Estimating the term structure of mortality: an application to actuarial studies

Marzieh Vahdani[1], **Ali Safdari-Vaighani**[2]

[1] Eco College of Insurance, Allameh Tabataba'i University
marzivahdani@gmail.com

[2] Department of the mathematics, Allameh Tabataba'i University
asafdari@atu.ac.ir

Abstract:
Insurance companies and pension funds which deal with human lifetime are interested in mortality forecasting to minimize the longevity risk. In this paper, we studied the mortality forecasting model based on the age-specific death rates by the usage of the state-space framework and Kalman filtering technique. To capture the volatility of time, the time varying trend has been added to the Lee-Carter (LC) model, which is the benchmark methodology in modeling and forecasting mortality since it was introduced in 1992. So, this model is a random walk with time varying drift (TV). We illustrated the performance of the proposed model using Iranian mortality data over the period 19502015. Numerical results show that, both models have good fitness and are tangent. So the TV model acts as well as the LC model, but the TV model has the advantages of fewer calculations and the time-varying drift which can be beneficial in time varying data sets.

Keywords: Mortality forecasting; Lee-Carter approach; State-space modeling; Kalman recursions.
MSC2010 Classifications: 62M10, 91B84, 35B45

[1] Corresponding author
Received: Received: 2021-04-14 Approved: 2021-06-22

Introduction

Life expectancy has increased significantly in recent decades all over the world. This increase, which is because of medical findings and improvement in standards of living, has economic consequences for life-related insurance companies and pension funds. To evaluate the patterns of mortality and forecast the future mortality, statistical methods are needed.

There are many studies which consider trends of mortality and try to forecast the future mortality with different viewpoints and methods. These studies were started in 17th century and have been continued in numerous methods. These methods were based simply on the collecting the number of deaths and births from available sources and they were lack of graphical presentations. Graunt (1662), Halley (1693) and De Moivre (1725) were among these studies.

More recently, Gompertz (1825) and Makeham (1860) let the force of mortality increase exponentially with age, adding a small constant, for better reflection of the age pattern of mortality at younger ages. Force of mortality represents the instantaneous rate of mortality at a certain age measured on an annualized basis. Later, others like Heligman Pollard (1980) try to fit curve to entire age range based on the ratio of the probability of dying at any age to that of the probability of not dying at that age. But none of these methods considered time effect in mortality rate. Most recent models fit curves to mortality rates in both the age and time dimension.

Lee and Carter [1] were among the first researchers who used stochastic trend methodology to count the time effect in mortality date. Lee and Carter (LC) explored the time series behavior of mortality movements between age groups by using a single latent factor which is responsible for describing the general level of log mortality. Log central death rates are modeled as the sum of a time invariant, age-specific constant, and the product of an age-specific time invariant component and the time-varying latent factor. The age-specific component represents the sensitivity of an individual age group to the general level of mortality changes [1].

Girosi and King [2] proposed a reformulation of the Lee and Carter model. They introduced a version of LC model, with a single latent factor, following a random walk with drift. Jong and Tickle [3] introduced a more flexible approach based on standard time series approaches to estimation and forecasting. The model introduced was the LC (smooth) model which is a smoothed version of the original LC model. They estimate the parameters by the means of the least square method and maximum likelihood estimation using Kalman filtering either. H'ari et al. [4], added the time factor to the drift of LC model in order to capture the volatility of time. The applied framework is the state-space framework that is a well-known method in time series. In this approach, the Kalman filtering technique is used to estimate the required parameters.

In this paper, we implement the LC model as well as the model with the time varying drift, called the TV model to Iranian mortality data to evaluate the function and accuracy of the TV model and the advantage of it according to the LC model. The available data range from 1950 to 2015 and include a peak due to Iran-Iraq war in 80s.

The paper is followed with the methodology section which describes the Lee-Carter model and Time-Varying model formulations. In section 3, we run and evaluate both methods using Iranian mortality data and see the results. At last a conclusion section will conclude the article, proposing future researches.

