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Mean-AVaR-Entropy optimization portfolio selection model in uncertain environments

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Abstract:
This paper investigates the complexities surrounding uncertain portfolio selection in cases where security returns are not well-represented by historical data. Uncertainty in security returns is addressed by treating them as uncertain variables. Portfolio selection models are developed using the quadratic-entropy of these uncertain variables, with entropy serving as a standard measure of diversification. Additionally, the study underscores the superior risk estimation accuracy of Average Value-at-Risk (AVaR) compared to variance. The research concentrates on the computational challenges of portfolio optimization in uncertain environments, utilizing the Mean-AVaR-Quadratic Entropy paradigm to meet investor requirements and assuage concerns. Two illustrative examples are provided to show the efficiency of the proposed models in this paper.

Keywords: Portfolio selection, Uncertain variables, Average Value-at-Risk, mean-AVaR-entropy, quadratic entropy.
Classification: MSC2010 or JEL Classifications: 91G10, 90C70, 90C90.

1 Introduction

The primary objective of the optimum portfolio selection theory is to maximize the profits of investors by considering a range of alternative investments based on their individual preferences. Initially, the mean-variance model developed by Markowitz served as the foundation for addressing the portfolio selection problem [33]. Numerous research have subsequently been conducted to explore portfolio optimization within the context of these two moments of the return distribution [45]. Subsequent

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to this, extensive research has been conducted to explore diverse methodologies that can be employed to model investment risk, aiming to achieve a more accurate estimation of risk. For instance, previous studies have utilised risk functions or Value at Risk (VaR) measures such as [2, 4, 5, 13, 24, 38, 39, 44, 47, 49]. However, VaR suffers from certain limitations. Firstly, it fails to provide information regarding the magnitude of losses exceeding the VaR level. Additionally, VaR does not satisfy the coherence criterion. Consequently, the concept of average value-at-risk (AVaR) has emerged as an alternative risk measure.

The proposal of AVaR is suggested as a potential solution to address the inherent issues associated with VaR. VaR, being a risk measure that lacks coherence in general, necessitates the introduction of AVaR as a more suitable alternative. In numerous articles, alternative terms such as conditional Value-at-Risk (CVaR), Tail Value-at-Risk (TVaR), or Expected Shortfall (ES) are employed to refer to the same concept [6, 23, 40, 42]. However, for the purpose of this discussion, we will adopt the term average value-at-risk (AVaR) as it more accurately captures the essence of the variable under consideration. Risk is derived from the presence of uncertainty, which may be categorised into two main types: objective uncertainty and subjective uncertainty. Stochasticity is a fundamental form of objective uncertainty, whereas probability theory serves as a mathematical discipline dedicated to the analysis of the characteristics and dynamics of random events. The conventional risk metric known as Value at Risk (VaR) or Average Value at Risk (AVaR) has typically been introduced within a stochastic framework. The present research introduces the concept of the credibilistic AVaR as a novel risk measure, offering a more advantageous alternative to VaR within the framework of uncertainty theory as proposed by Liu [27]. Numerous manuscripts addressing optimal portfolio selection problems have been published, highlighting the insufficiency of relying solely on average and variance, or alternative risk estimators such as AVaR, for determining the optimal portfolio allocation.

In conventional practice, it has been widely accepted that security returns exhibit stochastic behaviour, hence necessitating the application of probability theory as the primary means for achieving optimal portfolio selection. However, it is evident that the effectiveness of security measures is influenced by a range of factors, such as social, political, economic, human cognitive, and notably psychological factors. Research has demonstrated that historical data does not well capture short-term security returns. Empirical data suggests that the probability distribution of underlying asset returns exhibits greater peaks and heavier tails compared to the normal distribution. Furthermore, it is observed that the first two moments alone are inadequate in characterising this distribution. Several studies have employed fuzzy variables as a means to address the aforementioned problems [1,10,12,25,32]. However, the utilisation of fuzzy variables has been found to present certain paradoxes [19,30]. Consequently, the concept of uncertainty theory has garnered significant attention, leading many researchers to incorporate Liu's uncertain measurement theory into

their portfolio selection models [15–18, 34, 36, 50].
To assess diversification, entropy serves as a widely accepted measure of diversity [11, 14, 20–22, 37, 41, 46]. It is recognized that a higher entropy value in portfolio weights indicates a greater level of portfolio diversification. Previous literature has explored the use of entropy as an objective function in multi-objective model portfolio selection [20, 21, 41, 46]. Furthermore, Bera and Park [3] have presented asset allocation models that utilize entropy and cross entropy measures to generate well-diversified portfolios. When entropy is employed as an objective function to determine portfolio weights, the resulting weights are automatically non-negative. This means that an entropy-based model naturally avoids short-selling, which can be a desirable situation in portfolio selection for both theoretical and practical reasons [7, 8, 43].
Some studies recently applied different kinds of entropy with respect to portfolio optimization in uncertain environments [26, 31, 35, 48]. Dai in [9] defined quadratic entropy which has wider range and much easier to compute, compared with other traditional kinds of entropy. So in this manuscript Mean-AVaR-quadratic entropy is utilized to meet investor requirements and assuage concerns.

First, we verify the uncertain model in the framework of uncertain theory for portfolio selection by considering uncertain returns. Second, AVaR is considered as risk, and replaced Average Value-at-Risk instead of variance and add quadratic entropy to the mean-AVaR model in an uncertain environment and create a mean-AVaR-quadratic entropy uncertain portfolio optimization model. The uncertain mean-AVaR-quadratic entropy model will be formulated to get the basic opinion of accounting return, risk, entropy simultaneously in the portfolio optimization problem in general.

The present paper is structured in the following manner. In Section 2, a comprehensive understanding of uncertain variables will be acquired by a thorough examination of relevant knowledge. Following this, Section 3 will delve into the examination and AVaR and validation of quadratic entropy pertaining to three distinct categories of uncertain returns. In the fourth section, many models have been developed to address portfolio selection within the framework of mean-AVaR-quadratic entropy. In Section 5, two examples are employed to illustrate the efficacy of the suggested approach. In Section 6, a set of concluding remarks are presented.

2 Preliminaries

Consider Γ be a non-empty set, and define the σ -algebra L be a collection of all the events $\Theta \in L$ over Γ. It could be defined as a function that for each event Θ return $\mathcal{M}\{\Theta\}$ which indicates the belief degree which means that we believe Θ will occur. Liu [27] offered the following five axioms, in order to define uncertain measure in an axiomatic form, to ensure that the number $\mathcal{M}\{\Theta\}$ is not arbitrary and has special mathematical properties;

1: (Normality axiom) $\mathcal{M}(\Gamma) = 1$;
2: (Monotonicity axiom) $\mathcal{M}(\Theta_1) \leq \mathcal{M}(\Theta_2)$ every where $\Theta_1 \subseteq \Theta_2$;
3: (Duality axiom) $\mathcal{M}(\Theta) + \mathcal{M}(\Theta^c) = 1$ for every event Θ;
4: (Subadditivity axiom) For each sequence of events $\{\Theta_j\}$ that can be counted, we have

$$\mathcal{M}(\bigcup_{j=1}^{\infty} \Theta_j) \leq \sum_{j=1}^{\infty} \mathcal{M}(\Theta_j)$$

Definition 2.1. [28]. The set function $\mathcal{M}$ which satisfies the above axioms, is called an uncertain measure.

