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Revue of contingent capital pricing model using growth and barrier option approach with numerical application

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Abstract:
This paper investigates the effects of contingent capital, a debt instrument that automatically converts into equity if the value of the asset is below a predetermined threshold on the pricing process of a bank assets'. A traceable form of the contingent convertible bond is analyzed to find a closed-form solution for the price of this bond using barrier and growth options. We examine the interaction between growth options and financing policy in a dynamic business model. The contribution of this paper is to extend [10] and [22] research to include the evaluation of all aspects of banks' financial structure, with an emphasis on explicitly calculating the likelihood of the default event. The fundamental theorem of asset pricing and the first passage of time method have been used to generate closed formulas that are amenable to practical analysis. The potential benefits from contingent capital as financing and risk management instrument can be assessed through their contribution to reducing the probability of default. The appropriate choice of contingent capital parameters, the rate, and the conversion threshold can reduce shareholders incentives to change risk.

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1 Introduction

Non-binding financing has always been the ideal instrument for banks to insure investment against default and there by improve their credibility vis-Ã-vis depositors, creditors and regulators. The flexibility of these instruments should enable banks to cope with extreme events by relieving their financial structure of the rigid burdens of risky debt. Contingent convertible securities or CoCo bonds are a real interest in this area. They allow banks, thanks to their embedded option, to gain in flexibility and insurance against non-payment once exercised in a well specified financial distress circumstance. This becomes even more obvious when they are coupled with growth options. Moreover, they offer issuing companies the potential to create quasi-equity at a lower cost. CoCo bonds are rather reserved for large companies, mainly banks and insurance companies. The conversion takes place on the initiative of the issuer, under solvency conditions specified in advance, and the investor experiences the situation, with a significant capital loss, since the initial bond security becomes shares at a low price. So far, no CoCo bond has missed a coupon payment or experienced a trigger event. However, it is possible that a missed coupon payment or a trigger event may affect valuations process of bank assets.

In the aftermath of the last financial crisis of 2007, the volume and the quality of bank's capital have been so worsened that they were unable to generate significant new capital in the market and instead had to rely on governments to provide capital. CoCo bonds represent the third generation of hybrid securities, with an additional level of risk for investors. They are debt securities convertible into shares in case of emergency, if the credit worthiness of the issuing company drops below a predetermined threshold. CoCo bonds are designed to absorb the bank's losses during a period of financial distress, improving its capital position. They may absorb losses by automatically converting into shares or depreciating the nominal value when a pre-specified trigger event occurs. In the absence of a triggering event, the securities are hybrid financial instruments with debt-like characteristics. European banking regulator validated that CoCo bonds were sufficiently good instruments to enter into the calculation of the solvency ratio required by Basel III. With this idealization of CoCo bonds by the regulator, banks have an additional means to strengthen their balance sheets, beyond capital increases, profit-saving transactions and other asset disposals.

The issuer has the right to convert debts automatically and compulsorily into a number of shares when the market capital ratio of a company reaches a predetermined threshold. The CoCo bonds may raise interesting questions such as how does this hybrid instrument would affect the capital structure of a bank, the valuation of shareholders' capital, financial stability, regulatory oversight and investment decisions considering the capital structure constraints and incentive of risk taking as it was represented by [11] and [14]. All these questions have been addressed and analyzed by [2], [12] and [10]. In this paper, we follow [22] and [23] works with an emphasis on uncertainty, flexibility in a dynamic setting. [4] introduced the concept of convertibles that convert obligatorily in equity when the point of non-viability is reached. In 2010, [19] developed a structural credit risk model for valuing these instruments assuming that the return on a bank's assets follows a jump diffusion process. He concluded that contingent capital would be a viable low-cost method of mitigating financial risk if the trigger was sufficiently high. A structural model for the price of a CoCo bond was also presented in [1] where they studied the decisions of the capital structure mix. [7] analyzed the valuation of a CoCo bond with a capital ratio trigger and a partial conversion. They draw closed form expressions to value these instruments when corporate assets follow a geometric Brownian motion.

[10] investigated the impact of CoCo's introduction to the balance sheet and found that it can, if properly designed, reduce the probability of default. They model the dynamics and value of a CoCo bond by developing the [17] model to work with this debt instrument. [20] developed two methods of converting contingent convertibles to existing derivatives. They were based on equity derivatives and a standard debt. [24] examine investment and financing decisions for a company that has growth options, considering direct borrowings and shares in place as a means of financing. [21] suggest in a dynamic model of contingent claims to model [18]' debt overhang problem for a firm with a collection of growth options and assets in place. They argue that managers must adapt conservative leverage to reduce the impact of this problem when exercising future growth options. In the same context, [8] suggest that when debt priority is endogenous with the firm's capital structure, then there is an internal optimal priority structure that virtually eliminates sub-optimal investment incentives for investors.

In this article, we analyze how the effect of contingent capital reduces the bankruptcy costs of a bank that can be generated by a financial crisis using the real options approach. In fact, we first examine the formulas for closed forms of bank securities using barrier options. We also try to measure the probability of default of a bank that issued CoCo bond. We then study a specific case of a bank with existing assets and a growth option in a dynamic model where the investment cost is financed by stocks and contingent convertible bonds. We examine the interaction between investment and financing policies for a bank issuing CoCos in its capital structure. We try to clarify how the CoCo affects the value of the bank, the capital structure and the inefficiencies arising from the problem of debt overhang and under-investment. Numerical analysis demonstrates that contingent capital can be an effective tool to stabilize financial institutions. We find that with a sufficiently high CoCo conversion rate, the inefficiencies of the debt overhang problem and the incentives for shareholders to transfer risk can be completely eliminated. Our conclusion reveals that CoCo can stand for a reliable solution to ensure the bank and quickly recover the losses upon default, as mentioned in the literature.

The paper is organized as follows. Section 2 introduces the modeling. Section

3 presents the valuation of bank securities under the approach of barrier options and the probability of default. Section 4 presents the valuation of bank securities in the presence of a growth option. Section 5 provides numerical results. Section 6 concludes.

2 The analytic framework

2.1 The bank capital structure

We consider a capital structure of a bank that finance its growth option by issuing a contingent convertible bond and consists of three claims; deposits, CoCo Bonds and equity. We assume that the value of the bank assets V_t is governed by the following equation:

$$dV_t = \mu_v V_t dt + \sigma_v V_t dW_t \tag{1}$$

with μ_v denotes the drift of the process in the risk-neutral measure σ_v is the volatility and W_t is a standard Brownian motion.
Deposits are the largest debts in the capital structure of a bank. N^D is defined as the nominal value of the deposits. In the case of bank insolvency, the government regulator will choose one of these two events; either to force the liquidation, or to seize the bank. These two events imply that a default will occur either at maturity of the debt T if the value of the assets V_t is lower than the nominal value of the deposits N^D or at any time before the due date T such that $0 < t < T$ if the value of the assets V_t touches a default threshold V^D. [3] have modified the [17] model in a way that allows the bankruptcy of a bank before the maturity T of the debt, their model is of type stoppage contrary to the model of [17] which defines a single time fixed at the debt maturity. An event τ_D is defined as soon as a default occurs for all $\tau_D < T$, $\tau_D = inf t > 0, V_t \leq V^D$ and $V^D = N^D(1-\omega)$,$0 \leq \omega \leq 1$. This means that the default threshold is below the nominal value of deposits according to [3] and that the regulator has limited capacity to seize the bank in the event of insolvency. Note that the size of ω is related to the ability and willingness of the regulator to monitor and strengthen the solvency of banks. The model can be linked to the one where the default is only possible at maturity as in [17] when modifying the parameter of the regulatory input policy ω. If the default threshold is much lower than the nominal value of the deposits (ω is large enough), the default value of the model will only occur in general at maturity. Assuming that $\omega = 0$ represents a perfect ability of the regulator to seize the bank immediately when insolvency is reached and results in inability of shareholders to transfer depositors' wealth by changing the volatility of the asset.
Contingent convertible bonds are hybrid securities in the form of debt with the expected gains of a bond and the potential for loss of a stock. Assume N^C, the nominal value of the CoCo bond. If no conversion event occurs until maturity, the

holder of the CoCo bond receives N^C and the shareholders receive $V_t - N^C - N^D$ and, if the conversion takes place before T and the value of the bank assets touches a predefined conversion threshold denoted V^C then, $V^C = (1+\pi)(N^D + N^c)$with π, a debt ratio that measures the distance between the conversion threshold and the book value of the debt. Then the CoCo bond is converted into shares at the conversion rate β and, at maturity, the holder of the CoCo bond receives $\beta(V_t - N^D)$ and the original shareholders receive $(1-\beta)(V_t - N^D)$, except for the case when a default event occurs, the return is null for each holder of the CoCo bond and the original shareholders. A conversion initiation event τ_C is defined by $\tau_C = inft > 0, V_t \leq V^C$. When $\pi \geq 0$ implies that the conversion event will always occur before the default event, i. e $\tau_C \leq \tau_D$.