Methodology

Let D_{xt} be the number of people with age x that died in year t and E_{xt} be the exposure-to-risk at age x in year t. Then $m_x t$ is the logarithm of central death rate for age x in year t [5,6]. defined as:

$$m_{xt} = \ln\big(\frac{D_{xt}}{E_{xt}}\big) \tag{1}$$

where $x \in \{1, \ldots, na\}$, and $t \in \{1, \ldots, T\}$. Define

$$m_t = \begin{pmatrix} m_{1,t} \\ \vdots \\ m_{na,t} \end{pmatrix}$$

Then the model according to Lee and Carter [1] can be formulated as:

$$m_t = \alpha + \beta\gamma_t + \delta_t \tag{2}$$

Where m_t is the central death rate in year t. α and β are time invariant, age-specific constants. γ_t is the time-varying index of level of mortality which is one-dimensional underlying latent process and δ_t is a vector of (measurement) error terms. Taking expectation, yields the estimation of α as:

$$\alpha_x = m_x \tag{3}$$

And the parameters β and γ in (1) can be estimated via maximum likelihood. The optimum can be found easily via the singular value decomposition (SVD) of the matrix of centered age profiles, $U\Sigma V^t = m$. Then according to [3] $\hat{\beta} = U$ while $\hat{\gamma} = \Sigma V^t$. For better estimation of parameters, we need to run the second stage estimation of γ. As it is seen, LC estimation needs many calculations to result in parameter estimation. To reduce the calculations, and improve the estimation accuracy, we generalize the LC model to a time varying (TV) model. For this aim, the start point is the LC-reformulated model [2], which is defined as:

$$m_t = \theta + m_{t-1} + \zeta_t, \tag{4}$$

where ζ_t is the error term and $\theta = \beta c$. In the TV model $\theta = \beta c$ is considered as

$$\theta_t = a + Bx_t, \tag{5}$$

where

$$x_t =_{t-1} +\eta_t. \tag{6}$$

In the state-space framework, suppose that

$$Y_t :=_t= m_t - m_{t-1}, \quad t = 2, 3, \ldots, T.$$

Therefore, the *observation equation* is defined as

$$Y_t = a + Bx_t + \zeta_t, \tag{7}$$

and the *state equation* is defined as

$$x_t =_{t-1} +\eta_t, \tag{8}$$

where x_t is considered as a one-dimensional latent factor, and ζ_t is assumed to be zero. It means all the uncertainty of the model is defined in the η_t term, which $\eta_t \sim WN(0, Q)$. For estimating the unknown parameters, we need to maximize the likelihood function is given by

$$L(\theta|Y_1, \ldots, Y_{T-1}) = (2\pi)^{-na\times(T-1)/2} \prod_{t=1}^{T-1} \det(BB^t)^{-1/2} \exp\left(-\frac{1}{2}\sum_{t=1}^{T-1}(Y_t - \Gamma Y_{t-1})^t (BB^t)^{-1}(Y_t - \Gamma Y_{t-1})\right) \quad (9)$$

After computing B and Γ by the means of the two-stage iterative process introduced in [5], the parameter vector a is calculated by summing over $t = 2, 3, \ldots, T$. Then we have

$$a = \frac{1}{T-1}\sum_{i=2}^{T}(Y_i - Y_{i-1}) = \frac{(m_T - m_1) - \Gamma(m_{T-1} - m_0)}{T-1} \quad (10)$$

Therefore, the parameters of the TV model are estimated [8] . Moreover, the predictions of the death rates are based upon:

$$E(m_{T+\tau}|F_T) = m_T + \hat{a}\tau + \hat{B}\sum_{t=T+1}^{T+\tau} E(x_t|F_T) \quad (11)$$

$$E(x_{T+\tau}|F_T) = \hat{\Gamma}(\hat{x}_T) \quad (12)$$

Implementation and numerical results

As mentioned, life related insurance products face two kinds of risk. First institutions offering products based on lifetime of an individual, face risk because life time is uncertain. This kind of risk, known as micro- or pooling-risk can be reduced by increasing portfolio size.

But there exists another source of risk that cannot be reduced by increasing the number of people included. The macro- or longevity risk is the risk of increasing in life expectancy during time. Insurance companies and pension funds are of main institutions that are affected by this risk, and they always seek methods of mortality modeling which forecast mortality as accurate as possible.

If pension funds in Iran continue to use the static and adopted life table ”TD 88-90”, set for another country and is for many years ago, they will run budget deficit, as they usually do. In addition, the insurance companies suffer from statistic life tables. Statistic life tables do not reflect the mortality decrease, which will lower the premium. In a complete competitive market, the less the premium, the more the number of costumers. If one company cannot accurately forecast the mortality, and therefore do not lower the premium, it will lose its customers. A company with fewer and fewer customers will go bankrupt.

