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Research paper

Some applications of log-ergodic processes: ergodic trading model and call option pricing using the irrational rotation

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Abstract:
Due to the increasing popularity of futures trading among financial markets participants, the risk management of futures trading is of particular importance. In this paper, we study a futures trading strategy consisting of a long and a short positions by using the mean reversion property of positive log-ergodic financial processes. We introduce a model for estimating the ideal time for leaving a trading position on a stock. Also, using ergodic theorems, we investigate the European call option pricing problem using an stochastic irrational rotation on the unit circle. By means of the properties of log-ergodic processes, we use the time average of the stochastic process of risky assets instead of expectation in our calculations. Our findings indicate that the proposed model improves the accuracy of predicting optimal trading times and enhances the computational efficiency of option pricing.

Keywords: Call option, Futures trading, Irrational rotation, Log-ergodic process,
Classification: 37A30, 37H05, 60G10.

1 Introduction

Futures market is one of the most popular active markets internationally. The futures market, originating from the Djima Rice Exchange in 1710, has evolved significantly, now encompassing a wide range of financial products beyond its agricultural beginnings [6]. In 2006, the New York Stock Exchange partnered with the Amsterdam-Brussels-Lisbon-Paris Stock Exchange (Euronext Electronic Exchange) to form the first transcontinental stocks and options exchange. This, as well as the growth of Internet futures trading platforms created by developed companies, indicate the increasing trend of competition in the field of online futures and options services in the coming years [9]. In terms of trading volume, the National Stock Exchange of India in Mumbai is the largest stock futures exchange in the world [8].

Despite the fact that futures trading is one of the most popular types of securities trading in the world, the high risk of this type of trading is considered a deterrent

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factor for many risk-averse traders [7, 10]. Generally, it is difficult to predict the behavior of financial markets in terms of time, although a research using machine learning has been conducted in this field [11]. In the paper [12], Harrison Hong presented a model for the futures market and studied the returns and the trading pattern of its participants. Binh Do in the paper [13] has studied some trading models, consisting of long and short positions, on a pair of shares (risky assets). Researchers have studied futures trading in terms of risk and return. However, there are very few resources available on estimating the appropriate time to take a trading position. Rama Cont and his colleagues have described the risk and return of dynamic trading in the form of fluctuations of the market value of the basket from a reference level, and based on the price fluctuation, relative to that reference level, in a path-dependent manner, they have provided a framework for risk analysis of trading [14]. In this paper, we use the method of [14] and the concept of log-ergodic process [16] to model futures trading on a risky asset and estimate the ideal time to take a trading position.

The pricing of financial derivatives is one of the fundamental problems in mathematical finance [2]. In mathematical finance, we use the discount of an expectation for the pricing of financial derivatives [2]. In this paper, we assume that the price process of a risky asset follows a log-ergodic process [16]. Studying the price process of an asset as an ergodic dynamical system on the unit circle (Irrational rotation), we replace the time average of the price process by mathematical expectation in our calculations. Artur Avila and his colleagues have studied the behavior of the random walk process using the irrational rotation and have presented a new proof of the J. Beck central limit theorem [15].

The rest of the paper is organized as follows:

Some necessary concepts are introduced in section 2. In section 3, we present the futures trading model. Using the properties of the log-ergodic processes, we introduce the process of recurrence times and a time interval for leaving a trading position. In section 4, we define the irrational rotation on the unit circle and study it's properties. We study the European call option pricing problem using the irrational rotation in section 5. In section 6, we solve the ergodic partial differential equation of Black-Scholes presented in our recent work [16]. The results of the paper are presented in section 7.

2 preliminaries

Throughout the text, we use the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$, in which Ω is the space of events, $\mathcal{F}$ is a σ-algebra, $\mathbb{P}$ is an invariant probability measure (see [10]), and sub σ-algebra $\mathcal{F}_t$ represents the information of the financial market up to time t, which is generated by Wiener process W_t (i.e. $\mathcal{F}_t = \sigma\{W_s | s \leq t\}$). From [16] we have the following definitions.

Definition 2.1. (Log-ergodicity) We say that the positive stochastic process X_t

is logarithmically ergodic (log-ergodic), if its logarithm is mean ergodic. More precisely, the positive stochastic process X_t is log-ergodic if the process $Y_t = \ln(X_t)$ satisfies

$$\overline{< Y >} := \lim_{T\to\infty} \frac{1}{T}\int_0^T (1-\frac{\tau}{T})\mathbf{Cov}_{yy}(\tau)d\tau = 0, \quad \forall\tau \in [0,T]. \tag{1}$$

Where $\mathbf{Cov}_{yy}(\tau)$ is the covariance of Y_τ.

Definition 2.2. Let W_t be a standard Wiener process and $\beta > \frac{3}{2}$. For all $t, s \in [0,T]$, we define the ergodic maker operator of the process
$Y'_t = Y'_0 + D_t + R_t$ as

$$\xi^{\beta}_{\delta,W_\delta}[Y'_t] := 0\cdot Y'_0 + \frac{W_T}{T^\beta}\cdot D_\delta + \frac{1}{T^\beta}\cdot R_\delta = D^{T,W_T}_\delta + R^T_\delta, \tag{2}$$

where $\delta = t - s$ for $t > s$.

We introduce the definition of the inverse ergodic maker operator, which we use in section 6.

Definition 2.3. (Inverse Ergodic Maker Operator) For any mean ergodic random process $Z_\delta = D^{T,W_T}_\delta + R^T_\delta$ we define the IEMO as follows

$$\xi^{-\beta}_{t,W_t}[Z_\delta] = c + \frac{T^\beta}{W_T}\cdot D^{T,W_t}_t + T^\beta\cdot R^T_t, \quad t > 0, \tag{3}$$

where c is a constant.

Lemma 2.4. *For all $c > 0$ and any mean ergodic process $Y_t = D_t + R_t$, we have*

$$\xi^{-\beta}_{t,W_t}\left[\xi^{\beta}_{\delta,W_\delta}[c + Y_t]\right] = c + Y_t.$$

Proof. We have

$$\xi^{\beta}_{\delta,W_\delta}[c + Y_t] = 0\cdot c + \frac{W_T}{T^\beta}D_\delta + \frac{1}{T^\beta}R_\delta.$$

Now, from the definition 3 we have

$$\begin{aligned}\xi^{-\beta}_{t,W_t}[\frac{W_T}{T^\beta}D_\delta + \frac{1}{T^\beta}R_\delta] &= c + \frac{T^\beta}{W_T}\frac{W_T}{T^\beta}D_t + T^\beta\frac{1}{T^\beta}R_t = c + D_t + R_t\\ &= c + Y_t.\end{aligned}$$

□

Proposition 2.5. *For any finite mean ergodic process Z_δ, the coefficient c defined in 3 is unique.*

Proof. Assume that there exist numbers $c_1, c_2 > 0$. From lemma 2.4 for the finite processes $c_1 + Y_t$ and $c_2 + Y_t$ we have

$$\xi^{\beta}_{\delta,W_\delta}[c_1 + Y_t] = \xi^{\beta}_{\delta,W_\delta}[c_2 + Y_t]. \tag{4}$$

The equality 4 is true because the EMO drops the constants c_1 and c_2. Now, using 3 yields

$$\begin{aligned} \xi^{-\beta}_{t,W_t}\left[\xi^{\beta}_{\delta,W_\delta}[c_1 + Y_t]\right] &= \xi^{-\beta}_{t,W_t}\left[\xi^{\beta}_{\delta,W_\delta}[c_2 + Y_t]\right] \\ c_1 + Y_t &= c_2 + Y_t \\ c_1 &= c_2. \end{aligned}$$

□

Remark 2.6. Note that from uniqueness of the coefficient c in the definition 3 implies that IEMO is well-defined.

