios based on this type of evaluation. Although a portfolio may be deemed efficient, it doesn't necessarily compare as efficient against the efficient portfolio frontier. Thus, there's always another portfolio with potentially higher returns and/or lower risk. As depicted in Figure 2, units $\{a, b, c, g, f\}$, positioned on the efficient frontier, are indeed efficient, but not when juxtaposed with the efficient portfolio frontier. This implies the perpetual possibility of an investment offering lesser risk or greater returns. One approach to mitigate this disparity is to increase the number of portfolios, as outlined by Liu et al. (2015) in the following theorem.

Theorem 2.3. *Let $r = h(\sigma)$ be the portfolio frontier without risk-free assets and $r^* = h_m^*(\sigma)$ be the BCC frontier with m portfolio samples. Then $h_m^*(\sigma)$ converges to $h(\sigma)$ in probability when $m \to \infty$.*

To solve this problem, we present an input-orientated model that is firstly a linear model, and secondly, one where the efficiency of each portfolio can be calculated by comparing it to a portfolio on the actual efficient portfolio frontier.

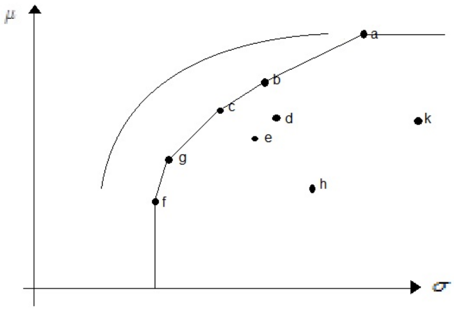

Figure 2: BCC frontier and efficiency portfolio frontier

3 Calculating portfolio efficiency relative to the efficient portfolio frontier using the linear model

In this section, we present the steps for obtaining a linear model for calculating the efficient portfolio.

The dual-Lagrangian model (7) is as follows:

$$\begin{array}{lll} \underset{k,f,h}{max} & \underset{\lambda}{inf} & \sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\lambda_j\sigma_{ij} - k\left(\sum_{j=1}^{n}\lambda_j\mu_j - \mu_o\right) + h\left(\sum_{j=1}^{n}\lambda_j - 1\right) - \sum_{j=1}^{n}\lambda_j f_j \\ s.t. & & k \geq 0 \\ & & f \geq 0 \end{array} \tag{10}$$

Provided the functions, objectives and constraints are continuously differentiable, the infimum occurs where the gradient is equal to zero.

$$\begin{array}{ll} \underset{k,f,h,\lambda}{max} & \sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\lambda_j\sigma_{ij} - k\left(\sum_{j=1}^{n}\lambda_j\mu_j - \mu_o\right) + h\left(\sum_{j=1}^{n}\lambda_j - 1\right) - \sum_{j=1}^{n}\lambda_j f_j \\ s.t. & \sum_{i}^{n}\lambda_j\sigma_{ij} - k\mu_j + h - f_j = 0 \quad j = 1,\dots,p \\ & k \geq 0 \\ & f \geq 0 \end{array} \tag{11}$$

According to Theorem 2.1, if λ^* is the optimal solution 7 then:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\lambda_j\sigma_{ij} - k\left(\sum_{j=1}^{n}\lambda_j\mu_j - \mu_o\right) + h\left(\sum_{j=1}^{n}\lambda_j - 1\right) - \sum_{j=1}^{n}\lambda_j f_j \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i^*\lambda_j^*\sigma_{ij} \tag{12}$$

We also know that for convex optimization problems, the duality gap is zero under a constraint qualification condition. Therefore, the optimal condition is that the values applicable in the inequality (12) are feasible in the two models (primal and dual).

$$\sum_{i=1}^{n}\lambda_j\sigma_{ij} - k\mu_j + h - f_j = 0 \quad j = 1,\ldots,p \tag{13}$$

$$\sum_{i=1}^{n}\lambda_i = 1 \tag{14}$$

$$\sum_{i=1}^{n}\lambda_i\mu_i \geq \mu_o \tag{15}$$

$$k \geq 0 \tag{16}$$

$$f \geq 0 \tag{17}$$

To change the inequality to an equality, we maximize the value on the left-hand side. According to (15) and (16), we then have $k\left(\sum_{i=1}^{n}\lambda_i\mu_i - \mu_o\right) = 0$ and we multiply equation (13) in λ_j.