The Lee-Carter [1] approach is the benchmark methodology in mortality forecasting since it was introduced. In many empirical applications the Lee-Carter approach results in a model that describes the log central death rates by means of linear trends. But it has sophisticate calculations, and multiple steps to result in estimation and predictions. Therefore, to implement the LC function, we use an online Lee Carter Mortality forecasting.

As described, based on the H'ari [4], we study a mortality forecasting model describing the time series behavior of age-specific death rates. Our model is an extended model of the Lee-Carter, which is reformulated by Girosi and King [2]. In this reformulation the log central death rates are directly modeled as random walks with drift. These drifts determine the long run forecasts. We extend this approach by allowing for a time varying trend for drifts, depending upon a few underlying latent factors, in order to capture the co-movements between the various age groups. By reformulating the model in a state-space framework, the Kalman filtering technique can be used to estimate the parameters by means of maximum likelihood estimate (MLE).

The reformulation of Girosi King [2] considers a drift for the random walk. These drift are constant over time. However, due to the volatility in (past) mortality data, the estimation of these trends, and, thus, the forecasts based on them, might be rather sensitive to the sample period employed. This change in drift is not considered in LC model. Our TV model allows for time-varying trends, depending on a few underlying factors, to make the estimates of the future trends less sensitive to the sampling period.

Another advantageous of our model is that we estimate the age specific parameters, the latent factor and its process in one single step, while the Lee-Carter model and its extensions estimate the parameters in multiple steps. In addition, the standard error estimation in our TV model can be

lcfit.demog.berkeley.edu

done in the same single step, while the standard error of the parameters in LC model is estimated in several steps which are complex in some cases, according to what Lee and Carter themselves say in [1]. This multi-steps approach makes the calculations more complicated and increases estimation error.

We applied a simplified model of H'ari's [4] model and illustrated our specification using Iranian mortality data over the period 19502015. The data are extracted from the United Nations dynamic life tables, and we use death rates that are obtained by the division of the number of death into the exposure- to-risk.

The data are prepared in EXCEL software to obtain death rates from death numbers existing in the UN site. Then the parameters of the model are estimated and predictions are done by the means of MATLAB codes, which are written for this purpose. At last curves and tables are inserted for result comparison.

When comparing the LC and TV results, both models have good fitness and are tangent, with the accuracy of one in ten thousand. So the TV model acts as well as the LC model. But the TV has the advantage of fewer calculations and the time-varying drift. In this section, the Iranian age-specific mortality data is used to illustrate the performance of the proposed model in comparison to the LC model. The data is the males life table for the years 1950 to 2015 with the age groups 1, 1-4, , 80-85, 85+ and is from World Population Prospects. Table 1 and figure1 show the forecasting results of both LC and TV methods for the period 2035-2040. It can be seen that the accuracy of the results is almost the same in all age groups. The curves of models are almost tangent that are not distinguishable in the curve, but the TV model has an advantage comparing to the benchmark methodology because the time varying trend which is added in TV model can be beneficial to decrease the sensitivity of the estimations of the future trends to the sampling period.

Besides we examined the sampling period sensitivity to the prediction results. Death rate trend of Iran shows that mortality rate for men had increased strongly during the period 1980-1985 which was coincident with the first 5 years of the Iran-Iraq war. We want to see how choosing the sample period will affect the mortality prediction, especially if the period

World Population Prospects: The 2012 Revision. United Nations, Population Division, Department of Economic and Social Affairs. Data is available at http://esa.un.org/unpd/wpp/index.htm, downloaded on 07.26.2015)

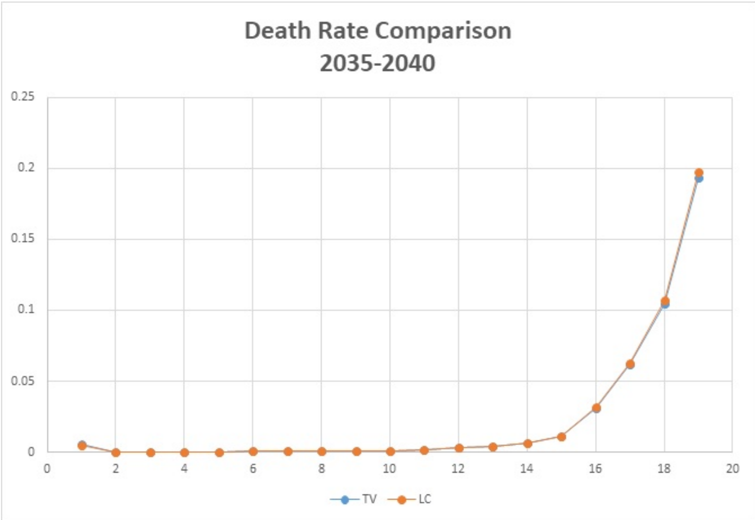

Figure 1: Death rate projections for the period 2035-2040 by the LC model and time varying model