2.2 The bank cash flow

We consider a bank with assets in place and a growth option. For any time, t assets in place generate cash flows of x, and following a geometric Brownian motion:

$$dx_t = \mu_x x_t dt + \sigma_x x_t dW_{x,t} \tag{2}$$

Where μ_x is the risk-adjusted is expected growth rate, σ_x is the volatility and $W_{x,t}$ is a standard Brownian motion. Following [8] and [21], we assume that the bank is entitled to exercise a growth option at any time to increase the size of its operations by paying an investment expense that is represented as a fixed unrecoverable cost noted, I. As soon as the bank exercises its growth option, its instantaneous cash flow gain increases from x to $(1 + \theta)x$, where $\theta > 0$ is constant that represents the growth ratio. We also assume that the bank manages its investment cost I, by issuing equity and contingent capital, CoCo instead of deposits. This issuance of the original debt has an indefinite maturity and a continuous and constant coupon payment noted,c_s, per unit of time until the bank is bankrupt. Similar to the deposits, the additional debt, the contingent convertible bond CoCo, has an infinite maturity and a coupon rate noted, c_c, perpetual in time until the conversion takes place. Assuming that the bank reimburses the coupon payments to the creditors, the latter is defined by the deductible taxes at a corporate tax rate, $\tau > 0$. It is also assumed that the default threshold and the conversion threshold are independent of time. In other words, as soon as the bank falls below the default threshold, the shareholders declare the bankruptcy, the value of the future cash flows of the bank is allocated to the owners of deposits, but a constant fraction, designated by α, with $0 < \alpha < 1$ will be lost because of the bankruptcy costs, it is called the bankruptcy loss rate. According to the financial supervisory authorities, we assume that the conversion threshold is determined exogenously. A conversion threshold can be defined numerically basing on a specific capital ratio. As soon as the triggering event occurs, a CoCo bond conversion triggered, the shareholders must distribute to CoCo bond holders a fraction β, where $0 < \beta < 1$, of the residual cash flows in the form of dividend, instead of paying coupon rates c_c.

3 Barrier option-based pricing of the bank securities and default probability

To assess the bank liabilities with a CoCo bond, a combination of differential barrier options (closed-form solution) is used. The particularity of this option is that the exercise can be activated or deactivated when the underlying reaches (or not) a given level (the barrier).

3.1 The value of the deposits

First, we consider the case of a capital structure with a CoCo bond to evaluate the deposit price. According to [17], the return for a deposit holder at maturity is equal to the yield of a zero-coupon bond and a European put option and the default occurs only at maturity, contrary to our case, where the default may occur before maturity and the options depend on the path (these options are part of the barrier options). The value of the deposits is dependent of the future cash flows in three cases. Case of no default: If the default does not occur during the lifetime of the option then the yield to maturity equal to N^D and as a result this return can be replicated by N^D units of a long position in an option of barrier "down and out" noted $DB^d out(V^D)$. The yield on maturity of this option is: $DB^d out(V^D) = N^D 1_{(\tau_D > T)}$. Case of default before maturity: If the default occurs before maturity then the yield at maturity can be replicated by $N^D(1-\omega)$ units in a "down and in" barrier option $DB^d in(V^D)$. The yield on maturity of this option is: $DB^d in(V^D) = N^D 1_{(\tau_D < T)}$. Case of default at maturity: If the default occurs at maturity, the deposit holders receive the remaining assets of the financial institution and therefore the loss can be replicated using a put option "down and out" noted $PB^d out(V^D, N^D)$. The yield on maturity of this option is, $PB^d out(V^D, N^D) = max(N^D - V_T, 0)1_{(\tau_D > T)}$. Hence the value of the deposits can be expressed as follows:

$$
\begin{aligned}
D^B &= N^D[DB^d out(V^D) + (1-\omega)DB^d in(V^D)] - PB^d out(V^D, N^D) \\
&= E^Q[exp^{-rT}(N^D 1_{(V_T > N^D)} + V_T 1_{(N^D > V_T)})1_{(\tau_D > T)} \\
&\quad + exp^{-r\tau_D} N^D (1-\omega) 1_{(\tau_D < T)}] \qquad (3)
\end{aligned}
$$

Where E^Q: the expectation in the neutral risk measure Q.

3.2 The CoCo bond and equity valuation

We consider the same cases as before to price the CoCo bond. The three mutually exclusive events be taken into account because of their different impact on the pricing process. Case of no conversion before maturity: In this case, the yield at maturity is replicated by N^C units in a "down and out" barrier option rated $DB^d out(V^C)$. The yield on maturity of this option is the same as the previous case where we replace N^D by N^C and V^D by V^C. Case of conversion and no default: The yield at maturity is replicated by β units of a barrier call option "down and in" denoted $CB^d in(V^C, N^D)$. The value of this option at maturity is equal to $CB^d in(V^C, N^D) = max(V_T - N^D, 0)1_{(\tau_D < T)}$. Case of conversion and default: In the event of conversion and default, the yield at maturity is replicated by β units of a "down and in" call barrier option denoted $CB^d in(V^D, N^D)$. In this case we have the same yield as in the case of conversion and no default except a different barrier. Taking into account all three cases, the value of CoCo bond will be given by:

$$\begin{aligned} C^B &= N^C DB^d out(V^C) + [CB^d in(V^C, N^D) - CB^d in(V^D, N^D)] \\ &= E^Q[exp^{-rT}(N^C 1_{(\tau_C > T)} + (V_T - N^D))1_{(\tau_C < T < \tau_D, V_T > N^D)}] \end{aligned} \quad (4)$$

The value of equity is made by the same mutually exclusive events in the case of CoCo bond and is therefore expressed as follows:

$$\begin{aligned} S^B &= CB^d out(V^C, N^D + N^C) + (1-\beta)[CB^d in(V^C, N^D) - CB^d in(V^D, N^D)] \\ &= E^Q[exp^{-rT}((V_T - N^D - N^C)1_{(\tau_C > T)} \\ &\quad +(1-\beta)(V_T - N^D))1_{(\tau_C < T < \tau_D, V_T > N^D)}] \end{aligned} \quad (5)$$

With $CB^d out(V^C, N^D + N^C) = max(V_T - N^D - N^C, 0)1(\tau_D > T)$. $CB^d out(V^C, N^D + N^C)$is a barrier call option down and out with an exercise price $(N^D + N^C)$and a barrier V^C.

3.3 The probability of default based on the first passage of time

Policymakers are interested in monitoring the probability of default of a financial institution because of the deleterious effect of default on the real economy. Indeed, an important motivation for the introduction of contingent capital is its ability to absorb losses and the resulting reduction in the bank's probability of default. This reduction is quantified by comparing the case of structure with and without contingent capital. The nominal values of the CoCo bond and the deposits are respectively N^C and N^D.

[3] assumed that the default occurs the first time the value of the bank's assets falls below a certain V^D barrier. The default event occurs for the first time at

$0 < t < T$ at which the value of the bank V_t falls below the level where the default event occurs for the first time at maturity. This is explained by the right of bond-holders to exercise a "security clause" which enables them to liquidate the bank if, at any moment, its value falls below the specified threshold. Thus, the default time is given by $\tau_D = inf t > 0, V_t \leq V^D = inf t > 0, ln(V^D/V_t) \geq 0$. Thus, the bankruptcy is defined in either following cases; the value of the assets touches the V^D barrier from above at any time before T, or conditional to always have above the V^Dbarrier, at maturity, the asset value is greater than V^D but lower than N^D. In fact, the default occurs before the maturity date if the asset value reaches the default threshold V^D or at maturity if the value of the asset is less than the face value of the N^D deposits.
The probability of default of a bank without CoCo bond is composed of two mutually exclusive events; either before maturity, the value of the asset is below the V^Ddefault threshold, or at maturity, the value of the asset is less than the total nominal value of the debt $N^(D)$. In this case, the depositors force the default if their claims are not fully respected. Thus, the probability of default can be expressed as follows:

$$PD(D) = Pr(\tau_D < T) + Pr(V_T < N^(D)\ \tau_D > T) Pr(\tau_D > T) \tag{6}$$

Following [3], let $m_t = min_(0 < t < T) V(t)$ the first time the asset value process crosses the bankruptcy barrier and is V_t a Brownian motion with $\mu_v t$ drift and variance $\sigma_v^2 t$.
Suppose $f(y)$ is the probability density of V_t and $g(y, x)$ is the joint probability density with $x = ln(V^D/V_t)$. The probability of default before maturity, that is, the portion of the value of the bond associated with the possibility of forced bankruptcy before the maturity date of the T bond given by:

$$\begin{aligned}
\Pr(\tau_D < T) &= \Pr(\min_{0<t<T} V(t) \leq x) \\
&= \Pr(V(t) \leq x) + \Pr(\min_{0<t<T} V(t) \leq x, V(t) > x) \\
&= \int_{-\infty}^{x} f(y)dy + \int_{x}^{+\infty} g(y,x)dy \\
&= \Phi(\frac{\ln \frac{V^D}{V_t} - (\mu - \sigma^2/2)}{\sigma\sqrt{T}}) + (\frac{V^D}{V_t})^{(2\mu/\sigma^2 - 1)} \Phi(\frac{\ln \frac{V^D}{V_t} + (\mu - \sigma^2/2)}{\sigma\sqrt{T}})
\end{aligned} \tag{7}$$

