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Research paper

Estimating the parameters of 3/2 stochastic volatility model with jump

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Abstract:
The financial markets reveal stylized facts that could not be captured by Black-Scholes partial differential equations (PDEs). In this research, we investigate 3/2 stochastic volatility to pricing options which is more compatible with the interpretation of implied volatility. Numerical study and calibrations show that the 3/2 model incorporating jumps effectively encompasses key market characteristics attributed. However, it requires more estimating parameters in comparison to the pure diffusion model. Stochastic volatility models with jumps describe the log return features of the financial market although more parameters are involved in estimations.

Keywords: Black-Scholes model, Stochastic volatility models, 3/2 model, 3/2 plus jump model.
MSC Classifications: 91G20, 60H30, 91G70

1 Introduction

In the modern financial markets, there is a growing demand for attractive financial products that can be accurately priced under real market conditions, accounting for their complexity beyond what a simplistic Black-Scholes model can offer. The Black-Scholes partial differential equation (PDE) has traditionally been employed for option pricing when the underlying asset is assumed to follow a continuous path. However, in many instances, such pure diffusion models prove inadequate in capturing the empirical observations related to stock price movements, as the Brownian paths they rely on are continuous everywhere. To address these limitations, stochastic volatility plus jump models have emerged as more flexible alternatives capable of effectively capturing the nuanced dynamics of financial markets. These models incorporate the concept of stochastic volatility, acknowledging that volatility is not constant but rather exhibits random variations over time. By considering both volatility and jump components, they can better account for the leptokurtic

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features observed in various financial datasets. In this study, we aim to develop a numerical scheme for estimating the parameters of the widely recognized 3/2 stochastic volatility model, while also incorporating the presence of jumps in the price process. By doing so, we seek to enhance our understanding and modeling capabilities of financial markets. Specifically, we focus on the normalized daily simple returns of the S&P500 index from January 3, 2012, to April 28, 2023, which exhibit distinct leptokurtic characteristics.

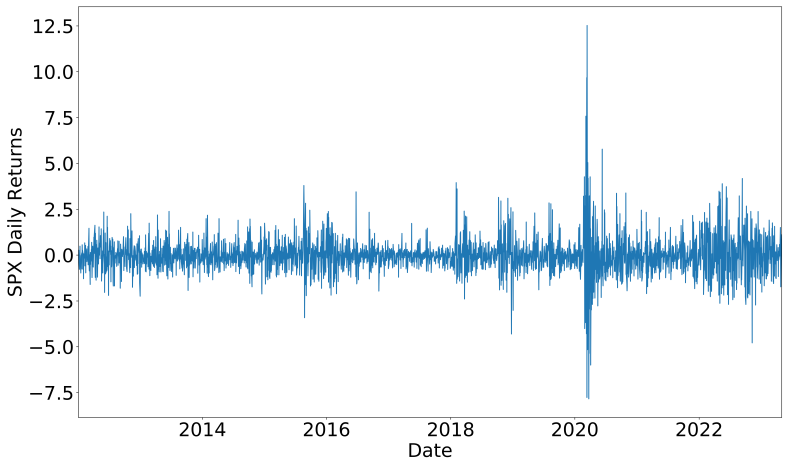

Figure 1: Normalized daily simple returns of SPX from Jan 3, 2012 to Apr 28, 2023

Our proposed framework allows us to construct an implied volatility surface, which provides valuable insights into the behavior of options with different maturities. Through our analysis, we find that the 3/2 stochastic volatility model with jumps yields a superior fit to the market data, as evidenced by its ability to align more closely with short-term options on the SPX index. By leveraging our numerical scheme, we can estimate the parameters of the stochastic volatility model and incorporate jump processes, thus enriching our understanding of market behavior and enabling more accurate pricing of complex financial products. This advancement in option pricing methodology holds significant implications for investors and financial institutions, as it allows for the development of innovative and attractive investment opportunities that align with the intricacies of real-world market conditions. In conclusion, our research highlights the limitations of the traditional Black-Scholes model and emphasizes the need for more sophisticated pricing methods that account for the complexities observed in actual market environments. By exploring stochastic volatility models with jump components, we demonstrate their effectiveness in capturing empirical observations and enhancing our ability to estimate option prices accurately. These findings contribute to the ongoing efforts to refine financial modeling techniques and provide valuable insights for practitioners and researchers in the field of quantitative finance.