$$\lambda_j\sum_{i=1}^{n}\lambda_i\sigma_{ij} - \lambda_j k\mu_j + \lambda_j h - \lambda_j f_j = 0 \quad j = 1,\ldots,p$$

Then

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\lambda_i\lambda_j\sigma_{ij} - k\sum_{j=1}^{n}\lambda_j\sum_{j=1}^{n}\mu_j + h\sum_{j=1}^{n}\lambda_j - \sum_{j=1}^{n}\lambda_j f_j = 0$$

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\lambda_j\lambda_j\sigma_{ij} = k\sum_{j=1}^{n}\lambda_j\sum_{j=1}^{n}\mu_j - h\sum_{j=1}^{n}\lambda_j + \sum_{j=1}^{n}\lambda_j f_j$$

With the help of equation (12) and the points mentioned above

$$k\sum_{j=1}^{n}\lambda_j\mu_j - h \leq \sum_{j=1}^{n}\sum_{i=1}^{n}\lambda_i^*\lambda_j^*\sigma_{ij}$$

and $k\lambda_j = w_j$ then

$$
\begin{aligned}
\max \quad & \varphi \\
s.t. \quad & \sum_i^n \lambda_i \sigma_{ij} - k\mu_j + h - f_j = 0 \quad j = 1, \dots, p \\
& \sum_i^n \lambda_j \mu_j - s = \mu_o \\
& \sum_i^n w_j \mu_j - k\mu_o = 0 \\
& \sum_i^n w_j \mu_j - h = \varphi \sigma_o \\
& \sum_j \lambda_j = 1 \\
& \sum_j w_j = k \\
& k \geq 0 \\
& f \geq 0 \\
& \lambda \geq 0 \\
& w \geq 0
\end{aligned}
\tag{18}
$$

where (σ_{oo}, μ_o) is the portfolio under evaluation
In the following theorem, we prove $\varphi^* = \theta^*$.

Theorem 3.1. *If φ^* is the optimum value in model (18) and θ^* is the optimum value in model (8), then $\varphi^* = \theta^*$.*

Proof. According to the fourth condition of model (18)
$\sum_i^n w_j^* \mu_j - h^* = \varphi^* \sigma_o$ then $\varphi^* \sigma_o = k^* \sum_{j=1}^n \lambda_j^* \mu_j - h^* = \sum_{j=1}^n \sum_{i=1}^n \lambda_i \lambda_j \sigma_{ij}$ and also according to equation (9) then $\varphi^* = \theta^*$.

□

According to Theorem 3.1, φ^* is efficiency.

Theorem 3.2. *The benchmark for the portfolio of under evaluation is as follows*

$$\left(\sum_{j=1}^n \sum_{i=1}^n \lambda_i^* \lambda_j^* \sigma_{ij}, \sum_{j=1}^n \lambda_j \mu_j \right) = (\varphi^* \sigma_o, \mu_o + s^*)$$

Proof. $h^* = \sum_i^n w_j^* \mu_j - \varphi^* \sigma_o$

and

$$\sum_i^n \lambda_j^* \sigma_{ij} - k^* \mu_i + \sum_i^n w_j^* \mu_j - \varphi^* \sigma_o - f_i^* = 0$$

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\lambda_i^*\lambda_j^*\sigma_{ij}-\varphi^*\sigma_o-\sum_{i=1}^{n}f_i^*\lambda_i^*=0$$

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\lambda_i^*\lambda_j^*\sigma_{ij}\leq\varphi^*\sigma_o$$

According to the complementary slackness theorem

$$\sum_{i=1}^{n}f_i^*\lambda_i^*=0$$

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\lambda_i^*\lambda_j^*\sigma_{ij}=\varphi^*\sigma_o$$

And according to the second condition of model (18)

$$\sum_{j=1}^{n}\lambda_j^*\mu_j-s^*=\mu_o\rightarrow\sum_{j=1}^{n}\lambda_j^*\mu_j-s^*=\mu_o+s^*$$

□

4 Empirical application

In this section, we take as an example the Chinese stock market data in Zhou et al. (2017) [26], with the following statistical properties.