Regarding the second event above,

$$\begin{aligned}
\Pr(V_T < N^D, \tau_D > T) &= \Pr(V_T < N^D, \min_{0<t<T} V(t) > x) \\
&= \int_x^{\ln(\frac{N^D}{V_t})} f(y)dy - \int_x^{\ln(\frac{N^D}{V_t})} g(y,x)dy \\
&= \int_x^{+\infty} f(y)dy - \int_{\ln(\frac{N^D}{V_t})}^{+\infty} f(y)dy - \int_x^{+\infty} g(y,x)dy \\
&- \int_{\ln(\frac{N^D}{V_t})}^{+\infty} g(y,x)dy \\
&= \Phi(\frac{\ln \frac{V_t}{V^D} + (\mu-\sigma^2/2)}{\sigma\sqrt{T}}) - \Phi(\frac{\ln \frac{V_t}{N^D} + (\mu-\sigma^2/2)}{\sigma\sqrt{T}}) \\
&- (\frac{V^D}{V_t})^{(2\mu/\sigma^2-1)}(\Phi(\frac{\ln \frac{V^D}{V_t} + (\mu-\sigma^2/2)}{\sigma\sqrt{T}}) + \Phi(\frac{\ln \frac{(V^D)^2}{V_t N^D} + (\mu-\sigma^2/2)}{\sigma\sqrt{T}}))
\end{aligned} \tag{8}$$

In summary, following the two probabilities of bankruptcy at maturity and anticipated bankruptcy, we obtain the following formula of the probability of default of a bank without a CoCo bond:

$$\begin{aligned}
PD(D) &= 1 - \Phi(\frac{\ln \frac{V_t}{(N^D+N^B)} + (\mu-\sigma^2/2)}{\sigma\sqrt{T}}) \\
&+ (\frac{V^D}{V_t})^{(2\mu/\sigma^2-1)}\Phi(\frac{\ln \frac{(V^D)^2}{V_t(N^D+N^B)} + (\mu-\sigma^2/2)}{\sigma\sqrt{T}})
\end{aligned} \tag{9}$$

For the case of a bank capital structure with a CoCo bond, either before maturity, the value of the asset is below the V^C conversion threshold, or at maturity, the value of the asset is less than the total nominal value of the debt $N^D + N^C$, the probability of default with a CoCo bond is:

$$\begin{aligned}
PD(C) &= \Phi(\frac{\ln \frac{V^D}{V_t} - (\mu-\sigma^2/2)}{\sigma\sqrt{T}}) + (\frac{V^D}{V_t})^{(2\mu/\sigma^2-1)}\Phi(\frac{\ln \frac{V^D}{V_t} + (\mu-\sigma^2/2)}{\sigma\sqrt{T}}) \\
&+ \Phi(\frac{\ln \frac{V_t}{V^D} + (\mu-\sigma^2/2)}{\sigma\sqrt{T}}) - \Phi(\frac{\ln \frac{V_t}{N^D} + (\mu-\sigma^2/2)}{\sigma\sqrt{T}}) \\
&- (\frac{V^D}{V_t})^{(2\mu/\sigma^2-1)}(\Phi(\frac{\ln \frac{V^D}{V_t} + (\mu-\sigma^2/2)}{\sigma\sqrt{T}}) + \Phi(\frac{\ln \frac{(V^D)^2}{V_t N^D} + (\mu-\sigma^2/2)}{\sigma\sqrt{T}})) \\
&= \Phi(\frac{\ln \frac{V^D}{V_t} - (\mu-\sigma^2/2)}{\sigma\sqrt{T}}) + (1 - \Phi(\frac{\ln \frac{V^D}{V_t} - (\mu-\sigma^2/2)}{\sigma\sqrt{T}})) \\
&- \Phi(\frac{\ln \frac{V_t}{N^D} + (\mu-\sigma^2/2)}{\sigma\sqrt{T}}) + (\frac{V^D}{V_t})^{(2\mu/\sigma^2-1)}\Phi(\frac{\ln \frac{(V^D)^2}{V_t N^D} + (\mu-\sigma^2/2)}{\sigma\sqrt{T}}) \\
&= 1 - \Phi(\frac{\ln \frac{V_t}{N^D} + (\mu-\sigma^2/2)}{\sigma\sqrt{T}}) + (\frac{V^D}{V_t})^{(2\mu/\sigma^2-1)}\Phi(\frac{\ln \frac{(V^D)^2}{V_t N^D} + (\mu-\sigma^2/2)}{\sigma\sqrt{T}})
\end{aligned} \tag{10}$$

We notice that his default probability is the same as without CoCo bond as soon as the value of the assets is less than $(N^D + N^C)$instead of N^D. So, it simply consists of replacing $N^(D)by(N^D + N^C)$ in the previous equation of the default probability of a bank with a CoCo bond.

4 Growth option-based bank securities pricing

The exercise of the growth option implies that bank's capital structure will be composed of three types of securities; equity, deposits and additional debt taking the form of CoCo bond. We define x^d as the default threshold after expansion investment, and x_c as the CoCo bond conversion threshold.

4.1 Pricing bank securities when the growth option is exercised

Based on the [4] and [16] models and the assumptions of [9] and [5], we examine, in section 4.1, the pricing of bank securities when the growth option is exercised using the fundamental theorem of asset pricing, the neutral risk fixation method and real option. The net worth of a bank is simply defined as the present value of the expected cash flows after investment. The value of the bank can be directly obtained by applying the fundamental theorem of asset pricing:

$$V(x) = E\left[\int_t^{\infty} \exp(-r(u-t))(1-\tau)x_u(1+\theta)du \mid x_t = x\right] = \frac{(1-\tau)(1+\theta)x}{r-\mu_x} \tag{11}$$

We consider a contingent capital as a derivative underlying the cash flow of the firm, where $F(x)$ represents its price, which is a function of the current level of cash-flow x. Since our model is homogeneous, $F(x)$ independent of time. Therefore, using the neutral risk fixation method, the function $F(x)$ must satisfy the following ordinary differential equation (ODE):

$$\frac{1}{2}\sigma_x^2 x^2 \frac{\partial^2 F(x)}{\partial x^2} + \mu_x \frac{\partial F(x)}{\partial x} - rF(x) + \xi = 0 \tag{12}$$

Where ξ represents a linear function of cash flows x generated by the contingent convertible bond, it can be expressed by the following relationship $\xi = ax+b$ with a and b being constant. Noting that, ξ, is an always a linear function of the cash flow x up to a stopping time $T_D = \inf\{t \geq 0 : x_t \in D\}$ responds. That is, the moment of the first passage of x from the domain D. After T_D, the asset will be unable to generate the cash flow. In other words, it disappears completely. Then, with some standard arrangement given to the function $F(x)$, the equilibrium price $F(.)$ of the asset is verified by the following equation:

$$F(x) = \frac{ax}{r-\mu_x} - \frac{b}{r} + B_1 x^{\gamma-} + B_2 x^{\gamma+} \tag{13}$$

Where $r > 0$ is the risk-free interest rate satisfying $r > {}_x$ as assumed in the literature for the value being limited. Letting B_1 and B_2 are constants to be determined by the boundary conditions designated outstanding for the contingent capital reserve and

$$\gamma^{\pm} = \frac{\left(\mu_x - \frac{1}{2}\sigma_x^2\right) \pm \sqrt{\left(\mu_x - \frac{1}{2}\sigma_x^2\right)^2 + 2\sigma_x^2 r}}{\sigma_x^2} \tag{14}$$

Where γ^- and γ^+ represent the positive and negative roots of the following quadratic equation:

$$\frac{1}{2}\sigma_x^2\gamma^2 + \left(\mu_x - \frac{1}{2}\sigma_x^2\right)\gamma - r = 0 \tag{15}$$

The bank's capital structure contains all securities, equity, deposit, and additional debts (CoCos). The cash flow can be expressed in the following form: $\xi = (1-\tau)((1+\theta)x - c_s - c_c)$. We define $a = (1-\tau)(1+\theta)$ and $b = -(1-\tau)(c_s+c_c)$, for all $x \geq x_c$, the cash flow level is greater than the conversion threshold. The value of the equity after expansion investment but before the conversion is given by the following formula:

$$\begin{aligned} S^G(x) =&(1-\tau)\left[\left(\frac{(1+\theta)x}{r-\mu_x} - \frac{(c_s+c_c)}{r}\right) - \left(\frac{(1+\theta)x_c}{r-\mu_x} - \frac{(c_s+c_c)}{r}\right)\left(\frac{x}{x_c}\right)\gamma^-\right] \\ &+ (1-\beta)S_c^G(x)\left(\frac{x}{x_c}\right)\gamma^- \end{aligned} \tag{16}$$