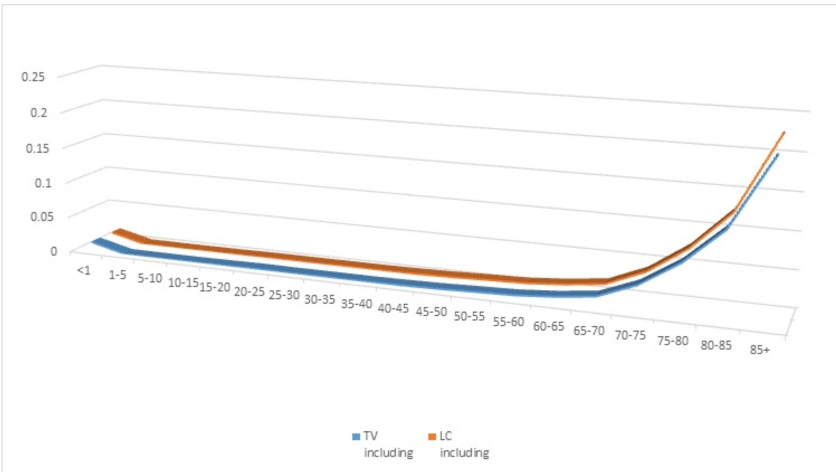

Figure 2: TV and LC comparison, including the death peak of war

includes a peak, like the case of war, or epidemic diseases. We seek a method to be less influenced by these peaks. Table 2 shows the mortality rate prediction for the years 2015-2020 based on the time varying (TV) model and Lee-Carter (LC) model, both in three cases. First the sample

age	TV model	LC model
<1	0.00525	0.00501
1-5	0.00014	0.00013
5-10	0.00008	0.00007
10-15	0.00010	0.00009
15-20	0.00037	0.00037
20-25	0.00114	0.00119
25-30	0.00079	0.00081
30-35	0.00087	0.00088
35-40	0.00069	0.00070
40-45	0.00094	0.00094
45-50	0.00156	0.00156
50-55	0.00284	0.00284
55-60	0.00367	0.00365
60-65	0.00639	0.00634
65-70	0.01112	0.01108
70-75	0.03105	0.03149
75-80	0.06192	0.06288
80-85	0.10501	0.10690
85+	0.19331	0.19772

Table 1: Death rate forecasting results for the period of 2035-2040.

period is 1980 to 2015 which includes the first years of war (including), with the increased rate of mortality. Then in the next step the rates are calculated by the time series of 1985 up to 2015 which excludes the first years of the war (excluding). At last the prediction based on the whole period (1950-2015) is presented (all). By our data set, according to figure 2 and figure 3, we cannot see a significant decrease in sensitivity of the result to the sample period.

Another advantage of the TV model is that the complexity of the computations is less than the LC model. The LC model estimates and predicts the parameters in multiple complex steps. But the TC model

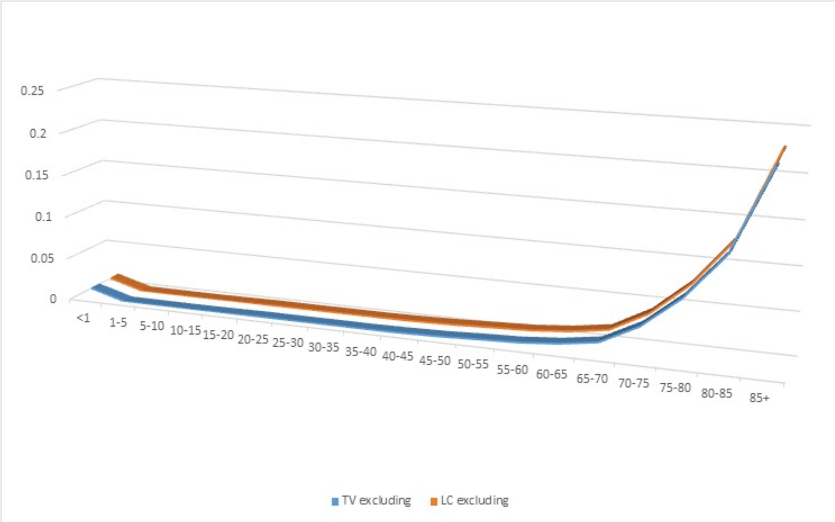

Figure 3: TV and LC comparison, excluding the death peak of war

using Calman Filtering technique, estimates in a simple step, using MLE function.