Definition 2.2. [28]. Consider Γ be a non-empty set, the σ-algebra L be a collection of all the events over Γ, and $\mathcal{M}$ be an uncertain measure according to the above definition, the triple $(\Gamma, L, \mathcal{M})$ is named an uncertain space.

5: (Product Measure Axiom) [28]. Let the triple $(\Gamma_k, L_k, \mathcal{M}_k)$ for $k = 1, 2, ..., n$, where $\Gamma = \Gamma_1 \times \Gamma_2 \times ...$ and $L = L_1 \times L_2 \times ...$ be uncertainty spaces, then it satisfyed in

$$\mathcal{M}(\prod_{k=1}^{\infty} \Theta_k) \leq \bigwedge_{k=1}^{\infty} \mathcal{M}_k(\Theta_k)$$

Where Θ_k, are arbitrary events and chosen from L_k for $k = 1, 2, ..., n$, respectively.

Definition 2.3. [28]. The uncertainty distribution for an uncertain variable such as η is defined by function $\Phi : \mathbb{R} \to [0, 1]$ that $\Phi(x) = \mathcal{M}\{\eta \leq x\}$.

Theorem 2.4. *[29] Let $\Phi_1, \Phi_2, ..., \Phi_n$ be uncertainty distributions of independent uncertain variables $\eta_1, \eta_2, ..., \eta_n$, respectively. If $f(t_1, t_2, ..., t_n)$ be increasing strictly. Then*

$$\eta = f(\eta_1, \eta_2, ..., \eta_n), \tag{1}$$

is an uncertain variable with uncertainty distribution

$$\Psi(t) = \sup_{f(t_1,t_2,...,t_n)=t} \Big(\min_{1 \leq i \leq n} \Phi_i(t_i) \Big), \quad t \in \mathbb{R}, \tag{2}$$

and following inverse function

$$\Psi^{-1}(\alpha) = f[\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), ..., \Phi_n^{-1}(\alpha)], \tag{3}$$

Where $\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), ..., \Phi_n^{-1}(\alpha)$ are unique for each $\alpha \in (0, 1)$.

Definition 2.5. [27]. The expected value of an uncertain variable η is defined by

$$E[\eta] = \int_0^{\infty} \mathcal{M}\{\eta \geq r\}dr - \int_{-\infty}^{0} \mathcal{M}\{\eta \leq r\}dr, \tag{4}$$

while at least one of the above integrals be finite.

Theorem 2.6. *[29]. Let a_1 and a_2 be real numbers and η_1 and η_2 be independent uncertain variables where them expected values are finite, then we have*

$$E[a_1\eta_1 + a_2\eta_2] = a_1E[\eta_1] + a_2E[\eta_2]. \tag{5}$$

Definition 2.7. [28](Liu 2009a) The entropy of an uncertain variable with an uncertainty distribution Φ is denoted as

$$H[\xi] = \int_{-\infty}^{\infty} S(\Phi(x))dx, \tag{6}$$

where $S(t) = -tlnt - (1-t)ln(1-t)$.

In order to quantify the uncertainty related to event A, the term $S(M\{A\})$ is utilized. To establish a precise definition of entropy, it is sufficient to choose a function $S(t)$ that exhibits an increasing trend on the interval $[0, 0.5]$ and a decreasing trend on the interval $[0.5, 1]$. Consequently, drawing inspiration from Vajda (1968)'s quadratic entropy, Dai endeavor to define a quadratic entropy within the context of uncertainty theory.

Definition 2.8. [9] The quadratic entropy, denoted as $Q[\xi]$, characterizes the uncertainty of an uncertain variable, represented by ξ, with an uncertainty distribution of Φ, and defined by

$$Q[\xi] = \int_{-\infty}^{\infty} (\Phi(t))(1-\Phi(t))dt, \tag{7}$$

The simplicity in calculating quadratic entropy is apparent when contrasted with traditional entropy.

Theorem 2.9. *[9] Let ξ be an uncertain variable taking values on the interval $[a, b]$. Then*

$$Q[\xi] \leq \frac{b-a}{4}$$

and equation holds if ξ has an uncertainty distribution $\Phi(t) = 0.5$ on $[a, b]$.

Proof. The theorem follows from the fact that the function $\Phi(t)(1-\Phi(t))$ reaches its maximum $\frac{1}{4}$ at $\Phi(t) = 0.5$. Quadratic entropy has a wider range in measuring the information deficiency. Firstly, we shows that for any given uncertain variable, its quadratic entropy is less than its entropy. □

Theorem 2.10. *[9] Let ξ be an uncertain variable. Then its entropy $H[\xi]$ is greater than its quadratic entropy $Q[\xi]$.*

Proof. For any $0 \leq t \leq 1$, we have

$$-ln\ t \geq \frac{1-t}{2}.$$

It implies that

$$-t\ ln\ t-(1-t)ln(1-t)\geq\frac{t(1-t)}{2}+\frac{(1-t)t}{2}\geq t(1-t).$$

For all x, we have $0\leq\Phi(x)\leq 1$. It follows that

$$\begin{aligned}Q[\xi]&=\int_{-\infty}^{+\infty}\Phi(x)(1-\Phi(x))dx\\&\leq\int_{-\infty}^{+\infty}-\Phi(x)ln\Phi(x)-(1-\Phi(x))ln(1-\Phi(x))dx\\&=H[\xi].\end{aligned}$$

The theorem is proved. □

The above theorem and example imply that quadratic entropy has a wider range, compared with traditional entropy, in measuring the information deficiency.

Theorem 2.11. *[9] Given that ξ represents an uncertain variable, and k is a real number. Then*

$$Q[\xi+k]=Q[k]. \tag{8}$$

In other words, the quadratic entropy remains unchanged regardless of the translations applied.