Theorem 2.7. *(Kac) Let f be a measure preserving transformation, and $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$. Define $\rho_A(\omega) = \min\{n \in \mathbb{N} | f^n(\omega) \in A\}$. Then,*

$$\mathbb{E}[\rho_A(\omega)] = \frac{1}{\mathbb{P}(A)}, \quad \forall \omega \in A.$$

Proof. For the proof and more details we refer the reader to [1]. □

3 Futures Trading Model: Time Estimation

3.1 The Setup

Suppose that the price process of a risky asset, S_t, be a log-ergodic positive stochastic process (with respect to $\xi^{\beta}_{\delta,W_\delta}[\cdot]$). Using [16], Poincaré recurrence theorem [1], and theorem 2.7 the first recurrence time (to the mean), τ_0 (in the time interval of length δ_0), exists and we define it as follows.

$$X_t = X_0 e^{Y_t}, \quad X_0 = x, \quad Z_\delta = \xi^{\beta}_{\delta,W_\delta}[Y_t], \quad Z_0 = 0, \tag{5}$$

$$\tau_0 := \inf\{t \in [0,s] \big| \delta_0 = s - 0, Z_{\delta_0} = 0\}, \quad s > 0. \tag{6}$$

We define the subsequent return times to the reference level for every $i \in \{1, 2, 3, \cdots\}$ with τ_i, such that $\tau_i < \tau_{i+1}$. We call the difference between every two consecutive recurrence times, the sojourn time and we denote it by δ_i.

Taking the level $Z_\delta = 0$ as the reference level, we divide the path of the Z_δ into two parts. One above the reference level, and one bellow the reference level. Then, we define the set of the sojourn times for the above the reference level as ϕ^+, and the set of sojourn times below the reference level as ϕ^- and we write

$$\phi^+ := \{\delta_i = \tau_{i+1} - \tau_i \big| Z_{\delta_i} > 0\}, \tag{7}$$

$$\phi^- := \{\delta_i = \tau_{i+1} - \tau_i \big| Z_{\delta_i} < 0\}, \quad \forall i \geq 0. \tag{8}$$

Therefore, we write the mean sojourn time of Z_δ as follows.

$$\overline{\phi}^+(z,\delta) := \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\delta_k, \quad \delta_k \in \phi^+,$$
$$\underline{\phi}^-(z,\delta) := \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\delta_k, \quad \delta_k \in \phi^-.$$

Obviously, we can also define the sets of all recurrence times and sojourn times, respectively, as follows.

$$\boldsymbol{\tau} = \{\tau_0, \tau_1, \tau_2, \cdots\},$$
$$\boldsymbol{\delta} = \phi^+ \cup \phi^- = \{\delta_0, \delta_1, \delta_2, \cdots\}, \quad \delta_i = \tau_{i+1} - \tau_i, \quad \forall i \geq 0.$$

For every time interval $[\tau_i, \tau_{i+1}]$, let $M_i = \max|Z_{\delta_i}|$. We denote the time when $|Z_\delta|$ reaches it's maximum along the path (after leaving the reference level), in every time interval $[\tau_i, \tau_{i+1}]$, by t_{M_i} and we call it the Order Execution Time (OET).

As the process Z_δ approaches zero, the reference level, we open a trading position (long, if the process gets below the reference level, and short if the process gets above the reference level). We interpret the time t_{M_i} as the time that we close the trade.

Let l and s be the long and short leverage coefficients, respectively. Then, we form a basket, V_t, consisting of only one long and one short positions. Therefore, we write the profit of this trade as follows.

$$\mathcal{V}_t = l\sum_{i\geq 1}\mathbf{1}_{[\phi^-]}|X_{\tau_i} - X_{t_{N_i}}| + s\sum_{i\geq 1}\mathbf{1}_{[\phi^+]}|X_{\tau_i} - X_{t_{M_i}}|, \tag{9}$$

where,

$$\mathbf{1}_{[\phi^\cdot]} := \begin{cases} 1, & \text{if} \quad \phi^\cdot \neq \emptyset, \\ 0, & \text{if} \quad \phi^\cdot = \emptyset. \end{cases}$$

3.2 The Process of Recurrence Times

Let Y_t be an Itô Markov stochastic process. Using [2], we have

$$Y_t = Y_0 + \int_0^t \sigma_s dW_s + \int_0^t \mu_s ds, \quad Y_0 = y.$$

Utilizing the ergodic maker operator (EMO), we get a mean ergodic process Z_δ, which is made from the process X_t, according to [16]. Therefore,

$$Y'_t = \ln(X_t) = Y'_0 + \int_0^t \sigma_s dW_s + \int_0^t \mu_s ds, \quad Y'_0 = y + \ln(x),$$
$$Z_\delta = \xi^\beta_{\delta,W_\delta}[Y'_t] = Z_0 + \frac{1}{T^\beta}\int_0^\delta \sigma_s dW_s + \frac{W_T}{T^\beta}\int_0^\delta \mu_s ds, \quad Z_0 = 0, \tag{10}$$

where μ_t and σ_t are the drift and volatility coefficients, which are adapted integrable functions of t and X, W_t is a standard Wiener process, and β is the inhibition degree parameter.

Theorem 3.1. *Consider the time interval $[0, T]$ and let X_t be a log-ergodic stochastic process. Then, the process of recurrence times, $\{\tau_i\}_{i\geq 0}$ satisfies the following dynamics.*

$$d\tau_i = -\frac{\left[\sigma_{\tau_i} + \int_0^{\tau_i} \mu_s ds\right]}{\mu_{\tau_i}} \frac{dW_{\tau_i}}{W_{\tau_i}}.$$
$$\tau_0 = \inf\{t \in [0, s] \big| \delta_0 = s - 0, Z_{\delta_0} = 0\}.$$

Proof. Consider the fixed path ω_0. When Z_δ meets it's mean along ω_0, at the time interval of length δ_i, we have

$$Z_{\delta_i} = \frac{1}{T^\beta} \int_0^{\delta_i} \sigma_s dW_s + \frac{W_T}{T^\beta} \int_0^{\delta_i} \mu_s ds = 0$$

According to [16] we know that the process Z_δ returns to it's mean in the time interval $[0, T]$ once at least. Therefore, we can consider δ_i as $\delta_T = T - 0 = 0$. Hence,

$$Z_T = \frac{1}{T^\beta} \int_0^T \sigma_s dW_s + \frac{W_T}{T^\beta} \int_0^T \mu_s ds = 0.$$

Using Itô lemma we have

$$\begin{aligned} dZ_T =& \big[\frac{-\beta}{T^{\beta+1}} \int_0^T \sigma_s dW_s - \frac{\beta W_T}{T^{\beta+1}} \int_0^T \mu_s ds + \frac{\mu_T W_T}{T^\beta}\big] dT \\ &+ \frac{1}{T^\beta} \big[\sigma_T + \int_0^T \mu_s ds\big] dW_T \\ =& \big[\frac{-\beta}{T} Z_T + \frac{\mu_T W_T}{T^\beta}\big] dT + \frac{1}{T^\beta} \big[\sigma_T + \int_0^T \mu_s ds\big] dW_T = 0. \end{aligned}$$

Since we considered $\delta = T - 0 = T$, we can write

$$dZ_\delta = \big[\frac{-\beta}{\delta} Z_\delta + \frac{\mu_\delta W_\delta}{\delta^\beta}\big] d\delta + \frac{1}{\delta^\beta} \big[\sigma_\delta + \int_0^\delta \mu_s ds\big] dW_\delta = 0.$$