Conclusion

Static life tables fail to predict true distribution of future mortality and lead to invalid death rate predictions; thus the need for a dynamic life table is evident. We run a dynamic model for predicting death rate, using the state-space framework and Kalman filtering technique, called a Time Varying model. Our model acts as accurate as the Lee-Carter model, and even more accurate. The other advantage of our proposed model, is the lower calculations. The third advantage is that the TV model, can reduces the volatility of time, means the peaks of deaths, which occur because of war or epidemic diseases (like covid-19 these days.) The existing model is suggested to be extended by more than one latent factors, and error terms with autoregressive moving average (ARMA) behavior instead of white noise. Thus, the model might show better performance.

	TV model			LC model		
age	including	excluding	all	including	excluding	all
<1	0.0123	0.0122	0.0127	0.0122	0.0122	0.01259
1-5	0.0004	0.0004	0.0005	0.00046	0.00044	0.00046
5-10	0.0002	0.0002	0.0002	0.00022	0.00022	0.00022
10-15	0.0002	0.0002	0.0002	0.0002	0.00021	0.00023
15-20	0.0005	0.0004	0.0006	0.00042	0.00043	0.00064
20-25	0.001	0.0009	0.0014	0.0009	0.00091	0.00141
25-30	0.0008	0.0008	0.0011	0.00073	0.00074	0.00112
30-35	0.0009	0.0009	0.0013	0.00087	0.00089	0.00126
35-40	0.0009	0.0009	0.0011	0.00089	0.00093	0.00114
40-45	0.0014	0.0014	0.0016	0.00136	0.00145	0.00159
45-50	0.0023	0.0024	0.0026	0.00231	0.0025	0.00259
50-55	0.004	0.0043	0.0045	0.00403	0.00435	0.00449
55-60	0.0055	0.0059	0.006	0.00552	0.006	0.00604
60-65	0.0096	0.0097	0.0101	0.00976	0.00987	0.0101
60-65	0.0096	0.0097	0.0101	0.00976	0.00987	0.0101
65-70	0.016	0.0163	0.0169	0.01634	0.01656	0.01687
70-75	0.0369	0.0389	0.0395	0.03847	0.04	0.03961
75-80	0.0708	0.0748	0.0748	0.07449	0.07722	0.07431
80-85	0.1182	0.1251	0.1238	0.1258	0.12959	0.12274
85+	0.2106	0.2216	0.2197	0.22638	0.22996	0.21807

Table 2: The central death rate forecast for the period 2015-2020 based on different sample period (including war 1980-2015; excluding war 1985-2015; all 1950-2015) and different methods (Time Varying Drift, and Lee-Carter)

Bibliography

[1] B.D. Lee, and R.L. Carter, *Modeling and forecasting U.S. mortality*, Journal of the American Statistical Association, 87 (1992), pp. 659- 671

[2] F. Girosi, and G. King, *Understanding the Lee-Carter Mortality Forecasting Method*, Working Paper, (2007), Harvard University.

[3] P.D. JONG, AND L., TICKLE, *Extending the Lee-Carter methodology of mortality projection*, PdD thesis, (2004), Department of Actuarial Studies, Macquarie University.

[4] N. H'ARI, A.D. WAEGENAERE, B. MELENBERG AND T.E. NIJMAN, *Estimating the term structure of mortality*, Insurance: Mathematics and Economics, 42 (2008), pp. 492504

[5] P.E. CAINES, *Linear stochastic systems*, John Wiley Sons, Inc., 1988.

[6] N.L. BOWERS, *Actuarial mathematics*, (1997), Society of Actuaries (SOA).

[7] D.C. DICKSON, M.R. HARDY AND H.R. WATERS, *Actuarial mathematics for life contingent risks*, (2009), New York: Cambridge University Press.

[8] P.J. BROCKWELL AND R.A. DAVIS, *Introduction to time series and forecasting*, (2002), Springer.