1.1 Model Descriptions

In this section, we intend to describe two models: 3/2, and 3/2 with jumps. For this purpose, we first write down the risk-neutral dynamics Q for each of these models. Then, we proceed to use the relationship between the probability density function and the characteristic function for pricing European option contracts. Therefore, it is necessary to obtain the characteristic functions for all three models and incorporate them into the option pricing formula.

(i) 3/2 model:

Let us assume that asset price under the risk-neutral martingale measure follows the 3/2 stochastic volatility model

$$\begin{aligned} \frac{dS_t}{S_t} &= (r-\delta)dt + \sqrt{V_t}dB_t^1 \\ dV_t &= \kappa V_t(\theta(t) - V_t)dt + \epsilon V_t^{\frac{3}{2}} dB_t^2 \end{aligned} \tag{1}$$

where r is the risk-free interest rate, δ is the dividend yield, ϵ is the volatility of volatility, κ and θ represent mean-reverting and expected long-term standard deviation of log returns, respectively. We have supposed correlated Brownian motions $dB_t^1.dB_t^2 = \rho dt$.

(ii) 3/2 plus jump model:

The risk-neutral dynamics of the stock price under the 3/2 model plus jump model is

$$\begin{aligned} \frac{dS_t}{S_t} &= (r-\lambda\bar{\mu})dt + \sqrt{V_t}dB_t^1 + d\left[\sum_{j=1}^{N_t}(\zeta_j - 1)\right] \\ dV_t &= \kappa V_t(\theta(t) - V_t)dt + \epsilon V_t^{\frac{3}{2}} dB_t^2 \end{aligned} \tag{2}$$

where dN denotes an $\mathbb{F}$-adapted Poisson process with constant λ and ζ_j is the relative jump size of the j^{th} jump. The distribution of ζ_j, $j = 1, 2, ..., N_t$, are assumed to be independent and normal with mean μ and variance σ^2 while μ, $\bar{\mu}$ and σ satisfy the following relation

$$\mu = \log(1+\bar{\mu}) - \frac{1}{2}\sigma^2$$

Proposition 1.1 ([1]). *In the defined model 1 with time-dependent mean-reversion level, the joint conditional Fourier-Laplace transform of log-forward* *price process* $X_T = \log(S_T e^{(r-\delta)(T-t)})$ *and the de-annualized realized variance* $(RV_T - RV_t) = \int_t^T V_s\, ds$ *is given by*

$$E^Q[e^{i\phi X_T - l(RV_T - RV_t)}|X_t, V_t] = e^{i\phi X_t}\frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}(\frac{2}{\epsilon^2 y(t,V_t)})^{\alpha} \tag{3}$$

$$\times M(\alpha;\gamma;\frac{2}{\epsilon^2 y(t,V_t)}) \tag{4}$$

where

$$y(t,V_t) = \left(\int_t^T e^{\int_t^{t'} P(u)\,du}\,dt'\right)V_t$$

$$\alpha = -(\frac{1}{2}-\frac{\tilde{q}}{\epsilon^2}) + \sqrt{(\frac{1}{2}-\frac{\tilde{q}}{\epsilon^2})^2 + 2\frac{l'}{\epsilon^2}}, \qquad \gamma = 2(\alpha + 1 - \frac{\tilde{q}}{\epsilon^2})$$

$$\tilde{q} = q + \rho\epsilon i\phi \qquad \text{and} \qquad l' = l + \frac{i\phi}{2} + \frac{\phi^2}{2}$$