Proof. Given that ξ is characterized by an uncertainty distribution Φ, it can be inferred that the uncertain variable $\xi+k$ will possess an uncertainty distribution $\Phi(t-k)$. This conclusion is based on the definition of quadratic entropy as

$$Q[\xi+k]=\int_{-\infty}^{\infty}(\Phi(t-k))(1-\Phi(t-k))dt=\int_{-\infty}^{\infty}(\Phi(t))(1-\Phi(t))dt=Q[\xi], \tag{9}$$

The theorem is proved. □

Theorem 2.12. *[9] Assume ξ is an uncertain variable with regular uncertainty distribution Φ. If the quadratic entropy $Q[\xi]$ exists, then*

$$Q[\xi]=\int_{0}^{1}(\Phi^{-1}(\alpha))(2\alpha-1)d\alpha. \tag{10}$$

Theorem 2.13. *[9] Let ξ and η be independent uncertain variables. Then for any real numbers a and b, we have*

$$Q[a\xi+b\eta]=|a|Q[\xi]+|b|Q[\eta]. \tag{11}$$

Theorem 2.14. *[9] Let ξ be an uncertain variable with expected value e and variance σ. Then*

$$Q[\xi]\leq\frac{\sigma}{\sqrt{3}}. \tag{12}$$

and the equality holds if and only if it is a linear uncertain variable $L(e-\sqrt{3}\sigma, e+\sqrt{3}\sigma)$.

Definition 2.15. Let $\lambda \in (0,1]$ be a confidence level and η be an uncertain variable, Then the function $VaR : (0,1] \to \mathbb{R}$ denotes Value-at-Risk of η, and defined by

$$VaR(\lambda) = sup\{x | \mathcal{M}\{\eta \geq x\} \geq \lambda\}.$$

Theorem 2.16. *For the risk confidence level* $\lambda \in (0,1]$*,*

$$VaR(\lambda) = \Phi^{-1}(1-\lambda),$$

where $\Phi^{-1}(1-\lambda)$ *denotes the inverse of uncertainty distribution function* $\Phi(\lambda)$*.*

Definition 2.17. Let $\lambda \in (0,1]$ be the confidence level for an uncertain variable η, then the function $AVaR : (0,1] \to \mathbb{R}$ denotes the average Value-at-Risk of η, and defined by

$$AVaR(\lambda) = \frac{1}{\lambda} \int_0^\lambda VaR(\gamma) d\gamma.$$

3 explanation of the problem

Consider for $\alpha < \beta$, $\eta \sim L(\alpha, \beta)$ be a linear uncertain variable.
i) The expected value of η is obtained as

$$E[\eta] = \frac{\alpha + \beta}{2} \tag{13}$$

ii) The Value-at-risk of η is obtained as

$$VaR(\lambda) = \lambda\alpha + (1-\lambda)\beta, \quad 0 \leq \lambda \leq 1. \tag{14}$$

iii) The Average Value-at-risk of η is obtained as

$$AVaR(\lambda) = \frac{\lambda\alpha}{2} + (1 - \frac{\lambda}{2})\beta, \quad 0 \leq \lambda \leq 1. \tag{15}$$

iV) The quadratic entropy of η is obtained as

$$Q(\eta) = \frac{\beta - \alpha}{6}. \tag{16}$$

Proof. i)It is known that the uncertainty distribution of the linear uncertain variable η is [27]

$$\Phi(\tau) = \begin{cases} 0, & \tau \leq \alpha, \\ \frac{\tau-\alpha}{\beta-\alpha}, & \alpha \leq \tau \leq \beta, \\ 1, & \tau \geq \beta. \end{cases} \tag{17}$$

So

$$\Phi^{-1}(\tau) = \tau\beta + (1-\tau)\alpha. \tag{18}$$

Using (4),

$$E[\eta] = \int_0^{+\infty} (1 - \Phi(\tau))d\tau - \int_{-\infty}^0 \Phi(\tau)d\tau. \tag{19}$$

then, if $\alpha \geq 0$,

$$E[\eta] = (\int_0^{\alpha} 1d\tau + \int_{\alpha}^{\beta} (1 - \frac{\tau - \alpha}{\beta - \alpha})d\tau + \int_{\beta}^{+\infty} 0d\tau) - \int_{-\infty}^0 0d\tau = \frac{\alpha + \beta}{2} \tag{20}$$

If $\beta \leq 0$,

$$E[\eta] = \int_0^{+\infty} 0d\tau - (\int_{-\infty}^{\alpha} 0d\tau + \int_{\alpha}^{\beta} \frac{\tau - \alpha}{\beta - \alpha}d\tau + \int_{\beta}^0 1d\tau) = \frac{\alpha + \beta}{2} \tag{21}$$

If $\alpha \leq 0 \leq \beta$,

$$E[\eta] = \int_0^{\beta} (1 - \frac{\tau - \alpha}{\beta - \alpha})d\tau + \int_{\alpha}^0 \frac{\tau - \alpha}{\beta - \alpha}d\tau = \frac{\alpha + \beta}{2} \tag{22}$$

Thus

$$E[\eta] = \frac{\alpha + \beta}{2} \tag{23}$$

ii)Using (18),

$$\Phi^{-1}(1 - \tau) = (1 - \tau)\beta + \alpha\tau. \qquad 0 \leq \tau \leq 1. \tag{24}$$

Then using theorem (2.16), the proof is obvious.
iii)Using (2.17) and (24), for $0 \leq \lambda \leq 1$,

$$\begin{aligned} AVaR(\lambda) &= \frac{1}{\lambda}\int_0^{\lambda} VaR(\gamma)d\gamma \\ &= \frac{1}{\lambda}\int_0^{\lambda} (\alpha\gamma + \beta(1 - \gamma)d\gamma \\ &= \frac{\lambda\alpha}{2} + (1 - \frac{\lambda}{2})\beta. \end{aligned}$$

iv) Using definition (2.8),

$$Q(\eta) = \int_{\alpha}^{\beta} (\frac{\tau - \alpha}{\beta - \alpha}.\frac{\beta - \tau}{\beta - \alpha})d\tau = \frac{\beta - \alpha}{6}.$$

□

Consider for $a < b < c$, $\eta \sim Z(a, b, c)$ be a Zigzag uncertain variable.
i) The expected value of η is obtained as

$$E[\eta] = \frac{a + 2b + c}{4} \tag{25}$$

ii) The Value-at-risk of η is obtained as

$$VaR(\lambda) = \begin{cases} (2\lambda - 1)a + (1-\lambda)2b, & 0 < \lambda \leq \frac{1}{2}, \\ 2b\lambda + (1-2\lambda)c, & \frac{1}{2} \leq \lambda < 1. \end{cases} \tag{26}$$

iii) AVaR of η is obtained as

$$AVaR(\lambda) = \begin{cases} (1 - \frac{\lambda}{2})2b + a(\lambda - 1), & 0 < \lambda \leq \frac{1}{2}, \\ \frac{1}{4\lambda}(2b - a) - \lambda c + b\lambda, & \frac{1}{2} \leq \lambda < 1. \end{cases} \tag{27}$$

iV) The quadratic entropy of η is obtained as

$$Q(\eta) = \frac{c-a}{6}. \tag{28}$$

Proof. i)It is known that the uncertainty distribution of the Zigzag uncertain variable η is [27]

$$\Phi(\tau) = \begin{cases} 0, & \tau \leq a, \\ \frac{\tau - a}{2(b-a)}, & a \leq \tau \leq b, \\ \frac{\tau + c - 2b}{2(c-b)}, & b \leq \tau \leq c, \\ 1, & \tau \geq c. \end{cases} \tag{29}$$