The first term determines the value of cash flow after the exercise of the growth option minus the payments of deposits and CoCo bond. The second term represents the actual value of the conversion trigger, this term refers to the value of equity if the level of cash flow x reaches the conversion threshold x_c. The third term gives the actual value of the equity $S_c^G(x)$ after the conversion adjusted by the fraction β, that the former shareholders will have lost if the conversion takes place. The amounts $(1-\beta)S_c^G(x)$ and $\beta S_c^G(x)$ go to the former and the new shareholders respectively. Further, we assume that $V_C^G(x)$ is the sum of the equity, deposits and the convertible contingent debt. The total value of the bank when growth and conversion options are exercised $V_C^G(x)$ is given by:

$$V_c^G(x) =V(x) + \frac{\tau c_c}{r}\left[1 - \left(\frac{x}{x_c}\right)^{\gamma^-}\right] + \frac{\tau c_s}{r}\left[1 - \left(\frac{x}{x^d}\right)^{\gamma^-}\right] - D^G - \alpha V\left(x^d\right)\left(\frac{x}{x^d}\right)^{\gamma^-} \tag{17}$$

17 refers to the sum of the net value of the bank and the tax shields related to the value of the convertible debt and the deposits minus the value of the deposit after expansion investment and the bankruptcy cost. When the CoCo bond is converted into equity and the growth option has already been exercised, the bank's capital structure owns only equity and deposits so that the cash flow of equity can be expressed in the following form $(1-\tau)((1+\theta)x - c_s)$, and therefore the value of equity after expansion investment when the conversion takes place is expressed as follows:

$$S_c^G(x) =(1-\tau)\left[\left(\frac{(1+\theta)x}{r-\mu_x} - \frac{c_s}{r}\right) - \left(\frac{(1+\theta)x^d}{r-\mu_x} - \frac{c_s}{r}\right)\left(\frac{x}{x^d}\right)^{\gamma^-}\right] \tag{18}$$

The first term of 18 represents the value of equity in the absence of default minus the value of equity in the event of default by the contingent claim value which pays 1 when the cash flow level x first reaches the x^d threshold from above $\left(\frac{x}{x^d}\right)^{\gamma^-}$

After the conversion, the bank has two types of securities only in its capital structure; equity and deposits. The perpetual loan designates the deposit with coupon c_s. The bankruptcy of the bank occurs when the value $V(x)$ falls below the default threshold after expansion investment x^d. In this case a fraction denoted α, of the future cash flow value of the bank will be lost due to the cost of bankruptcy, where, $0 < \alpha < 1$ is constant and therefore it represents the loss rate of bankruptcy. The present value of a contingent claim that pays a dollar when the level of cash flows x reaches the default threshold x^d for the first passage time $\left(\frac{x}{x^d}\right)^{\gamma^-}$. As a result, the value of the deposit after expansion investment is:

$$D^G = \frac{c_s}{r}\left(1 - \left(\frac{x}{x^d}\right)^{\gamma^-}\right) + (1-\alpha)V(x^d)\left(\frac{x}{x^d}\right)^{\gamma^-} \tag{19}$$

The first term refers to the value of the deposits if the bank does not reach the default minus this value in the event that the bank is bankrupt, that is the event when the value of cash-flow, x, reaches barrier x^d from above, plus the recovery rate $(1-\alpha)$ multiplied by the value of the bank in case of default, $V(x^d)$ and the bank default discount factor, $\left(\frac{x}{x^d}\right)^{\gamma^-}$.

The total value of the bank when the conversion option is not exercised is given by:

$$V^G(x) = V(x) + \frac{\tau c_c}{r}[1 - \left(\frac{x}{x_c}\right)^{\gamma^-}] + \frac{\tau c_s}{r}[1 - \left(\frac{x}{x^d}\right)^{\gamma^-}] - \alpha V(x^d)\left(\frac{x}{x^d}\right)^{\gamma^-}. \tag{20}$$

which is equal to the sum of the net value of the bank and the tax blocks for the deposits and convertible debt (CoCo) minus bankruptcy cost. The value of the CoCo bond, which is equivalent to the value of the coupon payments to the CoCo bond holders before the conversion takes place, plus the contingent claim value that the CoCo bond holders held at the time T_c is reached. According to [7], we can define this time by the following expression, $T_c = \inf\left\{t \geq 0 : \varphi V^G(x_t) \leq \frac{(c_s+c_c)}{r}\right\}$. The value of the CoCo bond can be expressed by the following expression:

$$C^G(x) = \frac{c_c}{r}\left[1 - \left(\frac{x}{x_c}\right)^{\gamma^-}\right] + \beta S_C^G(x_c)\left(\frac{x}{x_c}\right)^{\gamma^-} \tag{21}$$

When the level of cash flow x reaches the conversion threshold x_c, the shareholders of the bank decide to convert the contingent bonds into shares, so the CoCo bond holders receive the payoff following $C^G(x_c) = \beta S_c^G(x_c)$. Noting that, according to [2], we can define the conversion rate as follows, $\beta = min(\frac{\frac{c_c}{r}(1-\tau)}{S_c^G(x_c)}, 1)$.

4.2 Pricing the bank securities when the growth option is not exercised

To determine the value of shareholders before the exercise of the growth option, we need first to designate the investment threshold, noted x_i, which is also time-independent, as the default and conversion thresholds. For any designated investment threshold x_i, prior to the exercise of the investment option and before the bankruptcy, the shareholders receive a cash flow per unit of time equal to $\xi = (1-\tau)(x-c_s)$ that we can write $\xi = (1-\tau)x - (1-\tau)c_s$ and therefore define $a = (1-\tau)$ and $b = -(1-\tau)c_s$. If the cash flow level is below the investment threshold and above the default threshold before expansion investment, that is, it satisfies the following condition, $x_1^d < x < x_i$, $D = (x_1^d, x_i)$, the CoCo bond holders is $F(x) = S_1^G(x)$ which is the value held by the shareholders that must verify the boundary conditions: $S_1^G(x_1^d) = 0$ in default, shareholders get a net salvage value equal to zero and $S_1^G(x_i) = S^G(x_i) - (I - C^G(x_i)) = V_C^G(x_i) - I$ otherwise, shareholders get the net payoff at the time the option is exercised. In fact, if the bank is in default, the shareholders declare the bankruptcy so they receive a net salvage value of zero, otherwise they obtain a net gain equal, $V_C^G(x_i) - I$, at the time of the exercise which is equal to $S_1^G(x_i) - (I - C^G(x_i))$. To calculate deposits and equity values prior to investment, considering that the contingent values are determined simply through Laplace Transformations of the first passage time function density of x. And therefore, we get the following discount factors:

$$\Delta_d\left(x; x_1^d, x_i\right) \equiv E\left[e^{-r\left(T_1^d - t\right)} \mid T_1^d < T_i, x_t = x\right] = \left[\frac{x^{\gamma^+}(x_i)^{\gamma^-} - x^{\gamma^-}(x_i)^{\gamma^+}}{\left(x_1^d\right)^{\gamma^+}(x_i)^{\gamma^-} - \left(x_1^d\right)^{\gamma^-}(x_i)^{\gamma^+}}\right] \tag{22}$$

It is a discount factor that refers to the value of one dollar paid at the time of default before expansion investment.

$$\begin{aligned}\Delta_i\left(x; x_1^d, x_i\right) \equiv E\left[e^{-r(T_i - t)} \mid T_i < T_1^d, x_t = x\right] = &\left[\frac{x^{\gamma^+}\left(x_1^d\right)^{\gamma^-} - x^{\gamma^-}\left(x_1^d\right)^{\gamma^+}}{(x_i)^{\gamma^+}\left(x_1^d\right)^{\gamma^-} - (x_i)^{\gamma^-}\left(x_1^d\right)^{\gamma^+}}\right] \\ =&\Delta_i\left(x; x_i; x_1^d\right)\end{aligned} \tag{23}$$

It is a discount factor that refers to the value of one dollar paid at the time of the investment before the default. Where T_i denotes the investment stopping time, i.e.,
$T_i = \inf\{u \geq t; x_u \geq x_i\}$ and T_1^d denotes the default stopping time, i.e., $T_1^d = \inf\left\{u \geq t; x_u \leq x_1^d\right\}$. Noting that, $E[]$: designates a conditional expectation, so x_i and x_1^d respectively denote the random default and investment time delays. We obtain the equity value when the growth option is not exercised:

$$\begin{aligned}S_1^G(x) =&(1-\tau)\left(\frac{x}{r-\mu_x} - \frac{c_s}{r}\right) - (1-\tau)\left(\frac{x_1^d}{r-\mu_x} - \frac{c_s}{r}\right)\Delta_d\left(x; x_1^d, x_i\right) \\ &+ \left[V_C^G(x_i) - I - (1-\tau)\left(\frac{x_i}{r-\mu_x} - \frac{c_s}{r}\right)\right]\Delta_i\left(x; x_1^d, x_i\right)\end{aligned} \tag{24}$$