When Z_δ meets it's mean, at the time τ_i in the time interval of length δ_i , we have $Z_{\delta_i} = 0$. Take $\delta_i = \tau_i - 0 = \tau_i$. Therefore,

$$dZ_{\tau_i} = \big[\frac{\mu_{\tau_i} W_{\tau_i}}{\tau_i^\beta}\big] d\tau_i + \frac{1}{\tau_i^\beta} \big[\sigma_{\tau_i} + \int_0^{\tau_i} \mu_s ds\big] dW_{\tau_i} = 0.$$

Hence,

$$
\begin{aligned}
[\frac{\mu_{\tau_i} W_{\tau_i}}{\tau_i^{\beta}}]d\tau_i &= -\frac{1}{\tau_i^{\beta}}\big[\sigma_{\tau_i} + \int_0^{\tau_i} \mu_s ds\big] dW_{\tau_i} \\
\mu_{\tau_i} W_{\tau_i} d\tau_i &= -\big[\sigma_{\tau_i} + \int_0^{\tau_i} \mu_s ds\big] dW_{\tau_i} \\
\Rightarrow \quad d\tau_i &= -\frac{\big[\sigma_{\tau_i} + \int_0^{\tau_i} \mu_s ds\big]}{\mu_{\tau_i}} \frac{dW_{\tau_i}}{W_{\tau_i}}.
\end{aligned}
$$

Finally from 6 we write $\tau_0 = \inf\{t \in [0, s] \big| \delta_0 = s - 0, Z_{\delta_0} = 0\}$. □

An Example

Supposing a stock price process, S_t, follows the geometric Brownian motion

$$
\begin{aligned}
S_t &= S_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\}, \quad S_0 = s. \qquad (11) \\
Y'_t &= \ln(S_t) = Y'_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t, \quad Y'_0 = \ln(s).
\end{aligned}
$$

Where μ and σ are constants, and W_t is a standard Wiener process. Constructing the log-ergodic process Z_δ yields

$$
Z_\delta = Z_0 + \frac{(\mu - \frac{1}{2}\sigma^2)\delta W_T}{T^{\beta}} + \frac{\sigma W_\delta}{T^{\beta}}, \quad Z_0 = 0. \qquad (12)
$$

Using theorem 3.1 we have

$$
\begin{aligned}
d\tau_i &= -\frac{\big[\sigma + \int_0^{\tau_i} (\mu - \frac{1}{2}\sigma^2) ds\big]}{\mu - \frac{1}{2}\sigma^2} \frac{dW_{\tau_i}}{W_{\tau_i}}. \\
\tau_0 &= \inf\{t > 0 \big| t \in [t, t + \delta_0], Z_{\delta_0} = 0\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
d\tau_i &= -\frac{\big[\sigma + (\mu - \frac{1}{2}\sigma^2)\tau_i\big]}{\mu - \frac{1}{2}\sigma^2} \frac{dW_{\tau_i}}{W_{\tau_i}}. \\
&= -\Big[\frac{\sigma}{\frac{1}{2}\sigma^2 - \mu} + \tau_i\Big] \frac{dW_{\tau_i}}{W_{\tau_i}}.
\end{aligned}
$$

In the figures below, we have the plot of stock price process 11, figure 1, and the corresponding Z_δ process 12. Figure 2 illustrates the recurrence of Z_δ to its mean level over time. We can observe how interestingly process Z_δ tells us where to take a long or a short position on the stock. As an illustration, two sample short and long opportunities are shown by arrows.

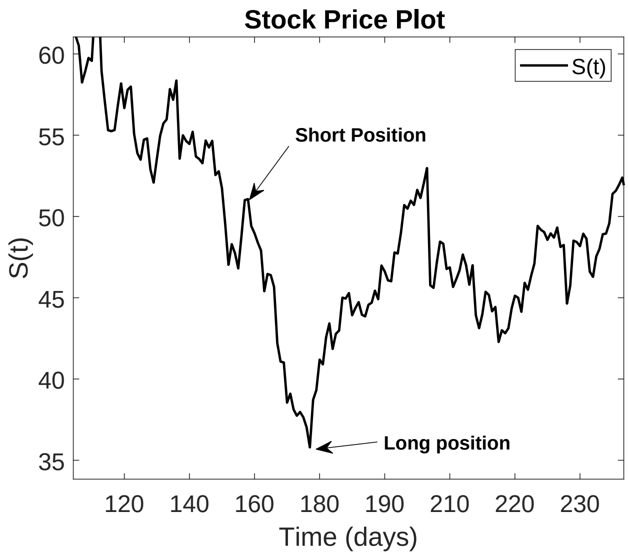

Figure 1: The plot of the stock price process S_t.

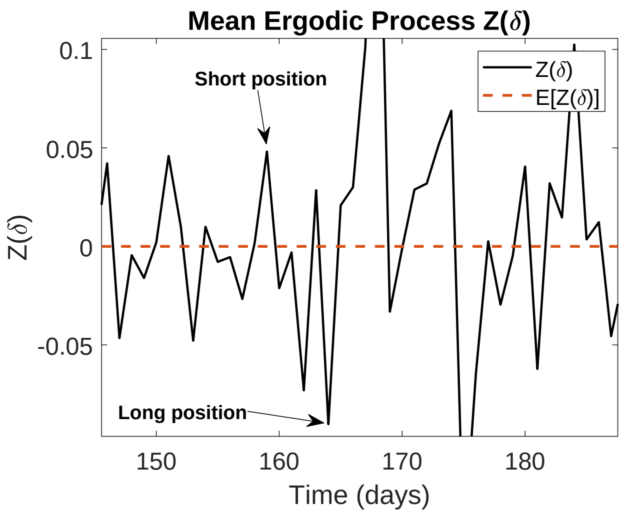

Figure 2: Corresponding mean ergodic Z_δ process of S_t.

As we compare the figures, we realize that there would be some errors in the plot of the Z_δ. Therefore, the model does not necessarily provide us with a perfect strategy. However, it reduces the trading risk.

3.3 Estimating t_{M_i}

Theorem 3.2. *The order execution time t_{M_i}, for all $i > 0$, satisfies the following relations*

$$\frac{\sigma_{t_{M_i}}}{\mu_{t_{M_i}}} < dt_{M_i} < d\tau_{i+1}.$$

Proof. Consider the fixed path ω_0. Without loss of generality consider the case that the path of Z_δ is bellow the reference level, $Z_\delta(\omega_0) = 0$. In each time interval $[\tau_i, \tau_{i+1}]$, of length δ_i, the absolute value of $Z_\delta(\omega_0)$ accepts a value in the interval $[0, |Z_{t_{M_i}}(\omega_0)|]$ $(\delta_{M_i} = t_{M_i} - 0)$. Therefore,

$$\begin{aligned}
&Z_{t_{M_i}}(\omega_0) \leq 0,\\
&Z_{t_{M_i}} = \frac{1}{{t_{M_i}}^\beta}\int_0^{t_{M_i}} \sigma_s dW_s + \frac{W_{t_{M_i}}}{{t_{M_i}}^\beta}\int_0^{t_{M_i}} \mu_s ds \leq 0,\\
&\quad \frac{1}{{t_{M_i}}^\beta}\int_0^{t_{M_i}} \sigma_s dW_s \leq -\frac{W_{t_{M_i}}}{{t_{M_i}}^\beta}\int_0^{t_{M_i}} \mu_s ds,\\
&\quad \int_0^{t_{M_i}} \sigma_s dW_s \leq W_{t_{M_i}}\int_0^{t_{M_i}} \mu_s ds, \quad \Rightarrow \quad \frac{\int_0^{t_{M_i}} \sigma_s dW_s}{\int_0^{t_{M_i}} \mu_s ds} \leq W_{t_{M_i}},\\
&\quad \frac{\sigma_{t_{M_i}} dW_{t_{M_i}}}{\mu_{t_{M_i}} dt_{M_i}} \leq dW_{t_{M_i}}, \quad \Rightarrow \quad dt_{M_i} \geq \frac{\sigma_{t_{M_i}}}{\mu_{t_{M_i}}}.
\end{aligned} \tag{13}$$