$M(\alpha;\gamma;Z)$ *is confluent hypergeometric function defined as*

$$M(\alpha;\gamma;Z) = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n}\frac{Z^n}{n!}$$

and

$$(x)_n = x(x+1)(x+2)...(x+n-1).$$

Proposition 1.2 ([2]). *In the model 2, the joint Fourier-Laplace transform of* $X_T = \log(S_T e^{(r-\lambda\bar{\mu})(T-t)})$ *and de-annualized realized variance* $(RV_T - RV_t) = (\int_t^T V_s\, ds + \sum_{j=1}^{N_T - N_t}(\zeta_j)^2)$ *is given by* $\int_t^T V_s\, ds$ *and*

$$E^Q[e^{i\phi X_T - l(RV_T - RV_t)}|X_t, V_t] = e^{i\phi X_t}\frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}(\frac{2}{\epsilon^2 y(t,v_t)})^{\alpha}$$
$$\times M(\alpha;\gamma;\frac{-2}{\epsilon^2 y(t,v_t)})exp[\lambda(T-t)(a-1)]$$

where

$$a = \frac{exp(\frac{-2l\mu^2 + 2i\mu\phi + \sigma^2\phi^2}{2+4l\sigma^2})}{\sqrt{1+2l\sigma^2}}$$

2 Main results

Here we have decided to perform least squares error fitting using the following method. Let X represent the remaining time to expiration, K represent the corresponding option strike prices, and other parameters specified at time T. Considering the calibration objective of minimizing the least squares error with the inclusion of a regularization function P (which can be, for example, the distance to the initial parameter), we will have:

$$\hat{\Theta} = \underset{\Theta \in U_\Theta}{\operatorname{argmin}} \sum_{i=1}^{N} \sum_{j=1}^{M} \omega_{ij} \left(C_{MP}(X_i, \tau_j) - C_{SV}(S_\tau, X_i, \tau_j, r_j, \Theta) \right)^2$$

where, $C_{MP}(X_i, \tau_j)$ denotes the market price of European call option with strike X_i and time to maturity τ_j, and C_{SV} is option price calculated by joint conditional Fourier-Laplace transform in proposition 2.1 and 2.2, which depend on the vector of model parameters $\Theta = (V_0, \kappa, \theta, \epsilon, \rho, \mu, \sigma, \lambda)$ (parameters μ, σ and λ belong to 3/2 plus jump model only). The estimate of parameters for both stochastic volatility models provided in Table 1, while the Truncated Newton Constrained (TNC) method was employed for minimization. By minimizing this objective function, we aim to find the best-fitting values for the parameters that minimize the sum of squared differences between the observed and estimated values while considering the regularization term. The regularization term helps prevent overfitting and provides a balance between fitting accuracy and complexity. This approach allows us to obtain a calibrated model that provides a close fit to the observed data while incorporating a regularization mechanism to control model complexity and improve generalization performance.

Table 1: Estimated parameters for the 3/2 models within 1-months from S&P500 options on June 10th, 2023.

Model	V_0	κ	θ	ϵ	ρ	μ	σ	λ
3/2	0.001	29.76	0.09	0.75	-0.5	-	-	-
3/2 plus jump	0.003	28.67	0.15	0.73	-0.47	-0.06	0.23	0.18

Conclusion

In this article, we have explored and compared the 3/2 model and the 3/2 plus jump model in the context of fitting the real market. Both of these models are used to describe the dynamics of financial markets and are widely utilized in option pricing and risk management. The 3/2 model is a stochastic volatility model that incorporates the mean reversion property of volatility. It assumes that the volatility of returns follows a square root process and allows for time-varying and time-dependent volatility. While this model is able to capture the volatility smile observed in the market, it has limitations in accurately capturing extreme events or sudden jumps in volatility. To address these limitations, the 3/2 plus jump model introduces a jump component to the 3/2 model. This jump component represents sudden and discontinuous movements in asset prices or volatility. By including