We use the ideas of [8] and [21] to interpret this equation. The first term refers to the net worth of equity, which is obtained by the value of the original asset in place

$\frac{(1-\tau)}{(r-\mu_x)}x$, minus the value of post-tax coupon payments of the deposit, $(1-\tau)\frac{c_s}{r}$, without the exercise of the growth option and without considering the probability of default. Then, the second term refers to the net worth of equity with a default rate x_1^d multiplied by the discount factor $\Delta_d(x; x_1^d, x_i)$. Finally, the third term is represented as a of the discount factor $\Delta_i(x; x_1^d, x_i)$ and the net profit of the exercise of the investment option, which is equal to the difference between the payoff when the growth option is exercised, $V_C^G(x_i) - I$, and the forgone net value to equity at the investment threshold x_i. If $x = x_1^d$, $\Delta_d(x_1^d) = 1$, $\Delta_i(x_1^d) = 0$ and $S_1^G(x_1^d) = 0$ and if $x = x_i$, $\Delta_d(x_i) = 0$, $\Delta_i(x_i) = 1$ and $S_1^G(x) = V_C^G(x_i) - I$.

In what follows, we determine the value of the deposit but before the exercise of the growth option and at the moment when the bank falls into default that we can designate by $D_1^G(x)$. In other words, when the cash flow level is below the investment threshold and above the pre-investment default level, that is, the current cash flow level x satisfies the following condition, $x_1^d < x < x_i$. To find the value of the deposit when the growth option is not exercised, $D_1^G(x)$, the cash flow value $\xi = c_s$, that is, $a = c_s$, $b = 0$ and $D = (x_1^d, x_i)$.

$$D_1^G(x) = \frac{c_s}{r}\left[1 - \Delta_d\left(x; x_1^d, x_i\right) - \left(\frac{x_i}{x^d}\right)^{\gamma^-}\Delta_i\left(x; x_1^d, x_i\right)\right] + (1-\alpha)\frac{1-\tau}{r-\mu_x}x_1^d\Delta_d\left(x; x_1^d, x_i\right)$$
$$+(1-\alpha)V\left(x^d\right)\left(\frac{x_i}{x^d}\right)^{\gamma^-}\Delta_i\left(x; x_1^d, x_i\right) \tag{25}$$

The first term is specified to measure the present value of the coupon payment of the deposit c_s. The second term is the liquidation value before expansion investment, which is represented as a discounted factor product $\Delta_d(x; x_1^d, x_i)$ and a residual value of the bank at the default threshold x_1^d. And finally, the third term also means the liquidation value but after the investment, which is represented as a product between the bank salvage value in case of default after expansion investment and the factor updated $(x_i/x^d)^(\gamma^-)\Delta_i(x; x_1^d, x_i)$. Now we determine the value of the bank when the growth option is not exercised $V_1(x)$ by the sum of the equity value $S_1^G(x)$ and the value of the deposit $D_1^G(x)$, it is given by the following formula:

$$\begin{aligned} V_1(x) \quad = \quad & \frac{1-\tau}{r-\mu_x}x + (\frac{1-\tau}{r-\mu_x}\theta x_i - I)\Delta_i(x; x_1^d, x_i) + \tau\frac{c_c}{r}[1 - (\frac{x_i}{x_c})^{\gamma^-}]\Delta_i(x; x_1^d, x_i) \\ & +\tau\frac{c_s}{r}[1 - \Delta_d(x; x_1^d, x_i) - (\frac{x_i}{x^d})^{\gamma^-}\Delta_i(x; x_1^d, x_i)] - \alpha[\frac{1-\tau}{r-\mu_x}x_1^d\Delta_d(x; x_1^d, x_i) \\ & +V_2^u(x^d)(\frac{x_i}{x^d})^{\gamma^-}\Delta_i(x; x_1^d, x_i)] \end{aligned} \tag{26}$$

The meaning of the equation above is as follows, the first term refers to the value of the non-indebted bank, the second term represents the value of the growth option, while the two last terms determine the values of the tax shield of the convertible debt and deposits, respectively, and the last term refers to the rate of loss in case of bankruptcy.

4.3 Optimal investment and financing policies

In the previous section, we discussed the pricing of a bank's securities in the presence of an investment threshold, two coupon rates of deposit and convertible debt. However, for this section we take them as decision variables in order to address the optimal investment and financing policies based on the previous findings. In the presence of rational and selfish shareholder behavior, we assume that all shareholders seek only their own interests because of their personal advantages by maximizing value $V_2^C(x_i)$ of their claims. Since bankruptcy must be determined in an endogenous way in order to maximize the value of equity, $S_1^G(x)$ must satisfy the smooth-pasting condition at the default threshold before investment, x_1^d, i.e., $\frac{\partial S_1^G(x)}{\partial x}|_{x=x_1^d} = 0$. After using this last condition, we obtain the optimal default threshold before the exercise of the growth option, it is written in the form solution of the following algebra equation:

$$\begin{aligned}&\frac{1-\tau}{r-\mu_x}x_1^d-(1-\tau)\left(\frac{x_1^d}{r-\mu_x}-\frac{c_s}{r}\right)\bar{\Delta}_d\left(x_1^d,x_i\right)\\&-\frac{1-\tau}{r-\mu_x}\theta x_i+\left[V_c^G\left(x_i\right)-I-(1-\tau)\left(\frac{x_i}{r-\mu_x}-\frac{c_s}{r}\right)\right]\bar{\Delta}_d\left(x_i,x_1^d\right)=0\end{aligned} \tag{27}$$

with $\bar{\Delta}_d\ (\ x_1^d,\ x_i) = \frac{\gamma^+(x_1^d)^{\gamma^+}(x_i)^{\gamma^-}-\gamma^-(x_1^d)^{\gamma^-}(x_i)^{\gamma^+}}{(x_1^d)^{\gamma^+}(x_i)^{\gamma^-}-(x_1^d)^{\gamma^-}(x_i)^{\gamma^+}}$

and $\bar{\Delta}_i\ (\ x_1^d,\ x_i) = \frac{(\gamma^--\gamma^+)(x_1^d)^{\gamma^-+\gamma^+}}{(x_1^d)^{\gamma^+}(x_i)^{\gamma^-}-(x_1^d)^{\gamma^-}(x_i)^{\gamma^+}}$

We determine the optimal investment threshold by applying the following smooth-pasting condition $\frac{\partial S_1^G(x)}{\partial x}|_{x=x_i} = \frac{\partial V_C^G(x_i)}{\partial x}|_{x=x_i}$ and therefore the optimal investment threshold is a solution of the following equation:

$$\begin{aligned}&\gamma^-(\frac{x_i}{x_c})^{\gamma^-}\frac{\tau c_c}{r}+\frac{(1-\tau)c_s}{r}\gamma^-[\frac{1+\theta}{r-\mu}x_2^d-\frac{c_s}{r}](\frac{x_i}{x_2^d})^{\gamma^-}-(1-\tau)(\frac{x_1^d}{r-\mu}-\frac{c_s}{r})\overline{\Delta}_i(x_i,x_1^d)\\&-\theta\frac{1-\tau}{r-\mu}x_i+[V_C^G(x_i)-I-(1-\tau)(\frac{x_i}{r-\mu}-\frac{c_s}{r})]\overline{\Delta}_d(x_i,x_1^d)=0;\end{aligned} \tag{28}$$

Once an investment threshold is determined, we examine the optimal financial structure. For this purpose, since the conversion threshold is supposed to be exogenous, it suffices to solve the problem of maximizing the value of equity according to: $C_c^* = argmax\{V_C^G(x_i)\}$. From Eq. 17, the optimal coupon rate C_c^* of the convertible bond is obtained by the solution of the following equation:

$$(c_s+c_c)^{\gamma^-+1}+B(\gamma^--1)(c_s+c_c)(x_i)^{\gamma^-}-B\gamma^-c_s(x_i)^{\gamma^-}=0 \tag{29}$$

where B $= \frac{r(1-\tau)(1+\theta)\phi}{r-\mu}$

Following [21] the deposits is already in place before investment when the growth option is exercised at the time T_i. Thus, the shareholders choose the coupon rate of the convertible bond c_c in order to maximize the value $V_C^G(x_i)$ and they choose the investment threshold x_i to satisfy the smooth-pasting condition Eq. 28. Since x_i and c_c are jointly obtained, the smooth-pasting condition does not allow us to consider the feedback effects between the endogenous investment threshold x_i and the coupon of the contingent bond c_c.