On the other hand, we have $\delta_i = \tau_{i+1} - \tau_i$ and

$$\begin{aligned}
&\tau_i < t_{M_i} < \tau_{i+1}, \quad \Rightarrow \quad 0 < t_{M_i} - \tau_i < \tau_{i+1} - \tau_i,\\
&\Rightarrow 0 < t_{M_i} - \tau_i < \delta_i.
\end{aligned} \tag{14}$$

Therefore from 13 we write,

$$\begin{aligned}
dt_{M_i} &\geq \frac{\sigma_{t_{M_i}}}{\mu_{t_{M_i}}}\\
dt_{M_i} - d\tau_i &\geq \frac{\sigma_{t_{M_i}}}{\mu_{t_{M_i}}} - d\tau_i.
\end{aligned}$$

Using 14 yields

$$\begin{aligned}
&\frac{\sigma_{t_{M_i}}}{\mu_{t_{M_i}}} - d\tau_i \leq dt_{M_i} - d\tau_i < d\delta_i = d\tau_{i+1} - d\tau_i\\
&\frac{\sigma_{t_{M_i}}}{\mu_{t_{M_i}}} \leq dt_{M_i} < d\tau_{i+1}.
\end{aligned}$$

□

4 Irrational Rotation

Definition 4.1. An irrational rotation is a map given by

$$R_\theta : [0,1] \to [0,1], \quad R_\theta = x + \theta \mod 1, \tag{15}$$

where θ is an irrational number.

The circle rotation can be thought of as a subdivision of a circle into two parts, which are then exchanged with each other. A subdivision into more than two parts, which are then permuted with one-another, is called an interval exchange transformation [1].

We can consider another notation for the irrational rotation, which is knows as the multiplicative notation.

Definition 4.2. (Irrational rotation) For the unit circle $\mathbb{S}^1$, let

$$R_\theta : \mathbb{S}^1 \to \mathbb{S}^1, \quad R_\theta(x) = xe^{2\pi i\theta}, \tag{16}$$

where θ is an irrational number. We call the mapping 16 an irrational rotation on the unit circle.

4.1 Stochastic Process θ_t

We consider θ_t as a stochastic process and dependent on the price process of a risky asset, X_t.

For all $t \geq 0$, let

$$0 < \theta_t \neq \frac{p}{q} < n, \quad n \in \mathbb{N}\backslash\{\infty\}.$$

Therefore,

$$0 < \theta_t^2 < n^2 \Rightarrow 0 < \mathbb{E}[\theta_t^2] < n^2. \tag{17}$$

For all $k \in \mathbb{N}$, the angles zero and $2k\pi$ overlap each other. But, we do not consider these angles to be the same.

From [2] we have the following relation for the European call option price.

$$C(t, X_t) = e^{-r(T-t)}\mathbb{E}^Q\big[\max[xe^{Y_t} - K, 0]\big],$$

where K is the strike price. We have

$$\begin{aligned}\ln(xe^{Y_t} - K) &= \ln(xe^{Y_t}) + \ln(1 - \frac{K}{xe^{Y_t}})\\ &= \ln(x) + Y_t + \ln(1 - \frac{K}{xe^{Y_t}}).\end{aligned}$$

Using the EMO yields

$$\xi^\beta_{\delta,W_\delta}[\ln(xe^{Y_t} - K)] = Z_\delta + \xi^\beta_{\delta,W_\delta}\Big[\ln(1 - \frac{K}{xe^{Y_t}})\Big]. \tag{18}$$

Lemma 4.3. *The natural logarithm,* $\ln(1 - \frac{K}{xe^{Y_t}})$*, is an irrational number.*

Proof. For a European call option to be exercised, we need to have $xe^{Y_t} > K$. Therefore, we have $0 < 1 - K/xe^{Y_t} < 1$. From Lindemann-Weierstrass theorem [4], it follows that e^a is non-algebraic, for every positive non-algebraic number a. Specifically, if a is a rational number, the e^a cannot be rational. Therefore, $\ln(1 - K/xe^{Y_t})$ is an irrational number, according to [3, 4]. $\square$

Now since 18 is irrational, we define the process $\theta(z,\delta)$ as follows

Definition 4.4. For $t>0$, we have

$$\theta(z,\delta) = Z_\delta + \frac{W_T}{T^\beta}\gamma_\delta, \quad \gamma_\delta = \ln\big(1 - \frac{K}{xe^{Y_\delta}}\big). \tag{19}$$

We call $\theta(z,\delta)$, the irrational angle process.

We denote the irrational rotation generated using $\theta(z,\delta)$ by $R_{\theta_t}(\cdot)$.

Proposition 4.5. *The process $\theta(z,\delta)$ is stationary.*

Proof. First, we evaluate the expectation of the process.

$$\mathbb{E}[\theta(z,\delta)] = \mathbb{E}[Z_\delta + \frac{W_T}{T^\beta}\gamma_\delta] = \mathbb{E}[Z_\delta] + \frac{\mathbb{E}[W_T]\gamma_\delta}{T^\beta} = 0.$$

Also we have

$$\begin{aligned}
\mathbb{V}ar[\theta(z,\delta)] &= \mathbb{E}[\theta(z,\delta)^2] - \mathbb{E}[\theta(z,\delta)]^2 = \mathbb{E}[Z_\delta^2 + \frac{W_\delta^2}{\delta^{2\beta}}\gamma_\delta^2]\\
=&\mathbb{E}[Z_\delta^2] + \frac{\gamma_\delta^2}{T^{2\beta-1}} = \frac{1}{T^{2\beta}}\int_0^\delta \sigma_s^2 ds + \frac{1}{T^{2\beta-1}}(\int_0^\delta \mu_s ds)^2 + \frac{\gamma_\delta^2}{T^{2\beta-1}}\\
\Rightarrow \mathbb{V}ar[\theta(z,\delta)] &= \frac{1}{T^{2\beta}}\Big[\int_0^\delta \sigma_s^2 ds + t\big[(\int_0^\delta \mu_s ds)^2 + \gamma_\delta^2\big]\Big]
\end{aligned}$$

Since Y_t is an Itô process we have $\int_0^\delta(\sigma_s^2+\mu_s)ds<\infty$. Therefore, $\mathbb{E}[\theta^2(z,\delta)]<\infty$.
For all $\delta,\epsilon>0$ we have

$$\mathbb{E}[\theta(z,\delta)] = \mathbb{E}[\theta(z,\delta+\epsilon)] = 0.$$

Finally, since the process $\theta(z,\delta)$ is only dependent on δ, the length of time intervals, the correlation function of $\theta(z,\delta)$ is also a function of δ. Hence, $\theta(z,\delta)$ is stationary. □

Proposition 4.6. *The process $\theta(z,\delta)$ is mean ergodic.*

Proof. We know that Z_δ is mean ergodic. Also, $\xi^\beta_{\delta,W_\delta}[\gamma_\delta]$ is mean ergodic, according to [16]. Therefore, from the properties of mean ergodic processes, [16], $\theta(z,\delta)$ is mean ergodic. □

4.2 The Properties of Irrational Rotation

Proposition 4.7. *If R_θ is an irrational rotation on the unit circle, with θ being an irrational number. Then,*

1. *The orbit of every $x\in[0,1]$ under R_θ is dense in the interval $[0,1]$.*

2. *R_θ is not topologically mixing.*

3. *R_θ is uniquely ergodic, with the lebesgue measure as the unique invariant probability measure.*