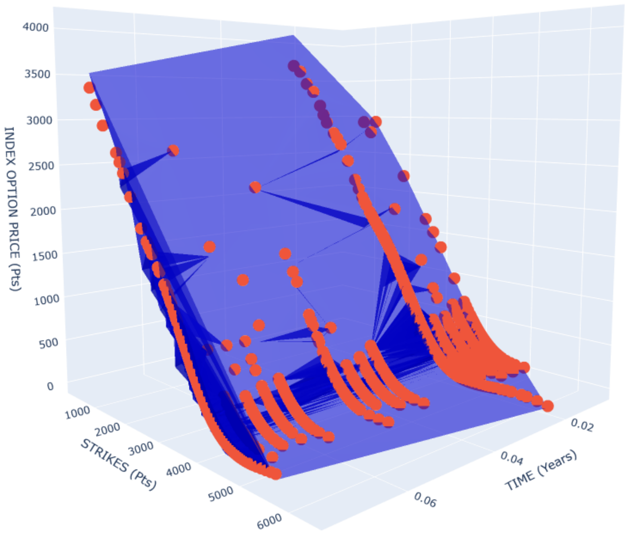

Figure 2: Calibration of 3/2 model with real market

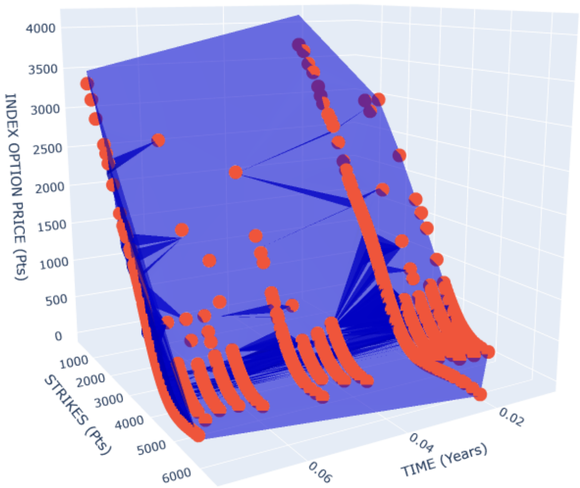

Figure 3: Calibration of 3/2 plus jump model with real market

jumps, this model aims to capture extreme market events and better fit real market data. In our comparison, we find that both the 3/2 model and the 3/2 plus jump model have strengths and weaknesses in fitting the real market. The 3/2 model is generally effective in capturing the volatility smile and overall trend of the market. It provides a reasonable representation of market dynamics and can be used for option pricing and risk management. However, the 3/2 model often falls short in accurately capturing extreme market events or sudden changes in volatility. This limitation can be addressed by the 3/2 plus jump model, which incorporates jumps to capture discontinuous movements in asset prices or volatility. The inclusion of the jump component enhances the model's ability to fit real market data and capture extreme events. In conclusion, while the 3/2 model provides a reasonable representation of market dynamics, the 3/2 plus jump model offers added flexibility and accuracy in fitting the real market. Depending on the specific needs and objectives of the analysis, either model can be used to effectively capture and analyze the dynamics of financial markets.

Bibliography

[1] G. G. Drimus, *Options on realized variance by transform methods: A non-affine stochastic volatility model*, Quantitative Finance., 11 (2009), 1679–1694.

[2] J. Baldeaux, A. Badran, *Consistent modelling of VIX and equity derivatives using a 3/2 plus jump model*, Research Paper Series 306, Quantitative Finance Research Center, University of Technology, Sydney, 2012.

[3] P. Carr, J. Sun *A new approach for option pricing under stochastic volatility*, Review of Derivatives Research, 10 (2007), 87–150.

[4] R. Cont, P. Tankov, *Financial modelling with jump processes*, CRC Press LLC, London, 2004.

[5] S. G. Kou, J. *Jump-diffusion models for asset pricing on financial engineering*, Handbooks in Operations Research and Management Science, 15 (2007), 73–116.

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