We determine the optimal solution coupon rate of the deposit C_s^* and in accordance with the coupon rate c_c, it suffices to solve the problem of maximizing the value of the bank, $V_1(x)$, as follows: $C_s^* = argmax\{V_1(x)\}$ with x represents the current cash flow level and the function $V_1(x)$ is given by Eq. 26. Unlike the coupon rate resolution C_c^*, the resolution of equation C_s^* is written in the form of a rather tedious algebra equation. And therefore, we cannot solve in the form of a solution as we did in Eq. 29, from which we have just given its maximization formula and in the numerical part that we want to study after this section, we directly provide a digital solution. To obtain the optimal coupon rate of the deposit C_s^*, the agent from the beginning must know all the other decision variables, the investment threshold x_i, the default threshold before expansion investment x_1^d, and so on the default threshold after investment expansion x^d are in functions of c_s. Considering the value of the bank $V_1(x)$, because it is also based on the coupon rate variable c_s compared to the current cash flow level x. Since the default is endogenously adopted by the shareholders in order to maximize the value of the equity market, the equity value of Eq. 18 must satisfy the smooth-pasting condition at the default threshold: $\frac{\partial S_c^G(x)}{\partial x}|_{x=x^d} = 0$. From this condition, we can deduce the expression of the optimal default threshold when the growth option is exercised $x^d = \frac{\gamma^-}{\gamma^- - 1}\frac{r-\mu_x}{r}\frac{c_s}{1+\theta}$. Similar to [7], the conversion of the CoCo bond will take place if a fraction, noted, φ, of the value of the unencumbered bank is also very small that the total sum of the debt. In other words, the conversion threshold x_c must satisfy the following expression: $\varphi V(x_c) \equiv \frac{c_s+c_c}{r}$. Hence the expression above will be $\varphi V(x_c) = \frac{(1-\tau)(1+\theta)x_c}{r-\mu_x}$ and therefore, the optimal conversion threshold is written in the following form: $x_c = \frac{r-\mu_x}{r}\frac{c_s+c_c}{1-\tau}\frac{1}{(1+\theta)\varphi}$. The expression of the conversion threshold means that the net value of the bank is greater than the existing debt at the moment when the conversion occurs as the financial regulation requires, since this fraction, $\varphi \in (0,1)$, may meet the existing standard accounting standard: the capital ratio of Tier1 which is valid only if an appropriate parameter value is taken. Obviously, the fraction φ is an index to measure a standard norm 2. Thus, more than this parameter, φ, is small, more than the regulatory norm becomes strict. However, since the conversion threshold is independent of the commercial risk, σ_x, this conversion rule will be improved. As a result, we result that the conversion threshold will be increased by the increase in the risk of the bank.

To obtain the probabilities of default and investment, following [8], we suppose that $\lambda = -(\mu_x - (\sigma_x^2)/2)$, and we rearrange the equations $\Delta_d(x; x_1^d, x_i)$ and

$\Delta_i(x; x_1^d, x_i)$ as follows:

$$\Delta_d\left(x; x_1^d, x_i\right) = \left(\frac{x}{x_1^d}\right)^{\frac{\hat{\lambda}}{\sigma_x^2}} \frac{\left(\frac{x_i}{x}\right)^{\sqrt{\left(\hat{\lambda}/\sigma_x^2\right)^2 + 2r/\sigma_x^2}} - \left(\frac{x_i}{x}\right)^{-\sqrt{\left(\hat{\lambda}/\sigma_x^2\right)^2 + 2r/\sigma_x^2}}}{\left(\frac{x_i}{x_1^d}\right)^{\sqrt{\left(\hat{\lambda}/\sigma_x^2\right)^2 + 2r/\sigma_x^2}} - \left(\frac{x_i}{x_1^d}\right)^{-\sqrt{\left(\hat{\lambda}/\sigma_x^2\right)^2 + 2r/\sigma_x^2}}} \tag{30}$$

$$\Delta_i\left(x; x_1^d, x_i\right) = \left(\frac{x}{x_i}\right)^{\frac{\hat{\lambda}}{\sigma_x^2}} \frac{\left(\frac{x_1^d}{x}\right)^{\sqrt{\left(\hat{\lambda}/\sigma_x^2\right)^2 + 2r/\sigma_x^2}} - \left(\frac{x_1^d}{x}\right)^{-\sqrt{\left(\hat{\lambda}/\sigma_x^2\right)^2 + 2r/\sigma_x^2}}}{\left(\frac{x_1^d}{x_i}\right)^{\sqrt{\left(\hat{\lambda}/\sigma_x^2\right)^2 + 2r/\sigma_x^2}} - \left(\frac{x_1^d}{x_i}\right)^{-\sqrt{\left(\hat{\lambda}/\sigma_x^2\right)^2 + 2r/\sigma_x^2}}} \tag{31}$$

Using the two equations above Eq. 30 and Eq. 31 respectively to calculate the default probability before investment and the probability of investment before default. Therefore, the probability of default before investment is expressed in the following form:

$$P_1^d(x) = \lim_{r\to 0} \Delta_d\left(x; x_1^d, x_i\right) = E\left[e^{-r\left(T_1^d - t\right)} \mid T_1^d < T_i\right] = \frac{(x_i)^{2\hat{\lambda}/\sigma_x^2} - x^{2\hat{\lambda}/\sigma_x^2}}{(x_i)^{2\hat{\lambda}/\sigma_x^2} - x_1^{d^2}/\sigma_x^2} \tag{32}$$

And the probability of investment before default is given by the following formula:

$$P_i(x) = \lim_{r\to 0} \Delta_i\left(x; x_1^d, x_i\right) = E\left[e^{-r(T_i - t)} \mid T_i < T_1^d\right] = \frac{(x)^{2\hat{\lambda}/\sigma_x^2} - x_1^{d^{2\hat{\lambda}}/\sigma_x^2}}{(x_i)^{2\hat{\lambda}}/\sigma_x^2 - x_1^{d^{2\hat{\lambda}}/\sigma_x^2}} \tag{33}$$

These probabilities of default and investment are simply the limits of the Laplace Transformation considering that the risk-free interest rate reaches zero. Noting that, by definition, we have only $P_i(x) = 1 - P_1^d(x)$

5 Numerical example

In order to quantify the effect of the introduction of contingent capital into the bank's capital structure, we calculate the probability of default for a capital structure with and without CoCo bond and perform an analysis of the risk change incentives. The introduction of contingent capital has two important stabilizing effects, the reduction of the default probability and the ability to choose terms that result in low levels of risk-taking incentives. The numerical example uses MATLAB software as its basis for operation.

5.1 Barrier option, default probabilities and risk incentives

The values of the basic parameters in this case are represented as follows : $T = 1$ year, $N^D = 100$, , $\beta = 0.5$, $\pi = 1\%$. The nominal value of a bond CoCo of 3% of deposits $N^C = 3$. The conversion occurs before the maturity of the debt as soon as the value of the assets of the bank is equal to $(1 + \pi)$ times the nominal value of the debt for the first time, $V^C = 104.03$. The leverage ratio, $LR = ((N^D + N^C)exp^(-rT))/V_T$: A base case value is chosen for leverage of 0.93. The interest rate r: A constant rate of 2.5%. Bank Assets (V_t): The implied bank asset value implied by a leverage ratio of 0.93 is 108.02. Asset risk (σ_v): The asset risk is equal to 5%. Regulatory entry policy (ω): It is assumed that the regulator seizes the bank for the first time as soon as the value of the assets is 3% lower than the nominal value of the deposits ($V^d = 97$).