4. *Let $[a, b] \in [0, 1]$. From the ergodicity of R_θ we have*

$$\lim_{N\to\infty} \sum_{n=0}^{N-1} \chi_{[a,b)}(R_\theta^n(x)) = b - a, \quad x \in [a, b].$$

Proof. For the proof and more details we refer the reader to [1]. □

Theorem 4.8. *The process R_{θ_t} is Markov.*

Proof. It suffices to prove the increments of R_{θ_t} are independent. We have

$$\forall t > 0, \quad R_{\theta_t}(0) = 0 \cdot e^{2\pi i\theta_t} = 0.$$

According to proposition 4.7, since the irrational rotation is ergodic and the orbit of every point is dense, it follows that for the time T and every $s, t \geq 0$, the angles θ_t and θ_s are independent from each other. Therefore, R_{θ_s} and R_{θ_t} are independent. Hence,

$$\begin{aligned}
&\mathbb{P}(R_{\theta_t} \leq x | R_{\theta_{t_1}}, \cdots, R_{\theta_{t_n}}) \\
=&\mathbb{P}(R_{\theta_t} - R_{\theta_{t_n}} + R_{\theta_{t_n}} \leq x | R_{\theta_{t_1}}, \cdots, R_{\theta_{t_n}}) \\
=&\mathbb{P}(R_{\theta_t} - R_{\theta_{t_n}} + R_{\theta_{t_n}} \leq x | R_{\theta_{t_n}}) = \mathbb{P}(R_{\theta_t} \leq x | R_{\theta_{t_n}})
\end{aligned}$$

□

We express the following theorem by using [5] to define the function ϕ, which is used in the Birkhoff ergodic theorem.

Theorem 4.9. *Let R_{θ_t} be an irrational rotation process and $\phi : [0, 1] \to \mathbb{R}$ be a continuous function, such that $\phi(0) = \phi(1)$. Then,*

$$\lim_{n\to\infty} \Big(\frac{1}{n} \sum_{k=0}^{n-1} \phi(R_{\theta_t}^k(x))\Big) = \int_0^1 \phi(y)dy, \quad \forall x \in [0, 1].$$

Proof. From [5] we define the function ψ_m as

$$\psi_m(x) = e^{2\pi imx} = \cos(2\pi imx) + i\sin(2\pi imx).$$

We have

$$\psi_m(R_{\theta_t}^k(x)) = e^{2\pi ik\theta_t + x} = e^{2\pi imk\theta_t}\psi_m(x).$$

For $m \neq 0$ we write

$$\Big|\frac{1}{n}\sum_{k=0}^{n-1}\psi_m(R^k_{\theta_t}(x))\Big| = \frac{1}{n}\cdot|e^{2\pi imx}|\cdot\Big|\sum_{k=0}^{n-1}e^{2\pi imk\theta_t}\Big|$$
$$= \frac{1}{n}\Big|\frac{1-e^{2\pi inm\theta_t}}{1-e^{2\pi im\theta_t}}\Big| \leq \frac{1}{n}\cdot\frac{2}{1-e^{2\pi im\theta_t}} \xrightarrow[n\to\infty]{} 0.$$

Therefore, if we take $\phi(x) = \sum_{m=-t_N}^{t_N}\theta_m\psi_m(x)$, in which $\theta_{-t_N}, \theta_{-t_{N+1}}, \cdots, \theta_{t_0}, \cdots, \theta_{t_N} \in \mathbb{Q}^{\complement}$, Then,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\phi(R^k_{\theta_t}(x)) = \theta_{t_0} = \int \phi(y)dy.$$

Finally, from [1] we know that the set of trigonometric polynomials is dense in the space of all periodic functions, which completes the proof. □

Remark 4.10. In the above theorem the angle θ_{-i}, for all $i > 0$, indicates clockwise orientation on the unit circle. Also t_0 indicates the time that we buy an option contract. Therefore, the angle $\theta_0 = \theta_{t_0}$ is not zero and it varies with respect to Z_{δ_0}. Hence, θ_{t_0} is a random irrational number.

Theorem 4.11. *Let $\theta(z,\delta)$ be a random angle process with respect to irrational rotation R_{θ_t} on $\mathbb{S}^1$, such that $\mathbb{E}[\theta_t] < \infty$. Then, $R_{\theta_t} = xe^{2\pi i\theta_t}$ is log-ergodic with respect to $\xi_{\delta,W_\delta}[\cdot]$.*

Proof. We have

$$Y_t = \ln(R_{\theta_t}) = \ln(x) + 2\pi i\theta(z,t), \quad Y_0 = \ln(x).$$

Let $\theta(z,\delta) = \theta_\delta$. Using EMO yields

$$Z_\delta = \xi^\beta_{\delta,W_\delta}[Y_t] = 0 + \frac{2\pi iW_T}{T^\beta}\theta_\delta, \quad Z_0 = 0.$$

Evaluating the covariance of Z_δ yields

$$\mathbf{Cov}_{zz} = \mathbb{E}[Z^2_\delta] = -\frac{4\pi^2}{T^{\beta-1}}\mathbb{E}[\theta^2_\delta]. \tag{20}$$

Now, it suffices to we prove

$$< Z >= \lim_{T\to\infty}\frac{1}{T}\int_0^T(1-\frac{\delta}{T})\mathbf{Cov}_{zz}(\delta)d\delta = 0. \tag{21}$$

Substituting 20 in 21 yields

$$\begin{aligned} < Z > &= \lim_{T\to\infty}\frac{1}{T}\int_0^T(\frac{\delta}{T}-1)\frac{4\pi^2}{T^{\beta-1}}\mathbb{E}[\theta^2_\delta]d\delta \\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\frac{\delta}{T}\frac{4\pi^2}{T^{\beta-1}}\mathbb{E}[\theta^2_\delta]d\delta - \lim_{T\to\infty}\frac{1}{T}\int_0^T\frac{4\pi^2}{T^{\beta-1}}\mathbb{E}[\theta^2_\delta]d\delta \\ &= \lim_{T\to\infty}\frac{4\pi^2}{T^{\beta+1}}\int_0^T\delta\mathbb{E}[\theta^2_\delta]d\delta - \lim_{T\to\infty}\frac{4\pi^2}{T^\beta}\int_0^T\mathbb{E}[\theta^2_\delta]d\delta. \end{aligned} \tag{22}$$

From 17 we know that $\mathbb{E}[\theta_\delta^2] = m < \infty$. Therefore, the integrals in 22 are finite. Hence, taking the limit as $T \to \infty$ yields $< Z >= 0$. □

The figure 3 demonstrates how the value of R_{θ_t} is varies with respect to stock price.

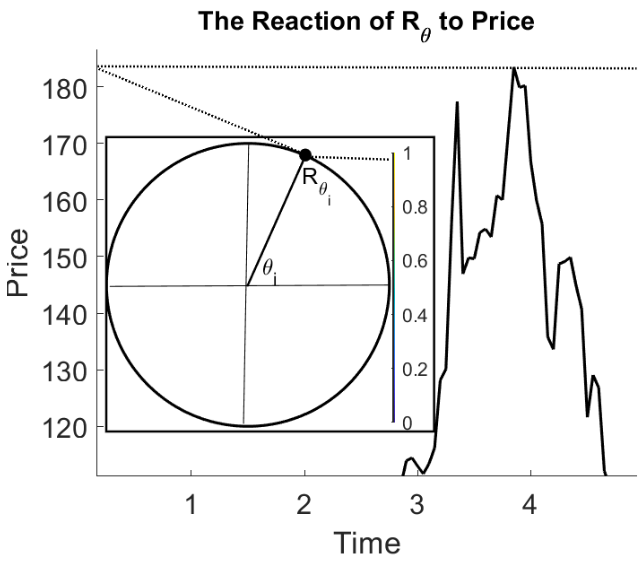

Figure 3: The behavior of the process R_{θ_t} on the unit circle relative to the stock price.