So

$$\Phi^{-1}(\tau) = \begin{cases} 2b\tau - a(2\tau - 1), & 0 \leq \tau \leq \frac{1}{2}, \\ 2b(1-\tau) + c(2\tau - 1), & \frac{1}{2} \leq \tau \leq 1. \end{cases} \tag{30}$$

Using (4),

$$E[\eta] = \int_0^{+\infty} (1 - \Phi(\tau))d\tau - \int_{-\infty}^{0} \Phi(\tau)d\tau.$$

then, if $a \geq 0$,

$$E[\eta] = (\int_0^{\alpha} 1d\tau + \int_a^b \frac{2b - \tau - a}{2(b-a)}d\tau + \int_b^c \frac{c-\tau}{2(c-b)}d\tau + \int_c^{+\infty} 0d\tau) - \int_{-\infty}^{0} 0d\tau = \frac{a+2b+c}{4}$$

If $a \leq 0 \leq b$,

$$E[\eta] = \int_0^b \frac{2b - a - \tau}{2(b-a)}d\tau + \int_b^c \frac{c-\tau}{2(c-b)}d\tau + \int_c^{+\infty} 0d\tau - \int_{-\infty}^{a} 0d\tau - \int_a^0 \frac{\tau - a}{2(b-a)}d\tau = \frac{a+2b+c}{4}.$$

If $b \leq 0 \leq c$,

$$E[\eta] = \int_0^c \frac{c-\tau}{2(c-b)}d\tau + \int_c^{+\infty} 0d\tau - \int_{-\infty}^{a} 0d\tau - \int_a^b \frac{\tau - a}{2(b-a)}d\tau - \int_b^0 \frac{\tau + c - 2b}{2(c-b)}d\tau = \frac{a+2b+c}{4}.$$

If $c \leq 0$,

$$E[\eta] = \int_0^{+\infty} 0 d\tau - \int_{-\infty}^{a} 0 d\tau - \int_a^b \frac{\tau - a}{2(b-a)} d\tau - \int_b^c \frac{\tau + c - 2b}{2(c-b)} d\tau - \int_c^0 1 d\tau = \frac{a + 2b + c}{4}.$$

Thus

$$E[\eta] = \frac{a + 2b + c}{4}.$$

ii)Using (30),

$$\Phi^{-1}(1-\tau) = \begin{cases} 2b(1-\tau) - a(2\tau - 1), & 0 < \tau \leq \frac{1}{2}, \\ 2b\tau + c(1 - 2\tau), & \frac{1}{2} \leq \tau < 1. \end{cases}$$

Then using theorem (2.16), the proof is obvious.
iii)Using (2.17) and (24), for $0 < \lambda \leq \frac{1}{2}$,

$$\begin{aligned} AVaR(\lambda) &= \frac{1}{\lambda} \int_0^{\lambda} VaR(\gamma) d\gamma \\ &= \frac{1}{\lambda} \int_0^{\lambda} 2b(1-\gamma) - a(2\gamma - 1) d\gamma \\ &= 2b(1 - \frac{\lambda}{2}) + a(\lambda - 1). \end{aligned}$$

and for $\frac{1}{2} \leq \lambda < 1$,

$$\begin{aligned} AVaR(\lambda) &= \frac{1}{\lambda} \int_0^{\lambda} VaR(\gamma) d\gamma \\ &= \frac{1}{\lambda} (\int_0^{\frac{1}{2}} 2b(1-\gamma) - a(2\gamma - 1) d\gamma + \int_{\frac{1}{2}}^{\lambda} ((1 - 2\gamma)c + 2b\gamma) d\gamma) \\ &= 2b(1 - \frac{\lambda}{2}) + a(\lambda - 1). \end{aligned}$$

iv) Using definition (2.8),

$$Q(\eta) = \int_a^b (\frac{\tau - a}{2(b-a)} . \frac{2b - a - \tau}{2(b-a)}) d\tau + \int_b^c (\frac{\tau + c - 2b}{2(c-b)} . \frac{c - \tau}{2(c-b)}) d\tau = \frac{c - a}{6}.$$

□

Consider a Normal uncertain variable $\eta \sim N(e, \sigma)$ where $e, \sigma > 0$.
i) The expected value of η is obtained as

$$E[\eta] = e. \tag{31}$$

ii) The VaR of η is obtained as

$$VaR(\lambda) = e - \frac{\sqrt{3}\sigma}{\pi} ln \frac{\lambda}{1 - \lambda} \tag{32}$$

iii) The AVaR of η is obtained as

$$AVaR(\lambda) = e\lambda - \frac{\sqrt{3}\sigma}{\pi}[\lambda ln\lambda + (1-\lambda)ln(1-\lambda)] \tag{33}$$

iv) The quadratic entropy of η is obtained as

$$Q(\eta) = \frac{\sqrt{3}\sigma}{\pi}. \tag{34}$$

Proof. i)It is known that the uncertainty distribution of the normal uncertain variable η is [27]

$$\Phi(\tau) = \left(1 + exp(\frac{\pi(e-\tau)}{\sqrt{3}\sigma})\right)^{-1}, \quad \tau \in \mathbb{R}, \tag{35}$$

So

$$\Phi^{-1}(\tau) = e - \frac{\sqrt{3}\sigma}{\pi} ln\frac{1-\tau}{\tau} \tag{36}$$

Using (4),

$$E[\eta] = \int_0^{+\infty} (1 - \Phi(\tau))d\tau - \int_{-\infty}^0 \Phi(\tau)d\tau.$$

Then,

$$E[\eta] = \int_0^{+\infty} 1 - \left(1 + exp(\frac{\pi(e-\tau)}{\sqrt{3}\sigma})\right)^{-1} d\tau - \int_{-\infty}^0 \left(1 + exp(\frac{\pi(e-\tau)}{\sqrt{3}\sigma})\right)^{-1} d\tau = e.$$

ii)Using (36),

$$\Phi^{-1}(1-\tau) = e - \frac{\sqrt{3}\sigma}{\pi} ln\frac{\tau}{1-\tau} \tag{37}$$

Then using theorem (2.16), the proof is obvious.
iii)Using (2.17) and (37),

$$\begin{aligned} AVaR(\lambda) &= \frac{1}{\lambda}\int_0^{\lambda} VaR(\gamma)d\gamma \\ &= \frac{1}{\lambda}\int_0^{\lambda} (e - \frac{\sqrt{3}\sigma}{\pi} ln\gamma + \frac{\sqrt{3}\sigma}{\pi} ln(1-\gamma))d\gamma \\ &= e\lambda - \frac{\sqrt{3}\sigma}{\pi}[\lambda ln\lambda + (1-\lambda)ln(1-\lambda)]. \end{aligned}$$

iv) Using definition (2.8),

$$Q(\eta) = \int_{-\infty}^{\infty} (1 + e^{\frac{\pi(e-\tau)}{\sqrt{3}\sigma}})^{-1}(1 - (1 + e^{\frac{\pi(e-\tau)}{\sqrt{3}\sigma}})^{-1})d\tau = \frac{\sqrt{3}\sigma}{\pi}.$$

□

4 Portfolio Selection Problem

Markowitz models had considered the security returns as random variables. As explained in the introduction, there are situations that returns of securities may be uncertain variables. In these cases, uncertain variables will be used to describe the security returns. Let η_i denotes uncertain return of the ith security, and x_i represents the proportion of investment in the ith security, and the given risk confidence level is denoted by $\lambda \in (0, 1]$. The investment return is determined by the expected value and risk by AVaR of a portfolio.