Figure 1 - Default probabilities with and without a CoCo Bond

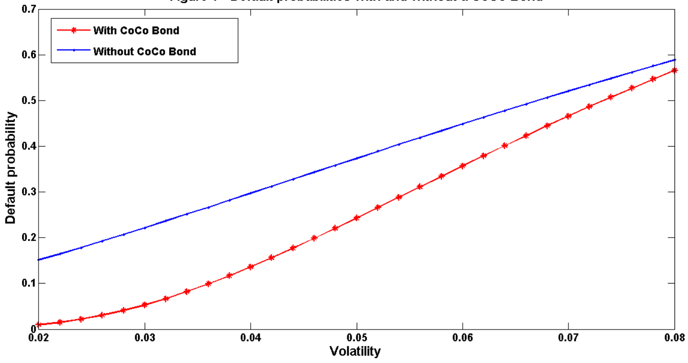


Figure 2 - Default probabilities of a CoCo bond with a range of LR values

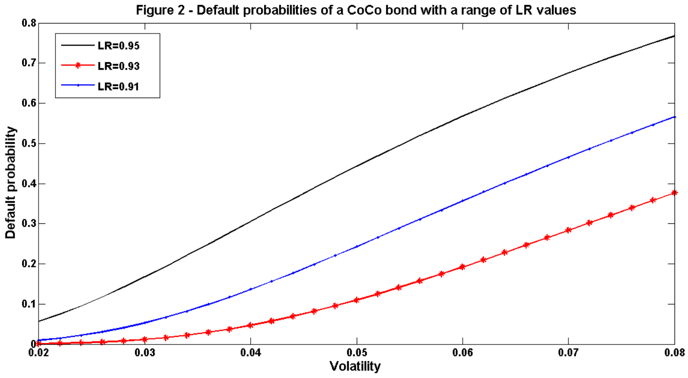


Figure 3 - Asset Value vs. Asset Risk for Different Conversion Rates

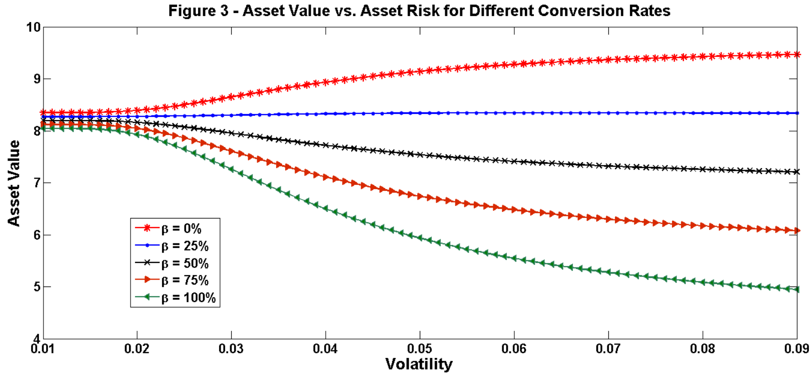


Figure 4 - Asset Value vs. Asset Risk for β=10% and π (0%, 1%, 2%)

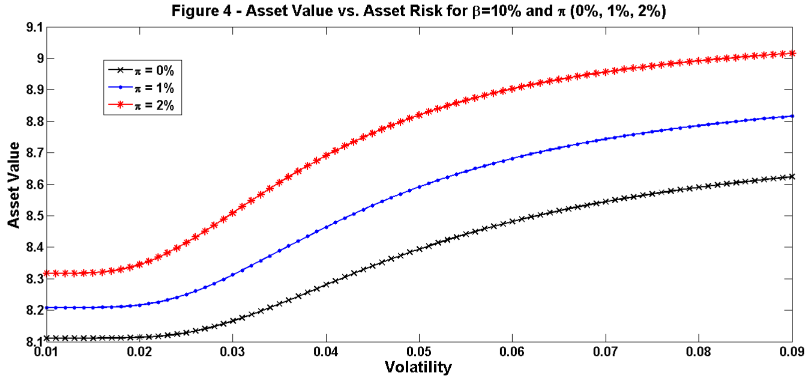


Figure 5 - Asset Value vs. Asset Risk for β=50% and π (0%, 1%, 2%)

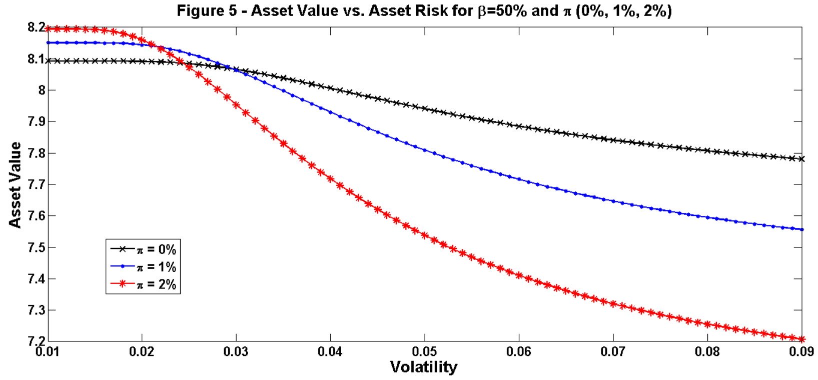
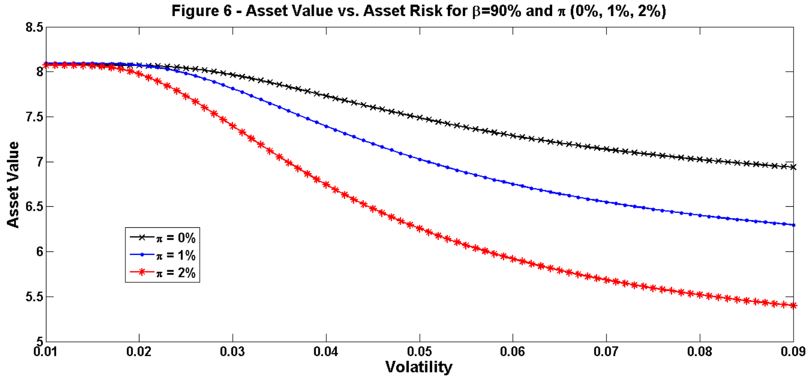


Figure 6 - Asset Value vs. Asset Risk for β=90% and π (0%, 1%, 2%)



From expressions 9 and 10, Figure 1 illustrates the default probability with and without CoCo for various asset volatilities. We show that default probability is lower with CoCo bond reflecting the stabilizing effect of CoCo's on the ultimate risk of a bank. However, for a capital structure with CoCo bond, the default probability with asset risk 5% equal to 36%, while a capital structure without CoCo bond gives respective probabilities of 45%.

We graph the default probability with CoCo for a range of LR in Figure 2 using equation 10. Figure 2 shows that the probability of default for contingent capital depends on the leverage ratio as long as it is intended to preserve the stability of the banking system. The leverage ratio and volatility are an increasing function of the probability of default. Equations 3, 4, and 5 are used to analyze how the conversion threshold affects changes in risk incentives in Figures 3, 4, 5, and 6. Figure 3 shows the effect of the conversion rate on the risk motivation of the shareholder. The asset value decreases as the conversion rate increases. The choice of asset risk by the shareholder is highly dependent on the conversion rate. For a relatively low conversion rate ($\beta = 0\%, 25\%$), asset value increases with asset risk. Intuitively, an increase in asset risk results in a larger increase in the value of the assets due to the lower conversion rate of the loss of value of assets due to the dilution of the shares during the conversion. The inverse relationship is present for a relatively high conversion rate ($\beta = 75\%, 100\%$). Indeed, a high level of the conversion rate leads to a reduction in the value of the assets. The asset value is almost insensitive to asset risk for the intermediate levels of the conversion rate ($\beta = 50\%$). For example, the asset value is close to constant with respect to asset risk when the conversion rate equals 50%. The same figure shows that the conversion leads to a reduction of the debt but also a dilution of the current shareholders. If the dilution is low (low conversion rate), the ancient shareholders are motivated to increase the risk and to reduce the dilution risk in the mean time for high conversion rate. Figures 4, 5 and 6 examine the effect of the conversion threshold on risk incentives changes. The conversion threshold has little impact on the relationship between incentives and the conversion rate, which is the dominant factor that affects risk-taking motivation. The influence of the CoCo bond on investment and financing policies reduces inefficiencies related to problems of debt overhang and underinvestment. Figure 4 shows that when the conversion rate is relatively low around 10%, shareholders will try to choose the maximum level of the asset risk for any level of the conversion threshold. Figure 5 shows the change in asset value at asset risk for a conversion rate of 50%, for which the effect of asset risk on the asset value is low for the three levels of the threshold conversion. Figure 6 considers a relatively high conversion rate around 90%. In this case, the relationship between the value of the assets and the asset risk is down for the three conversion thresholds; thus, shareholders prefer the minimum level of asset risk. For intermediate levels of the conversion rate, risk-taking incentives remain low for the different conversion ratios.

5.2 Growth option, debt-overhang and optimal bank value

The values of the basic parameters in this case are displayed as follows: the current liquidity rate $x_0 = 20$, risk free interest rate $r = 0.06$, the cost of bankruptcy $\alpha = 0.25$, the rate of return $\mu_x = 0.01$, volatility $\sigma_x = 0.25$ and the effective tax rate $\tau = 0.15$. The cash flow growth ratio $\theta = 1$ therefore the investment cost to exercise the growth option is $I = 200$. Thus, we take the conversion rate $\beta = 0.4$ according to [12]. As a last resort, the capital adequacy ratio that is equal to $1 - \varphi = 0.05$ according to [7].