5 European Call Option Pricing Using The Irrational Rotation

In order to study the problem of option pricing using the irrational rotation, we need to consider a corresponding value of K' in the interval $[0, 1]$ for the strike price K. Therefore, we consider ϕ such that $\phi(K') = K$.

Theorem 5.1. *The European call option price with strike price K and the exercise time T is given by*

$$C(R_{\theta_t}, K) = e^{-rt}\Big(\frac{W_T}{T^\beta}\ln(1 - \frac{K}{S_{t_0}}) - K\Big),$$

where R_{θ_t} is the irrational rotation on the unit circle $\mathbb{S}^1$.

Proof. From [2] we write

$$\begin{aligned} C(R_{\theta_t}, K) =& e^{-rt}\mathbb{E}^Q[\phi(R_{\theta_t}(x)) - \phi(K')] \\ =& e^{-rt}\mathbb{E}^Q[\phi(xe^{2\pi i\theta_t}) - K] \\ =& e^{-rt}\big[\mathbb{E}^Q[\phi(xe^{2\pi i\theta_t})] - K\big] \\ =& e^{-rt}\Big(\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\phi(R^k_{\theta_t}(x)) - K\Big) \\ =& e^{-rt}\Big(\int\phi(y)dy - K\Big). \end{aligned} \tag{23}$$

From 4.9 we have

$$\phi(y = x) = \sum_{m=-t_N}^{t_N}\theta_m\psi_m(x),$$
$$\psi_m(x) = e^{2\pi imx} = \cos(2\pi imx) + i\sin(2\pi imx).$$

Therefore,

$$\begin{aligned} \int\phi(y)dy =& \int\theta_0\psi_0(y)dy \\ =& \int_0^1\theta_0 dy = \theta_{t_0} = \frac{W_T}{T^\beta}\ln(1 - \frac{K}{S_{t_0}}). \end{aligned} \tag{24}$$

Substituting 24 in 23 yields

$$C(R_{\theta_t}, K) = e^{-rt}\Big(\frac{W_T}{T^\beta}\ln(1 - \frac{K}{S_{t_0}}) - K\Big),$$

where S_{t_0} is the stock price at the time we buy the option contract. □

6 Ergodic Partial differential equation of BlackScholes

From [16] we have the following proposition.

Proposition 6.1. *(Ergodic Partial differential equation of BlackScholes) Under the assumptions of the Black-Scholes model, the European call option price, $C(Z_\delta, \delta)$, relative to the stock price variation $Z_\delta = z$, with respect to short rate r, inhibition degree parameter β, and the strike price K satisfies in the following partial differential equation.*

$$\begin{aligned} &\frac{\partial C}{\partial\delta} + rz\frac{\partial C}{\partial z} + \frac{1}{2}B_\delta^2\frac{\partial^2 C}{\partial z^2} - rC = 0, \\ &\textit{for}\quad 0 < |z| < \infty, \quad 0 < \delta < \delta_T = T - 0, \\ &\textit{where}\quad B_\delta = \frac{q}{\delta^{\beta-1}} + \frac{\sigma}{\delta^\beta}, \quad q = \mu - \frac{1}{2}\sigma^2, \end{aligned} \tag{25}$$

together with initial conditions $C(0,\delta) = 0$ and $C(z,\delta_T) = (|z| - \ln(K))^+$.

The goal is to solve this equation in this section.

Proposition 6.2. *The PDE 25 has the representation of the form*

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\eta\frac{\partial^2 U}{\partial y^2}, \tag{26}$$

where $0 < \tau < T$ *and* $-\infty < y < \infty$.

Proof. From [16] we have

$$\begin{aligned}
Z_\delta &= \frac{1}{T^\beta}\Big[(\mu - \frac{1}{2}\sigma^2)\delta W_T + \sigma W_\delta\Big] \\
Z_\delta &= \frac{1}{T^\beta}\Big[(\mu - \frac{1}{2}\sigma^2)\delta - (\mu - \frac{1}{2}\sigma^2)\delta + (\mu - \frac{1}{2}\sigma^2)\delta W_T + \sigma W_\delta\Big] \\
T^\beta Z_\delta &= Y_\delta + (\mu - \frac{1}{2}\sigma^2)[W_T - \delta] \\
Y_\delta &= T^\beta Z_\delta - (\mu - \frac{1}{2}\sigma^2)[W_T - \delta]
\end{aligned}$$

Therefore, we consider the following change of variables.

$$\delta = T - \tau, \quad y = T^\beta z - (\mu - \frac{1}{2}\sigma^2)[W_T - \delta] \tag{27}$$

Using 27 we have

$$\frac{\partial C}{\partial \delta} = \frac{\partial C}{\partial \tau}\frac{\partial \tau}{\partial \delta} = -\frac{\partial C}{\partial \tau}, \tag{28}$$

$$\frac{\partial C}{\partial z} = \frac{\partial C}{\partial y}\frac{\partial y}{\partial z} = T^\beta\frac{\partial C}{\partial y}, \tag{29}$$

$$\begin{aligned}
\frac{\partial^2 C}{\partial z^2} &= \frac{\partial}{\partial z}(\frac{\partial C}{\partial z}) = \frac{\partial}{\partial z}(T^\beta\frac{\partial C}{\partial y}) = \frac{\partial}{\partial y}(T^\beta\frac{\partial C}{\partial y})\frac{\partial y}{\partial z} \\
&=(\frac{\partial^2 C}{\partial y^2} - \frac{\partial C}{\partial y})T^{2\beta}.
\end{aligned} \tag{30}$$

Substituting 28, 29, and 30 in 25 yields

$$-\frac{\partial C}{\partial \tau} + rzT^\beta\frac{\partial C}{\partial y} + \frac{1}{2}B_\delta^2(\frac{\partial^2 C}{\partial y^2} - \frac{\partial C}{\partial y})T^{2\beta} - rC = 0 \tag{31}$$

Now take

$$\begin{aligned}
C(y,\tau) &= U(y,\tau)e^{ay+b\tau}, \qquad (32)\\
\frac{\partial C}{\partial \tau} &= (bU + \frac{\partial U}{\partial \tau})e^{ay+b\tau}, \\
\frac{\partial C}{\partial y} &= (aU + \frac{\partial U}{\partial y})e^{ay+b\tau}, \\
\frac{\partial^2 C}{\partial y^2} &= [a^2U + 2a\frac{\partial U}{\partial y} + \frac{\partial^2 U}{\partial y^2}]e^{ay+b\tau}.
\end{aligned}$$

Substituting the above relations in 31 yields

$$\begin{aligned} bU + \frac{\partial U}{\partial \tau} =& rzT^{\beta}\big[aU + \frac{\partial U}{\partial y}\big] + \frac{B_{\delta}^2 T^{2\beta}}{2}\big[(a^2 - a)U + (2a-1)\frac{\partial U}{\partial y} + \frac{\partial^2 U}{\partial y^2}\big] - rU, \\ \Rightarrow \frac{\partial U}{\partial \tau} =& \big[rzT^{\beta} + \frac{B_{\delta}^2 T^{2\beta}}{2}(2a-1)\big]\frac{\partial U}{\partial y} + \frac{B_{\delta}^2 T^{2\beta}}{2}\frac{\partial^2 U}{\partial y^2} \\ &+ \big[rzT^{\beta}a + \frac{B_{\delta}^2 T^{2\beta}}{2}(a^2 - a) - r - b\big]U. \end{aligned}$$