One of the problems in portfolio optimization is minimizing the Average Value at Risk (AVaR) in order to reduce risk at a given expected return level ϑ that investors find acceptable. To assess diversification, entropy serves as a widely accepted measure of diversity. It is recognized that a higher entropy value in portfolio weights indicates a greater level of portfolio diversification, so quadratic entropy is admissible more than ϱ. In this case, the portfolio optimization model can be displayed as

$$\text{minimize} \quad AVaR[\sum_{i=1}^{n} x_i\eta_i] \tag{38}$$

$$\text{subject to} \begin{cases} E[\sum_{i=1}^{n} x_i\eta_i] \geq \vartheta, \\ Q[\sum_{i=1}^{n} x_i\eta_i] \geq \varrho, \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \geq 0, \qquad i = 1, 2, ..., n. \end{cases}$$

Alternatively, another portfolio selection problem can be maximizing expected return on the limitation that the risk which denotes by AVaR does not overpass a preset risk level ϖ and quadratic entropy is admissible more than ϱ in advance. This optimization model becomes as

$$\text{maximize} \quad E[\sum_{i=1}^{n} x_i\eta_i] \tag{39}$$

$$\text{subject to} \begin{cases} AVaR[\sum_{i=1}^{n} x_i\eta_i] \leq \varpi, \\ Q[\sum_{i=1}^{n} x_i\eta_i] \geq \varrho, \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \geq 0, \qquad i = 1, 2, ..., n. \end{cases}$$

This optimization problem can be formulated in some other different kinds, such as maximizing entropy or multi-objective programming model as

$$\begin{array}{ll} \text{maximize} & E[\sum_{i=1}^{n} x_i\eta_i] \\ \text{maximize} & Q[\sum_{i=1}^{n} x_i\eta_i] \\ \text{minimize} & AVaR[\sum_{i=1}^{n} x_i\eta_i] \end{array} \tag{40}$$

$$\text{subject to}\begin{cases}\sum_{i=1}^{n} x_i = 1.\\ x_i \geq 0, \qquad i = 1, 2, ..., n.\end{cases}$$

In order to solve this problem, consider $w_i, i = 1, 2, 3, 4$ be positive real numbers which indicate the weights of the three appropriated objectives, and $w_i \in [0, 1]$, so this multi-objective model can be transformed into a single-objective optimization model as

$$\text{minimize} \quad w_1 AVaR[\textstyle\sum_{i=1}^{n} x_i\eta_i] - w_2 E[\sum_{i=1}^{n} x_i\eta_i] - w_3 Q[\sum_{i=1}^{n} x_i\eta_i] \tag{41}$$

$$\text{subject to}\begin{cases}\sum_{i=1}^{n} x_i = 1.\\ x_i \geq 0, \qquad i = 1, 2, ..., n.\end{cases}$$

Note that if x^* be an optimal solution of model (41), then x^* will be a pareto optimal solution of multi-objective programming model (40).

Theorem 4.1. *Let $\eta_i \in L(\alpha_i, \beta_i)$ for $i = 1, 2, ..., n$ be a linear uncertain variable, and $0 < \lambda \leq 1$. Then model (38) can be changed to the crisp equivalent as following form*

$$\text{minimize} \quad \sum_{i=1}^{n} x_i\left(\frac{\lambda\alpha_i}{2} + (1 - \frac{\lambda}{2})\beta_i\right) \tag{42}$$

$$\text{subject to}\begin{cases}\sum_{i=1}^{n} x_i\left(\dfrac{\alpha_i + \beta_i}{2}\right) \geq \vartheta,\\ \sum_{i=1}^{n} x_i\left(\dfrac{\beta_i - \alpha_i}{6}\right) \geq \varrho,\\ \sum_{i=1}^{n} x_i = 1,\\ x_i \geq 0, \qquad i = 1, 2, ..., n.\end{cases}$$

and model (39) can be changed to the crisp equivalent as following form

$$\text{maximize} \quad \sum_{i=1}^{n} x_i\left(\frac{\alpha_i + \beta_i}{2}\right) \tag{43}$$

$$\text{subject to}\begin{cases}\sum_{i=1}^{n} x_i\left(\frac{\lambda\alpha_i}{2} + (1 - \dfrac{\lambda}{2})\beta_i\right) \leq \varpi,\\ \sum_{i=1}^{n} x_i\left(\dfrac{\beta_i - \alpha_i}{6}\right) \geq \varrho,\\ \sum_{i=1}^{n} x_i = 1,\\ x_i \geq 0, \qquad i = 1, 2, ..., n.\end{cases}$$

Proof. Since, all of uncertain variables are linear in this mean that $\eta_i \in L(\alpha_i, \beta_i)$ for $i = 1, 2, ..., n$. Moreover, the expected value have obtained as $E[\sum_{i=1}^{n} x_i\eta_i] =$

$\sum_{i=1}^{n} x_i(\frac{\alpha_i+\beta_i}{2})$ and the variance $AVaR[\sum_{i=1}^{n} x_i\eta_i] = \sum_{i=1}^{n} x_i(\frac{\lambda\alpha_i}{2} + (1-\frac{\lambda}{2})\beta_i)$ and the quadratic entropy $Q[\sum_{i=1}^{n} x_i\eta_i] = \frac{\beta_i-\alpha_i}{6}$. Substituting the above formulas into model (38) and (39), the theorem will be proved. □

Theorem 4.2. *Let $\eta_i \in N(e_i, \sigma_i)$ for $i = 1, 2, ..., n$ be a Normal uncertain variable, and $0 < \lambda \leq 1$. Then model (38) can be changed to the crisp equivalent as following form*

$$\text{minimize} \quad \sum_{i=1}^{n} x_i(e_i\lambda - \frac{\sqrt{3}\sigma_i}{\pi}[\lambda ln\lambda + (1-\lambda)ln(1-\lambda)]) \tag{44}$$

$$\text{subject to} \begin{cases} \sum_{i=1}^{n} x_i e_i \geq \vartheta, \\ \sum_{i=1}^{n} x_i(\frac{\sqrt{3}\sigma_i}{\pi}) \geq \varrho, \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \geq 0, \qquad i = 1, 2, ..., n. \end{cases}$$

and model (39) can be changed to the crisp equivalent as following form

$$\text{maximize} \quad \sum_{i=1}^{n} x_i e_i \tag{45}$$

$$\text{subject to} \begin{cases} \sum_{i=1}^{n} x_i(e_i\lambda - \frac{\sqrt{3}\sigma_i}{\pi}[\lambda ln\lambda + (1-\lambda)ln(1-\lambda)]) \leq \varpi, \\ \sum_{i=1}^{n} x_i(\frac{\sqrt{3}\sigma_i}{\pi}) \geq \varrho, \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \geq 0, \qquad i = 1, 2, ..., n. \end{cases}$$