Figure 7 - The sensitivity of debt overhang to cash flow before expansion investment under the different financing policies and CoCo conversion rates

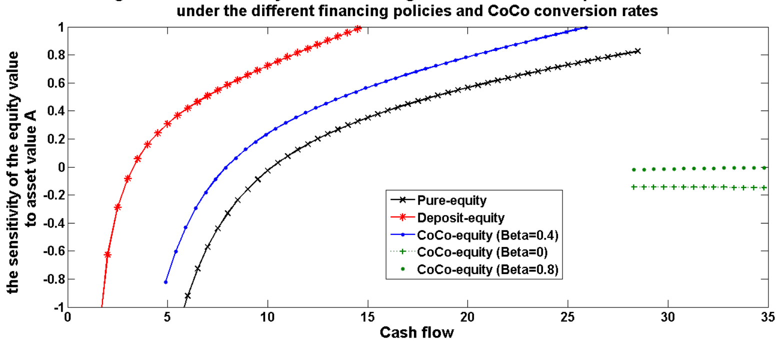


Figure 8 - The sensitivity of debt overhang to cash flow after expansion investments under the different financing policies and conversion rates of CoCo

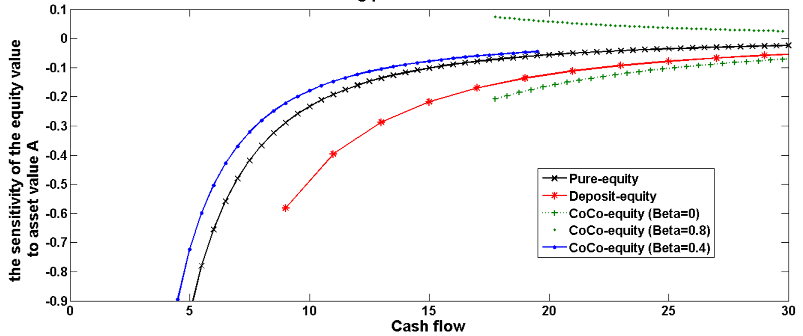


Figure 9 - The value of the optimal bank befor expansion investment for different asset volatilities and financing policies

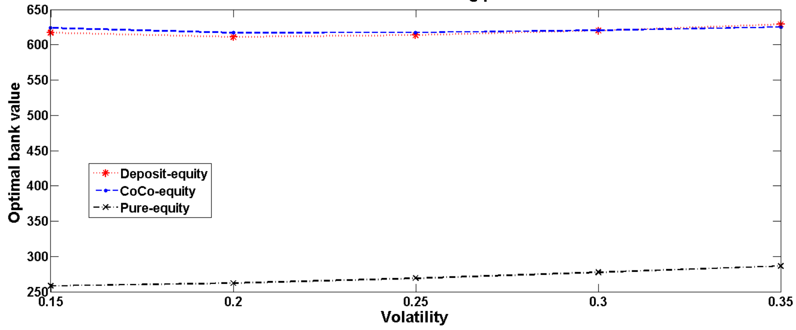
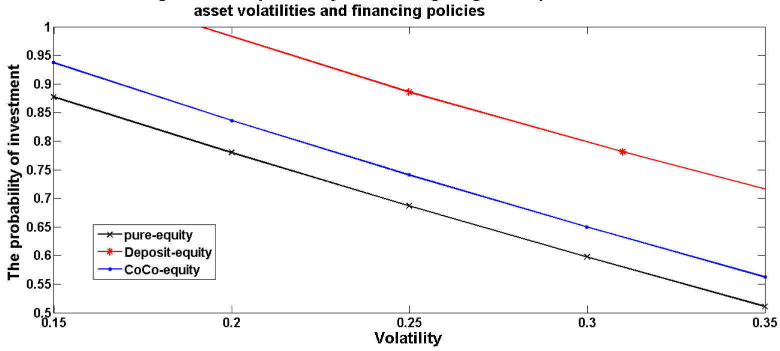


Figure 10 - The probability of exercising the growth option for different asset volatilities and financing policies



We plot in Figures 7 and 8 the sensitivity of the debt overhang to cash flow before and after expansion under various financing policies and conversion ratio values based on equations 16 and 18.

When the growth option is not exercised the incentive to equity financing is independent of the conversion rate. Indeed, from Eq. 28 and Eq. 29, the conversion rate has no impact on the investment threshold and the optimal coupon rate of the CoCo bond. Thus, the default thresholds and the optimal coupon rate of the deposit will be invariable with the conversion rate and the initial leverage ratio. The higher the deposit ratio, the more shareholders are forced to inject capital in order to provide some liquidity and cover their positions against a financial deterioration of the bank. Figure 7 illustrates the effect of the debt overhang problem for different situations. In fact, the debt overhang problem under CoCo-equity financing is more severe than that of deposit-equity financing, but it is less severe under pure-equity financing. The severity of the problem of debt overhang will be greater when the level of cash flows is lows, but it rapidly collapses when the level of cash flows becomes higher. Thus, the closer this level of cash flows to the investment threshold, the more the incentive to re-inject capital is needed.

Figure 8 exhibits different situations where the growth option has been exercised. Noticeably, there is an immediate problem of debt overhang, except if the CoCo bond is issued with a sufficiently high conversion rate. This is because the bank has no other growth option to exercise after the investment. The conversion rate has a significant effect on the problem of debt overhang if the growth option is exercised. The conversion of CoCo has two completely opposite effects on the value of equity; the value of equity is significantly reduced due to the dilutive effect since during the conversion a large part of the equity should be distributed to CoCo holders. The value of equity can be increased by the effect of the debt since shareholders will no longer have the right to continue to pay CoCo coupon payments after the conversion. Generally, the use of deposit as a means of financing always gives rise to the existence of a debt overhang problem for the bank, [18]. The higher the conversion rate, the greater the existence of the dilution effect. In fact, if the conversion rate is chosen sufficiently high, the dilutive effect of the share may dominate the effect of the debt and, consequently, the shareholders benefit from new injections of own funds to avoid conversion, because the conversion is more expensive for existing shareholders.

In Figure 9, we use equation 26 to show how asset volatility affects the value of the bank prior to investment. The value of the bank first decreases and then increases with the volatility of the assets before expansion investment regardless of the financing policies. The value of the bank always decreases with the volatility of assets, [12]. CoCo-equity financing leads to a maximum bank value, which corresponds to that of deposit-equity financing, but much higher than that of pure-equity financing. The reason for this unlike deposit-equity financing, is that CoCo-equity financing has more capacity to absorb the costs of bankruptcy but its tax benefits are decreasing. The alternative case assets if the deposit is protected, the value of the bank under CoCo-equity financing is generally higher than that under deposit-equity financing, [12].

For various asset volatilities and various financing strategies, we represent the effect of asset volatility on the likelihood of exercising the growth option before a bank goes bankrupt using equation 33, as shown in Figure 10. The probability of implementing the investment expansion under CoCo-equity financing is lower than deposit-equity financing, but is greater than the financing pure-equity. The investment threshold under the CoCo-equity financing framework is always between those financed by pure- and deposit-equity. [15] find that shareholders underinvest when exercising the growth option under pure-equity financing because shareholders incur costs in order to exercise this option, but in return they share advantages at the level of investment with the debtors. Whereas, [8] argue that shareholders over-invest in deposit-equity financing because shareholders can benefit in this case from full benefits of premature investment, but in return they share the investment costs with the new custodians, leaving the highest losses that can be generated by the bankruptcy to the original debtors.

6 Conclusion

Contingent capital is a financial instrument that automatically converts into equity when a trigger occurs in the event of financial distress and can therefore absorb losses in the event of going concern. The main reason for introducing CoCo into the legal system was to improve the loss-absorbing capacity before the bankruptcy of an institution. In this paper, we analyze the effect of the inclusion of contingent capital in the capital structure of financial institutions. We provided closed-ended solutions for CoCo bond prices and other bank liabilities by replicating payments by sets of barrier options on the one hand and by growth options before and after expansion investment as part of different financing policies on the other hand. We examined the interaction between a bank's investment and financing strategies in a dynamic model, which has an existing asset and a growth option. The bank optimally chooses the investment threshold and capital structure by negotiating the tax benefits of the deposit and the investment benefits with the costs of bankruptcy.

Next, we demonstrated that there are two channels through which contingent capital can be efficient to stabilize the banking sector. First, banks that issue contingent capital are significantly less likely to default. Second, contingent capital can be designed to reduce incentives for risk-taking. We found that a bank, with a capital structure including contingent capital, has a lower probability of default than a bank with a capital structure without CoCo bond. In addition, contingent capital design has a significant impact on risk-taking motivation. For relatively low conversion rates, shareholders are encouraged to increase asset risk, while a high conversion rate implies a willingness to reduce risk. The intuition for this effect is that increasing the asset risk level makes conversion more likely. It is important to note that the conversion rate has a significant impact on risk incentives, and that for intermediate levels of the conversion rate, incentives to change the risk can be virtually eliminated.

We found that when the investment threshold is close to the level of cash flows, regardless of financing policies, the inefficiencies resulting from debt overhang are reduced and even eliminated altogether, but in general these inefficiencies do not appear after expansion investment. In particular, if the conversion rate is sufficiently high, the inefficiencies disappear completely. However, the conversion rate does not affect inefficiencies prior to the investment expansion. In addition, CoCo-equity financing is able to absorb most of the risk faced by the issuing bank as well as reducing conflicts of interest between shareholders and depositors. CoCo-equity financing reduces the problem of underinvestment under pure-equity financing and the over-investment problem under equity-based financing. We have shown that the introduction of contingent capital into banks' capital structures represents a possibility to significantly reduce incentives to increase bank risk, reduce the rate of bank failure and reduce the need for an expensive provision of capital. Therefore, contingent capital is an efficient tool for stabilizing the financial system for financial institutions.

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