Now take

$$a = \frac{1}{2} - \frac{rz}{B_{\delta}^2 T^{\beta}}, \quad \text{and} \quad b = \frac{4rzT^{\beta} - B_{\delta}^2 T^{2\beta}}{8} - \frac{r^2 z^2}{2B_{\delta}^2} - r.$$

Therefore, we have

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\underbrace{B_{\delta}^2 T^{2\beta}}_{\eta}\frac{\partial^2 U}{\partial y^2}, \tag{33}$$

With initial condition

$$U(y,0) = C(z,T)e^{-ay} = \max[|z| - \ln(K)].$$

□

6.1 Solving the Heat Equation

Proposition 6.3. *The heat equation 26 has a solution of the form*

$$U(y,\tau) = \int_{-\infty}^{\infty} K(y-\gamma,\tau) f(\gamma) d\gamma.$$

Proof. Using Fourier transform we have

$$\begin{aligned} \mathcal{F}\{U(y,\tau)\} = \hat{U}(\omega,\tau) =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(y,\tau)e^{-i\omega y} dy \\ \Rightarrow \mathcal{F}\{U_{\tau}(y,\tau)\} =& \frac{\partial \hat{U}(\omega,\tau)}{\partial \tau} = \hat{U}_{\tau}(\omega,\tau), \\ \mathcal{F}\{U_{yy}(y,\tau)\} =& -\omega^2 \mathcal{F}\{U(y,\tau)\} = -\omega^2 \hat{U}(\omega,\tau). \end{aligned}$$

From 33 we have

$$\begin{aligned} \mathcal{F}\{U_{\tau}\} &= \mathcal{F}\{\frac{B_{\delta}^2 T^{2\beta}}{2} U_{yy}\} \\ \hat{U}_{\tau}(\omega,\tau) &= -\frac{B_{\delta}^2 T^{2\beta}}{2}\omega^2 \hat{U}(\omega,\tau). \end{aligned} \tag{34}$$

The equation 34 is partial differential equation with solution

$$\hat{U}(\omega,\tau) = c(\omega)\exp\{-\frac{B_{\delta}^2 T^{2\beta}}{2}\omega^2 \tau\}.$$

Using Fourier transform for the initial condition $U(y,0) = f(x)$ yields

$$\mathcal{F}\{U(y,0)\} = \hat{U}(\omega,0) = \hat{f}(\omega), \quad \hat{U}(\omega,0) = c(\omega).$$

Therefore,

$$\begin{aligned}\hat{U}(\omega,\tau) &= \hat{U}(\omega,0)\exp\{-\frac{B_\delta^2 T^{2\beta}}{2}\omega^2\tau\} \\ &= \hat{f}(\omega)\exp\{-\frac{B_\delta^2 T^{2\beta}}{2}\omega^2\tau\}.\end{aligned}$$

Now using inverse Fourier transform we have

$$\begin{aligned}U(y,\tau) = \mathcal{F}^{-1}\{\hat{U}(\omega,\tau)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{U}(\omega,\tau)e^{i\omega y}dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{-\frac{B_\delta^2 T^{2\beta}}{2}\omega^2\tau}e^{i\omega y}d\omega \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Big(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(\gamma)e^{-i\omega\gamma}d\gamma\Big)e^{i\omega y-\frac{B_\delta^2 T^{2\beta}}{2}\omega^2\tau}d\omega \\ &= \int_{-\infty}^{\infty}\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega(y-\gamma)-\frac{B_\delta^2 T^{2\beta}}{2}\omega^2\tau}d\omega\Big)f(\gamma)d\gamma.\end{aligned}$$

Now taking

$$K(y-\gamma,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega(y-\gamma)-\frac{B_\delta^2 T^{2\beta}}{2}\omega^2\tau}d\omega. \tag{35}$$

yields

$$U(y,\tau) = \int_{-\infty}^{\infty}K(y-\gamma,\tau)f(\gamma)d\gamma.$$

□

Lemma 6.4. *If $\gamma = 0$, then for $K(y,\tau)$ we have*

$$K(y,\tau) = \frac{1}{\sqrt{2\pi\tau B_\delta^2 T^{2\beta}}}e^{-\frac{y^2}{2\tau B_\delta^2 T^{2\beta}}}.$$

Proof. Taking $\gamma = 0$ in 35 yields

$$\begin{aligned}K(y,\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega y-\frac{B_\delta^2 T^{2\beta}}{2}\omega^2\tau}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\frac{y^2}{2\tau B_\delta^2 T^{2\beta}}}e^{\left(\frac{\sqrt{B_\delta^2 T^{2\beta}}}{\sqrt{2}}\omega\sqrt{\tau}-\frac{iy}{\sqrt{2\tau B_\delta^2 T^{2\beta}}}\right)^2}d\omega.\end{aligned}$$

Let

$$\lambda = \frac{\sqrt{B_\delta^2 T^{2\beta}}}{\sqrt{2}}\omega\sqrt{\tau} - \frac{iy}{\sqrt{2\tau B_\delta^2 T^{2\beta}}} \Rightarrow d\lambda = \sqrt{\frac{\tau B_\delta^2 T^{2\beta}}{2}}d\omega.$$

Since $\int_{-\infty}^{\infty} e^{-\lambda^2} d\lambda = \sqrt{\pi}$,

$$\begin{aligned} K(y,\tau) &= \frac{1}{2\pi} e^{-\frac{y^2}{2\tau B_\delta^2 T^{2\beta}}} \int_{-\infty}^{\infty} e^{-\lambda^2} \frac{\sqrt{2}}{\sqrt{\tau B_\delta^2 T^{2\beta}}} d\lambda \\ &= \frac{1}{2\pi} \frac{\sqrt{2}}{\sqrt{\tau B_\delta^2 T^{2\beta}}} e^{-\frac{y^2}{2\tau B_\delta^2 T^{2\beta}}} \int_{-\infty}^{\infty} e^{-\lambda^2} d\lambda \\ &= \frac{1}{\sqrt{2\pi\tau B_\delta^2 T^{2\beta}}} e^{-\frac{y^2}{2\tau B_\delta^2 T^{2\beta}}}. \end{aligned}$$

□

6.2 European Option Price: Solving the Equation

Theorem 6.5. *The European call option price with respect to stock price X, price variations z, the EMO $\xi^{\beta}_{\delta,W_\delta}[\cdot]$, and the strike price K satisfies the following equation.*

$$C(z,\tau) = e^{-r\tau} e^{\frac{y(\lambda-2)+\frac{1}{4}p(\lambda-2)^2}{2\lambda}} \big[|z| - \ln(K)\big] N[d],$$

where

$$d = \frac{\ln\Big[X/\ln(K)\Big]}{\sqrt{2p\lambda}}, \quad p = r|z|\tau T^{\beta}, \quad \lambda = \frac{T^{\beta} B_\delta^2}{r|z|}.$$

Proof. Using propositions 6.2, 6.3, lemma 6.4, and 32 we write

$$e^{ay+b\tau} = \exp\Big\{\frac{\tau B_\delta^2 T^{2\beta} y - 2\tau T^{\beta} r|z|y + r|z|\tau^2 B_\delta^2 T^{3\beta} - \frac{1}{4} B_\delta^4 T^{4\beta}\tau^2 - r^2 z^2 \tau^2 T^{2\beta}}{2\tau B_\delta^2 T^{2\beta}}\Big\}.$$