Proof. Since, all of uncertain variables are Normal in this mean that $\eta_i \in N(e_i, \sigma_i)$ for $i = 1, 2, ..., n$. Moreover, the expected value have obtained as $E[\sum_{i=1}^{n} x_i\eta_i] = \sum_{i=1}^{n} x_i e_i$ and the variance $AVaR[\sum_{i=1}^{n} x_i\eta_i] = \sum_{i=1}^{n} x_i(e_i\lambda - \frac{\sqrt{3}\sigma_i}{\pi}[\lambda ln\lambda + (1-\lambda)ln(1-\lambda)])$ and the quadratic entropy $Q[\sum_{i=1}^{n} x_i\eta_i] = \sum_{i=1}^{n} x_i(\frac{\sqrt{3}\sigma_i}{\pi})$. Substituting the above formulas into model (38) and (39), the theorem will be proved. □

5 Numerical Example

Example 5.1. Suppose that there are 10 stocks which their monthly return rates are estimated by experienced experts and they are Linear uncertain variables. Table 1 represents the simulated expected values of these stocks. An investor would like to create an optimal portfolio, and he wishes to minimize The Average Value at Risk, so solving model (44) to obtain the optimal portfolio is the main concern.

Table 1: data of securities which are linear uncertain variables for example 1.

stocks	x_1	x_2	x_3	x_4	x_5
	$L(0.4, 0.9)$	$L(1.7, 2.4)$	$L(0.3, 0.7)$	$L(0.1, 0.6)$	$L(0.5, 1.5)$
stocks	x_6	x_7	x_8	x_9	x_{10}
	$L(2, 3.3)$	$L(0.9, 1.4)$	$L(2.1, 2.3)$	$L(0.8, 1)$	$L(1, 1.2)$

Consider that in investor's mind, the minimum expected return that can accept is 2.5, and the Avarage Value at Risk not allowed to exceed 0.6, $\lambda = 0.1$ and minimum expected entropy that can accept be 0.08, which shows the level of portfolio diversification. Then the model (44) will be as follows:

$$\begin{aligned} \text{minimize} \quad & 0.875x_1 + 2.365x_2 + 0.68x_3 + 0.575x_4 + 1.45x_5 \\ & +2.235x_6 + 1.37x_7 + 2.29x_8 + 0.99x_9 + 1.19x_{10} \end{aligned} \tag{46}$$

$$\text{subject to} \begin{cases} 0.65x_1 + 2.05x_2 + 0.5x_3 + 0.35x_4 + x_5 + 2.65x_6 + 1.15x_7 + 2.2x_8 \\ +0.9x_9 + 1.1x_{10} \geq 2.5, \\ 0.083x_1 + 0.116x_2 + 0.066x_3 + 0.083x_4 + 0.166x_5 + 0.216x_6 \\ +0.083x_7 + 0.033x_8 + 0.033x_9 + 0.033x_{10} \geq 0.08, \\ x_1 + x_2 + ... + x_{10} = 1, \\ x_i \geq 0, \qquad i = 1, 2, ..., 10. \end{cases}$$

and the model (45) will be as follows:

$$\begin{aligned} \text{maximize} \quad & 0.65x_1 + 2.05x_2 + 0.5x_3 + 0.35x_4 + x_5 + 2.65x_6 \\ & +1.15x_7 + 2.2x_8 + 0.9x_9 + 1.1x_{10} \end{aligned} \tag{47}$$

$$\text{subject to} \begin{cases} 0.875x_1 + 2.365x_2 + 0.68x_3 + 0.575x_4 + 1.45x_5 + 2.235x_6 + 1.37x_7 \\ +2.29x_8 + 0.99x_9 + 1.19x_{10} \leq 0.6, \\ 0.083x_1 + 0.116x_2 + 0.066x_3 + 0.083x_4 + 0.166x_5 + 0.216x_6 \\ +0.083x_7 + 0.033x_8 + 0.033x_9 + 0.033x_{10} \geq 0.08, \\ x_1 + x_2 + ... + x_{10} = 1, \\ x_i \geq 0, \qquad i = 1, 2, ..., 10. \end{cases}$$

The optimal solution of model (46) is $x_3^* = 0.06976744$, $x_6^* = 0.9302326$ and $x_i^* = 0$, $\quad i = 1, 2, 4, 5, 7, 8, 9, 10$, so the optimal value of objective function is 2.126512. This means that for minimizing the risk with the expected value rather than 2.5 and expected entropy in order to portfolio diversification more than 0.08, the investor must allocate his capital according to x^*.

The optimal solution of model (47) is $x_3^* = 0.1968709$, and $x_4^* = 0.8005215$, $x_6^* = 0.002607562$ and $x_i^* = 0$, $\quad i = 1, 2, 5, 7, 8, 9, 10$, so the optimal value of objective function is 0.385528. This means that for maximizing the expected return with given constraints, the investor must allocate his capital according to x^*.

Example 5.2. Suppose that there are 10 stocks which their monthly return rates are estimated by experienced experts and they are Zigzag uncertain variables. Table 2 represents the simulated expected values of these stocks. An investor would like to create an optimal portfolio, and he wishes to minimize The Average Value at Risk, so solving model (44) to obtain the optimal portfolio is the main concern. Consider that in investor's mind, the minimum expected return that can accept

Table 2: data of securities which are zigzag uncertain variables for example 2.