$$E\left(e\right)=\left[1.0077\quad 1.0057\quad 1.0024\quad 1.0070\quad 1.0063\right]$$

$$cov(e)=\begin{pmatrix}0.0081&0.0042&0.0034&0.0016&0.0007\\0.0042&0.0150&0.0030&0.0014&0.0012\\0.0034&0.0030&0.0086&0.0016&0.0012\\0.0016&0.0014&0.0016&0.0047&0.0014\\0.0007&0.0012&0.0012&0.0014&0.0043\end{pmatrix}$$

If we evaluate these stocks using model (18), the result is in table 1:

Table 1: Valuation with model (18) and model (8)

Portfolio	A	B	C	D	E
φ^*	1	0.1637	0.2855	0.6482	0.5712
θ^*	1	0.1637	0.2855	0.5769	0.5711

According to the figures in Table (1), only portfolio A is efficient and located on the frontier.

If you multiply the value of the variance of D by the value of its return from model (18) (benchmark variance), the result is $\sigma^2=0.0030$ $(\sigma=0.0030)$, but if

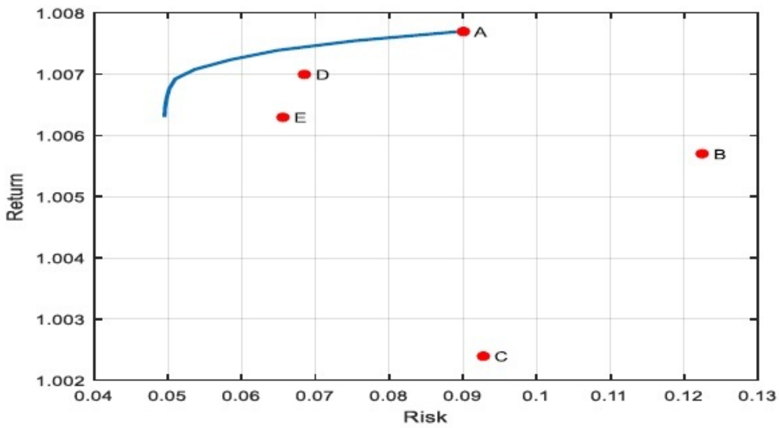

Figure 3: Efficiency portfolio frontier

you multiply it by the return obtained from model (8), the result is $\sigma^2 = 0.0027$ ($\sigma^2 = 0.052$).This difference is small and may be due to software error.

One of the benefits of model (18) is that desired portfolios are produced from existing stock $\left(\sum_{j=1}^{n}\sum_{i=1}^{n}\lambda_i^*\lambda_j^*\sigma_{ij}, \sum_{j=1}^{n}\lambda_j\mu_j\right)$ such that $\sum_j \lambda_j = 1$, can be evaluated using model (18) and compare its proximity to the efficient portfolio frontier with the other portfolios.

Based on research, linear models generally run faster than nonlinear models on large data processing and analysis and can help improve the performance and efficiency of data analysis.

5 Conclusion

The model presented in this paper is a linear model used to evaluate the performance of portfolio, which we transform into a linear model using the concept of the dual Langrangian. In previous studies, DEA models like the BCC serve to evaluate the performance of a portfolio using a linear model. However, these required a large number of portfolios to ensure the convergence of the efficient frontier model to the portfolio efficient frontier (Theorem (2.3))

To evaluate each portfolio with the model proposed it is not necessary to increase the number of portfolios. In fact, this model can evaluate portfolio performance with only the available assets, and this increases the speed of calculation. Nonetheless, one problem with this model is that we only evaluate portfolio performance using an input orientation. Another is that it only considers the indexes of variance and mean.

Bibliography

[1] M. Branda, M. Kopa, *DEA-risk efficiency and stochastic dominance efficiency of stock indices*, FinancUver-Czech J Econ Financ, (2012), 62(2), 10624.

[2] W. Briec, K. Kerstens, and O. Jokung,*Mean-Variance-Skewness Portfolio Performance Gauging: A General Shortage Function and Dual Approach*, Management Science, (2006), 53, pp. 153-149.

[3] F. Choobineh, D. Branting, *A Simple Approximation for Semi variance*, European Journal of Operational Research, (1986), Vol. 27, pp. 364-370.

[4] G. Hanoch, H. Levy , *The efficient analysis of choices involving risk*, Review of Economic Studies, (1969), 36 (3) 335346.