Let $p = r|z|\tau T^{\beta}$, Then

$$\begin{aligned} e^{ay+b\tau} &= \exp\Big\{\frac{\tau B_\delta^2 T^{2\beta} y - 2py + p\tau B_\delta^2 T^{2\beta} - \frac{1}{4} B_\delta^4 T^{4\beta}\tau^2 - p^2}{2\tau T^{2\beta} B_\delta^2}\Big\} \\ &= \exp\Big\{\frac{B_\delta^2 T^{\beta} \frac{p}{r|z|} y - 2py + pB_\delta^2 T^{\beta} \frac{p}{r|z|} - \frac{1}{4} B_\delta^4 T^{2\beta} \frac{p^2}{r^2 z^2} - p^2}{2\frac{p}{r|z|} T^{\beta} B_\delta^2}\Big\} \\ &= \exp\Big\{\frac{p\big[\frac{B_\delta^2 T^{\beta}}{r|z|} y - 2y\big] + p^2\big[\frac{B_\delta^2 T^{\beta}}{r|z|} - \frac{1}{4}\frac{B_\delta^4 T^{2\beta}}{r^2 z^2} - 1\big]}{2p\frac{T^{\beta} B_\delta^2}{r|z|}}\Big\} \end{aligned}$$

Now let $\lambda = \frac{T^{\beta} B_\delta^2}{r|z|}$. Therefore,

$$\begin{aligned} e^{ay+b\tau} &= \exp\Big\{\frac{p\big[\lambda y - 2y\big] + p^2\big[\lambda - \frac{1}{4}\lambda^2 - 1\big]}{2p\lambda}\Big\} \\ &= \exp\Big\{\frac{y(\lambda-2) + \frac{1}{4}p(\lambda-2)^2}{2\lambda}\Big\} \\ &= \exp\Big\{(\lambda-2)\frac{y + \frac{1}{4}p(\lambda-2)}{2\lambda}\Big\}. \end{aligned}$$

Therefore,

$$C(z,\tau) = e^{-r\tau}\frac{1}{\sqrt{2\tau\pi B_\delta^2 T^{2\beta}}}e^{\frac{y(\lambda-2)+\frac{1}{4}p(\lambda-2)^2}{2\lambda}}\int_{-\infty}^{\infty} e^{-\frac{(y-\gamma)^2}{2p\lambda}}\max[|z|-\ln(K),0]d\gamma$$

Since $|z| = \left|\xi^{\beta}_{\delta,W_\delta}[\gamma]\right|$, and $\xi^{-\beta}_{t,W_t}\left[\xi^{\beta}_{\delta,W_\delta}[\gamma]\right] = \gamma$ we have

$$\begin{aligned}
C(z,\tau) =& e^{-r\tau}\frac{1}{\sqrt{4\pi p\lambda}}e^{\frac{y(\lambda-2)+\frac{1}{4}p(\lambda-2)^2}{2\lambda}}\int_{\xi^{-\beta}_{t,W_t}[\ln(K)]}^{\infty} e^{-\frac{(y-\gamma)^2}{2p\lambda}}(\xi^{\beta}_{\delta,W_\delta}[\gamma]-\ln(K))d\gamma\\
=& e^{-r\tau}e^{\frac{y(\lambda-2)+\frac{1}{4}p(\lambda-2)^2}{2\lambda}}\frac{1}{\sqrt{4\pi p\lambda}}\\
&\underbrace{\Big[\int_{\xi^{-\beta}_{t,W_t}[\ln(K)]}^{\infty} e^{-\frac{(y-\gamma)^2}{2p\lambda}}\xi^{\beta}_{\delta,W_\delta}[\gamma]d\gamma - \ln(K)\int_{\xi^{-\beta}_{t,W_t}[\ln(K)]}^{\infty} e^{-\frac{(y-\gamma)^2}{2p\lambda}}d\gamma\Big]}_{I_1-I_2}. \quad (36)
\end{aligned}$$

Let $M = \frac{y-\gamma}{\sqrt{2p\lambda}}$. We have $dM = \frac{-d\gamma}{\sqrt{2p\lambda}}$. Hence,

$$I_1 = |z|\int_{\frac{\ln(X)-\ln(\xi^{-\beta}_{t,W_t}[\ln(K)])}{\sqrt{2p\lambda}}}^{-\infty} e^{-\frac{1}{2}M^2}(-\sqrt{2p\lambda})dM, \quad (37)$$

$$I_2 = \ln(K)\int_{\frac{\ln(X)-\ln(\xi^{-\beta}_{t,W_t}[\ln(K)])}{\sqrt{2p\lambda}}}^{-\infty} e^{-\frac{1}{2}M^2}(-\sqrt{2p\lambda})dM. \quad (38)$$

Substituting 37 and 38 in 36 yields

$$\begin{aligned}
C(z,\tau) =& e^{-r\tau}e^{\frac{y(\lambda-2)+\frac{1}{4}p(\lambda-2)^2}{2\lambda}}\frac{1}{\sqrt{4\pi p\lambda}}\big[|z|-\ln(K)\big]\\
&\int_{-\infty}^{\frac{\ln(X)-\ln(\xi^{-\beta}_{t,W_t}[\ln(K)])}{\sqrt{2p\lambda}}} e^{-\frac{1}{2}M^2}(\sqrt{2p\lambda})dM\\
=& e^{-r\tau}e^{\frac{y(\lambda-2)+\frac{1}{4}p(\lambda-2)^2}{2\lambda}}\big[|z|-\ln(K)\big]\\
&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\ln(X)-\ln(\xi^{-\beta}_{t,W_t}[\ln(K)])}{\sqrt{2p\lambda}}} e^{-\frac{1}{2}M^2}dM
\end{aligned}$$

Using lemma 2.4 yields $\xi^{-\beta}_{t,W_t}[\ln(K)] = \ln(K)$. Therefore,

$$C(z,\tau) = e^{-r\tau}e^{\frac{y(\lambda-2)+\frac{1}{4}p(\lambda-2)^2}{2\lambda}}\big[|z|-\ln(K)\big]N\Big[\frac{\ln(X)-\ln[\ln(K)]}{\sqrt{2p\lambda}}\Big].$$

Now taking $d = \frac{\ln(X)-\ln[\ln(K)]}{\sqrt{2p\lambda}}$ yields.

$$C(z,\tau) = e^{-r\tau}e^{\frac{y(\lambda-2)+\frac{1}{4}p(\lambda-2)^2}{2\lambda}}\big[|z|-\ln(K)\big]N[d],$$

where

$$p = r|z|\tau T^{\beta}, \quad \lambda = \frac{T^{\beta}B_\delta^2}{r|z|}, \quad B_\delta = \frac{q}{\delta^{\beta-1}} + \frac{\sigma}{\delta^{\beta}}, \quad q = \mu - \frac{1}{2}\sigma^2.$$

□

7 Conclusion

Choosing the right time to leave a trading position is always of much interest for market participants. In this paper, we presented a model using log-ergodic processes to estimate a time interval for exiting a trading position with profit.

We provided a novel approach to European call option pricing by incorporating stochastic irrational rotation. We substituted the time average of a mean ergodic process with the expectation in our calculations, since the irrational rotation is ergodic. Compared to the traditional Black-Scholes model, this method offers improved accuracy in capturing market dynamics and enhanced computational efficiency due to the use of mean ergodic processes.

Additionally, we solved the ergodic partial differential equation of Black-Scholes, introduced in our recent work [16]. We have found a unique solution to this equation which includes the inhibition degree parameter β.

All in all, the financial markets can be studied using the theory of dynamical systems, as they actually have the property of change of state with respect to time. Studying the financial markets using ergodic theory have two main advantages: First, a new approach to solving and modeling financial and economics problems is provided to us. Second, the use of time average instead of the expectation in some of the calculations, makes it easier to study the markets in the long run. On the other hand, the existence of only one descriptive parameter, β, shows that our work is in early stages, and there are much work to do in the future studies for a flawless model to be made.

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