stocks	x_1	x_2	x_3	x_4	x_5
	$Z(-0.3, 2, 2.5)$	$Z(-0.3, 2.8, 3.2)$	$Z(-0.4, 2.5, 4)$	$Z(-0.2, 3, 3.5)$	$Z(-0.2, 2.5, 3)$
stocks	x_6	x_7	x_8	x_9	x_{10}
	$Z(-0.6, 3, 4)$	$Z(-0.1, 2, 2.5)$	$Z(-0.4, 3, 4)$	$Z(-0.1, 1.9, 3)$	$Z(-0.2, 2.1, 2.5)$

is 2.6, and the Average Value at Risk not allowed to exceed 4.2, $\lambda = 0.2$ and and minimum expected entropy that can accept be 0.6, which shows the level of portfolio diversification. then the model (38) will be as follows:

$$\begin{aligned} \text{minimize } \; & 3.84x_1 + 5.28x_2 + 4.82x_3 + 5.56x_4 + 4.66x_5 \\ & +5.88x_6 + 3.68x_7 + 5.72x_8 + 3.5x_9 + 3.94x_{10} \end{aligned} \tag{48}$$

$$\text{subject to} \begin{cases} 1.55x_1 + 2.125x_2 + 2.15x_3 + 2.325x_4 + 1.95x_5 + 2.35x_6 + 1.6x_7 + 2.4x_8 \\ +1.675x_9 + 1.625x_{10} \geq 2.6, \\ 0.467x_1 + 0.583x_2 + 0.733x_3 + 0.617x_4 + 0.533x_5 \\ +0.767x_6 + 0.433x_7 + 0.733x_8 + 0.517x_9 + 0.45x_{10} \geq 0.6 \\ x_1 + x_2 + ... + x_{10} = 1, \\ x_i \geq 0, \qquad i = 1, 2, ..., 10. \end{cases}$$

and the model (39) will be as follows:

$$\begin{aligned} \text{maximize } \; & 1.55x_1 + 2.125x_2 + 2.15x_3 + 2.325x_4 + 1.95x_5 + 2.35x_6 \\ & +1.6x_7 + 2.4x_8 + 1.675x_9 + 1.625x_{10} \end{aligned} \tag{49}$$

$$\text{subject to} \begin{cases} 3.84x_1 + 5.28x_2 + 4.82x_3 + 5.56x_4 + 4.66x_5 + 5.88x_6 \\ +3.68x_7 + 5.72x_8 + 3.5x_9 + 3.94x_{10} \leq 4.2, \\ 0.467x_1 + 0.583x_2 + 0.733x_3 + 0.617x_4 + 0.533x_5 \\ +0.767x_6 + 0.433x_7 + 0.733x_8 + 0.517x_9 + 0.45x_{10} \geq 0.6 \\ x_1 + x_2 + ... + x_{10} = 1, \\ x_i \geq 0, \qquad i = 1, 2, ..., 10. \end{cases}$$

The optimal solution of model (48) is $x_3^* = 0.3842593$, $x_9^* = 0.6157407$ and $x_i^* = 0, \quad i = 1, 2, 4, 5, 6, 7, 8, 10$, so the optimal value of objective function is 3.5. This means that for minimizing the risk with the expected value rather than 2.6 and expected entropy in order to portfolio diversification more than 0.6, the investor must allocate his capital according to x^*. The minimum relevant risk is 3.5.
The optimal solution of model (49)is $x_3^* = 0.5303030$, and $x_9^* = 0.4696970$, and $x_i^* = 0, \quad i = 1, 2, 4, 5, 6, 7, 8, 10$, so the optimal value of objective function is 1.682. This means that for maximizing the expected return with given constraints, the investor must allocate his capital according to x^*. The maximum relevant return is 1.682.

6 Conclusion

This study calculates the Average Value at Risk (AVaR) and quadratic entropy of uncertain variables by the use of theorems and proofing techniques. Additionally, it describes uncertain models in relation to the mean-AVaR-quadratic entropy framework, which is used for optimal portfolio selection.
Portfolios are constructed according on investor preferences in an uncertain environment and considering the use of Average Value at Risk (AVaR) as a measure of investment risk and quadratic entropy as portfolio diversification. The models that have been acquired have been converted into linear programming problems in certain instances involving uncertain variables. The outcomes derived from the developed models for addressing portfolio selection challenges including uncertain returns are expected to hold significant value in the fields of economics and financial mathematics, encompassing both theoretical and practical applications.

Data availability and conflict of interest statement

This work is of theoretical nature and has not analyzed or generated any datasets.
The authors have no conflicts of interest to declare in relation to this article.
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Author contribution statement

F.O., L.T. and K.N. contributed in designing the model and computational framework, organizing the research and performing numerical simulations and reviewing the results and writing of the manuscript.

Bibliography

[1] M. Abdi, S.B. Ebrahimi and A.A. Najafi, *An online portfolio selection algorithm using beta risk measure and fuzzy clustering*, Journal of Mathematics and Modeling in Finance (JMMF), 3:2, (2023) PP. 63–76.

[2] S. Basak and A. Shapiro, *Value-at-risk based risk management: optimal policies and asset prices*, Rev. Financ. Stud., 14 (2001), PP. 371–405.

[3] A.K. Bera and S.Y. Park, *Optimal portfolio diversification using the maximum entropy principle*, Economet. Rev., 27(2008), PP. 484-512.

[4] Campbell, R., Huisman, R. and Koedijk, K. *Optimal portfolio selection in a value-at-risk framework*, J. Bank. Finance, 25 (2001), PP. 1789–1804.

[5] F.Y. Chen, *Analytical VaR for international portfolios with common jumps*, Comput. Math. with Appl., 62 (2011), PP. 3066–3076.

[6] Y. Chen, J. Wang, and W. Zhang, *Tail Distortion Risk Measure for Portfolio with Multivariate Regularly Variation*, Communic in Maths and Stats., 10 (2022), PP. 263–285.

[7] D. Cuoco and H. Liu, *Optimal consumption of a divisible durable good*, J. Econ. Dynam. Control., 24 (2000), PP. 561-613.

[8] P. Chou, W. Li and G. Zhou, *Portfolio optimization under asset pricing nomalies*, Jpn. World Econ., 18 (2006), PP. 121-142.

[9] W. Dai, *Quadratic entropy of uncertain variables*, Soft Comput., 22 (2018), PP. 5699-5706. https://doi.org/10.1007/s00500-017-2602-y

[10] X. Deng, X. He and C. Huang, *A new fuzzy random multi-objective portfolio model with different entropy measures using fuzzy programming based on artificial bee colony algorithm*, Eng Comput., 39(2) (2022), PP. 627–49.

[11] A. Dionisio, R. Menezes and D.A. Mendes, *Uncertainty analysis in financial markets: Can entropy be a solution?*, In Proceedings of the 10th Annual Workshop on Economic Heterogeneous Interacting Agents (WEHIA 2005), University of Essex, Colchester, UK, PP. 13-15, June, 2005.

[12] E. Garajova and M. Hladik, *On the optimal solution set in interval linear programming*, Comput Optim Appl., 72 (2019), PP. 269–292.

[13] W.W. Hogan and J.M. Warren, *Toward the development of an equilibrium capitalmarket model based on semivariance*, J. Financ. Quant. Anal., 9 (1974), PP. 1–11.

[14] R.E. Hoskisson, M.A. Hitt, R.H. Johnson and D. Moesel, *Construct validity of an objective (entropy) categorical measure of diversification strategy*, Strat. Manag. J., 14 (2006), PP. 215-235.

[15] R.P. Huang, Z.S. Xu, S.J. Qu, X.G. Yang and M. Goh, *Robust Portfolio Selection with Distributional Uncertainty and Integer Constraints*, J. Oper. Res. Soc. China, (2023). https://doi.org/10.1007/s40305-023-00466-4

[16] X. Huang, *Mean-risk model for uncertain portfolio selection*, Fuzzy Optim Decis Making., 10 (2011), PP. 71–89.