[5] G. Iazzolino, M. E. Bruni, S. Veltri, D. Morea, and G. Baldissarro,*The impact of ESG factors on financial efficiency: An empirical analysis for the selection of sustainable firm portfolios*, Corporate Social Responsibility and Environmental Management, (2023), 30(4), 1917-1927.

[6] M. C. Jensen , *The performance of mutual funds in the period*, 19451968. Journal of Finance, (1968), 23, 389416.

[7] T. Joro, P. Na ,*Portfolio performance evaluation in a meanvarianceskewness framework*, European Journal of Operational Research, (2006), 175(1), 446461.

[8] P. D. Kaplan, R. H. Alldredge ,*Semi variance in Risk-Based Index Construction: Quantidex Global Indexes*, The Journal of Investing,(1997), Vol. 6, pp. 82-87.

[9] K. Kerstens, A. Mounir , and I. V. Woestyne ,*Geometric Representation of the Mean-Variance Skewness Portfolio Frontier Based Upon the Shortage Function*, European Journal of Operational Research, (2011), 210, pp. 81-94.

[10] Q. Y. E. Lim, Q. Cao, and C. Quek, *Dynamic portfolio rebalancing through reinforcement learning*, Neural Computing and Applications, (2022), 34(9), 7125-7139.

[11] W. Liu, Z. Zhou, D. Liu, and H. Xiao,*Estimation of portfolio efficiency via DEA*, Omega, (2015), 52, 107118.

[12] S. Lozano, E. Gutiérrez ,*Data envelopment analysis of mutual funds based on second-order stochastic dominance*, European Journal of Operational Research, (2008), 189(1), 230244.

[13] S. O. Lozza, E. Angelelli, and D. Toninelli ,*Set-portfolio selection with the use of market stochastic bounds*, Emerging Markets Finance and Trade, (2011), 47(5): 524.

[14] Y. Ma, R. Han, and W. Wang, *Portfolio optimization with return prediction using deep learning and machine learning*, Expert Systems with Applications, (2021), 165, 113973.

[15] H. Markowitz, *Portfolio selection*, The Journal of Finance, (1952), 7(1), 7791.

[16] H. Markowitz, *Portfolio Selection: Efficient Diversification of Investments*, New York: Yale University Press, John Wiley,(1992), Vol. 119, pp. 165-166.

[17] M. R. Morey, R. C. Morey ,*Mutual fund performance appraisals: a multi-horizon perspective with endogenous benchmarking*, Omega, International Journal of Management Science,(1999), 27, 241258.

[18] W. F. Sharpe , *Mutual fund performance*, Journal of Business,(1966), 39(1), 119138.

[19] W.F. Sharpe, J.A. Gordon, and V.B. Jeffry, *Investment*, 5th. Ed, Prentice HallCollege Div,(1995).

[20] H. Soleimani, H. R. Golmakani, and M. H. Salimi ,*Markowitz-based portfolio selection with minimum transaction lots, cardinality constraints and regarding sector capitalization using genetic algorithm*, Expert Systems with Applications,(2009), 36, 50585063.

[21] J. Tobin, *Liquidity Preference as Behavior Toward Risk*, Review of Economic Studies, (1958), 25, pp.65-86.

[22] J. Treynor ,*How to rate management of investment funds*, Harvard Business Review,(1965), 43(1), 6375.

[23] A. L. Turner, and E. J. Weigel, *Daily Stock Market Volatility: 1928-1989*, Management Science,(1992), 38, pp. 1586-1609.

[24] P. WOLFE, *A duality theorem for non-linear programming*, Quarterly of applied mathematics, (1961), 19(3), 239-244.

[25] A. YOSHIMOTO , *The meanvariance approach to portfolio optimization subject to transaction costs*, Journal of the Operations Research Society of Japan,(1996), 39(1), 99117.

[26] Z. ZHOU,H. XIAO,Q. JIN AND W. LIU,*DEA frontier improvement and portfolio rebalancing: An application of China mutual funds on considering sustainability information disclosure*, European Journal of Operational Research,(2017), 121.

How to Cite: Fatemeh Fattahi[1], Farhad Hosseinzadeh Lotfi[2], Andrew C. Worthington[3], *A comparison of the linear model and the efficient frontier for the evaluation of portfolio performance*, Journal of Mathematics and Modeling in Finance (JMMF), Vol. 4, No. 1, Pages:83–96, (2024).