[17] X. Huang, *Mean-variance models for portfolio selection subject to experts estimations*, Expert Syst Appl., 39 (2012a), PP. 5887–5893.

[18] X. Huang, *A risk index model for portfolio selection with returns subject to experts estimations*, Fuzzy Optim Decis Making, 11 (2012b), PP. 451–463.

[19] X. Huang and L. Qiao, *A risk index model for multi-period uncertain portfolio selection*, Inform Sci., 217 (2012), PP. 108–16.

[20] P. Jana, T.K. Roy and S.K. Mazumder, *Multi-objective mean-variance-skewness model for Portfolio Optimization*, AMO., 9 (2007), PP. 181-193.

[21] P. Jana, T.K. Roy and S.K. Mazumder, *Multi-objective possibilistic model for portfolio selection with transaction cost*, J. Comput. Appl. Math., 228 (2009), PP. 188-196.

[22] J.N. Kapur and H.K. Kesavan, *Entropy Optimization Principles with Applications*, Academic Press: San Diego, CA, USA, 1992.

[23] H. Konno, K. Tanaka and R. Yamamoto, *Construction of a portfolio with shorter downside tail and longer upside tail*, Comput Optim Appl., 48(2011), PP. 199–212.

[24] B. Li, Y. Sun, G. Aw and K.L. Teo, *Uncertain portfolio optimization problem under a minimax risk measure*, Appl. Math. Model., 76 (2019), PP. 274–281.

[25] J. Li and J. Xu, *Multi-objective portfolio selection model with fuzzy random returns and a compromise approach-based genetic algorithm*, Inform Sci., 220 (2013), PP. 507–521.

[26] SH. Li, J. Peng, B. Zhang and D. Ralescu, *Mean-Variance-Entropy Portfolio Selection Models with Uncertain Returns*, Int. J. of Manag. Fuzzy Syst., 7(3)(2021), PP. 47–54. https://doi.org/10.11648/j.ijmfs.20210703.12

[27] B. Liu, *Uncertainty Theory*, seconded, Springer-Verlag, Berlin, 2007.

[28] B. Liu, *Some research problems in uncertainty theory*, J Uncertain Syst., 3(1) (2009), PP. 3–10.

[29] B. Liu, *Uncertainty theory: a branch of mathematics for modeling human uncertainty*, Springer, Berlin, 2010.

[30] B. Liu, *Why is there a need for uncertainty theory?*, J Uncertain Syst., 6 (2012), PP. 3–10.

[31] J. Liu, J. Xie, H. Ahmadzade and M. Farahikia, *A new measure of indeterminacy for uncertain variables with application to portfolio selection*, J. Intell. Fuzzy Syst., 40,3 (2021), PP. 5289-5293. https://doi.org/10.3233/JIFS-202073

[32] R. Mehralizade and A. Mehralizade, *LR mixed fuzzy random portfolio choice based on the risk curve*, Int J Uncertain Fuzz., 30(02) (2022), PP. 231–61.

[33] H. Markowitz, *Portfolio selection.* J Finance., 7(1) (1952), PP. 77–91.

[34] A.R. Nazemi, B. Abbasi and F. Omidi, *Solving portfolio selection models with uncertain returns using an artificial neural network scheme.* Appl. Intell., 42 (2015), PP. 609–621.

[35] Y. Ning, H. Ke and Z. Fu, *Triangular entropy of uncertain variables with application to portfolio selection*, Soft Comput., 19 (2015), PP. 2203-2209. https://doi.org/10.1007/s00500-014-1402-x

[36] F. Omidi, B. Abbasi and A.R. Nazemi, *An efficient dynamic model for solving a portfolio selection with uncertain chance constraint models*, J CAM., 319 (2017), PP. 43–55.

[37] G.C.Philippatos and N. Gressis, *Conditions ofequivalence among E-V, SSD and E-H portfolio selection criteria: The case for uniform, normal and lognormal Distributions*, Manag. Sci., 21 (1975), PP. 617-625.

[38] P.Y. Qian, Z.Z. Wang and Z.W. Wen, *A Composite Risk Measure Framework for Decision Making Under Uncertainty.* J. Oper. Res. Soc. China., 7 (2019), PP. 43–68. https://doi.org/10.1007/s40305-018-0211-9

[39] R.T. Rockafellar and S. Uryasev, *Conditional value-at-risk for general loss distributions*, J. Bank. Finance., 26 (2002), PP. 1443–1471.

[40] M. Salahi, T. Khodamoradi and A. Hamdi, *Mean-standard deviation-conditional value-at-risk portfolio optimization*, Journal of Mathematics and Modeling in Finance (JMMF), 3:1, (2023) PP. 83–98.

[41] B. Samanta and T.K. Roy, *Multi-objective Portfolio Optimization Model*, Tamsui Oxf. J. Math. Sci., 21 (2005), PP. 55-70.

[42] Sh. Shahbeyk, *Robustness in Mean-Variance Portfolio Optimization*, Journal of Mathematics and Modeling in Finance (JMMF), 2 (2022), PP. 195-204.

[43] N. Shi, M. Lai, S. Zheng and B. Zhang, *Optimal algorithms and intuitive explanations for Markowitz portfolio selection model and Sharpes ratio with no short-selling*, Sci. China Math. Phys. Astron. Technol. Sci., 51 (2008), PP. 2033-2042.

[44] Y. Sun, G. Aw, K.L. Teo and G. Zhou, *Portfolio optimization using a new probabilistic risk measure*, J. Ind. Manag. Optim., 11 (2015), PP. 1275–1283.

[45] Y. Takano and R. Sotirov, *A polynomial optimization approach to constant rebalanced portfolio selection.* Comput Optim Appl., 52 (2012), PP. 645–666.

[46] I. Usta and Y.M. Kantar, *Analysis of multi-objective portfolio models for the istanbul stock exchange*, In Proceedings of the 2nd International Workshop on Computational and Financial Econometrics (CFE08), Neuchatel, Switzerland, PP. 19-21 June, 2008.

[47] J. Yang, H. Lian and W. Zhang, *A Class of Structured High-Dimensional Dynamic Covariance Matrices*, Communic in Maths and Stats., 2023.

[48] M. Yin, W. Qian and W. Li, *Portfolio selection models based on Cross-entropy of uncertain variables*, J. of Intell. & Fuzzy Sys., 31(2)(2016), PP. 737–747, https://doi.org/10.3233/JIFS-169006

[49] Y. Yoshida, *Maximization of Returns under an Average Value-at-Risk Constraint in Fuzzy Asset Management*, Procedia Com. Sci., 112 (2017), PP. 11–20.

[50] Y. ZHU, *Uncertain optimal control with application to a portfolio selection model*, Cybern Syst., 41 (2010), PP. 535–547.

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