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A high order numerical method for Ito stochastic Volterra integral equations

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Abstract:
The main purpose of this paper is to propose a high order numerical method based on the finite difference methods for solving nonlinear Itô stochastic Volterra integral equations (SVIEs) of the second kind. To develop the method, a fourth-order implicit finite difference method and the explicit Milstein method are implemented for the discretization of non-stochastic and stochastic integral parts, respectively. To solve the original SVIEs, the proposed method has the deterministic fourth-order and strong stochastic first-order accuracy. The convergence analysis of the proposed method is proved. The finite difference method under consideration requires solving a 2×2 system of equations at each step for one-dimensional SVIE. Therefore, the proposed method is very simple to implement and does not require a lot of computational cost. Some numerical examples are prepared to indicate the verity and efficiency of the new method. Moreover, the comparative numerical results show that this method is more accurate than those existing methods given in the literature.

Keywords: Finite difference, Itô stochastic Volterra integral equations, fourth order.
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1 Introduction

Integral equations (IEs) of Volterra type appear in a variety of science and technology fields such as plasma physics [6]. The Volterra IEs (VIEs) of the second kind are a special type of integral equations and are often evolved in many engineering domains such as petrol industry. Recently, there has been an increasing interest to investigate the theory for Volterra equations [5], Fredholm equations [32], integro-differential equations [26], fractional calculus [10, 19] and numerical methods to

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solve these types of equations [4, 5, 11]. Also, the stochastic integral equations appear in many fields of problems such as the study of the growth of biological populations [15], the stochastic formulation of problems in reactor dynamics [7,20], and in many other problems arising in the general areas of biology, physics and engineering. Moreover, in recent years, there is an increasing requirement to inquire the behavior of even more sophisticated dynamical systems in physical, medical, engineering and financial applications [9,14,18]. The main motivation of this paper is to construct a high-order numerical method to solve the following Itô stochastic VIEs (SVIEs):

$$y(t) = f(t) + \int_0^t k^1(s,t)u^1(y(s))ds + \int_0^t k^2(s,t)u^2(y(s))dB(s), \quad t \in [0,T], \quad (1)$$

where $f(t)$ and kernels $k^1(s,t), k^2(s,t), u^1, u^2$ are known L_2 functions, while $y(t)$ is the unknown L_2 function and $B(t)$ is a Brownian motion process. Since the exact solution of the SVIEs is often unavailable, we have to use approximate and numerical methods for solving these type of problems, generally. To solve the SVIEs given by Eq. (1), the researches have introduced various types of approximate methods such as wavelets approximation technique [23,24], collocation method [22], iterative technique [28,29] and operational matrix method [13,14,18]. In order to achieve a highly accurate solution for the SVIEs (1), the computational cost of these methods is very expensive. For instance, to develop an operational matrices based method with step size $h = \frac{1}{m}$, it must be solved a m-dimensional system of nonlinear algebraic equations. Thus the computational cost of this method will be increased by selecting a tiny step size of h. The finite difference methods are a class of efficient numerical procedures to solve the SVIEs [16]. These methods can be implemented for solving the SVIEs in two forms implicit and explicit. The implicit finite difference methods are effective for the numerical solution of the stiff SVIEs and stiff SDEs. It must be noted that a straightforward formulation of a fully implicit finite difference method for SVIEs often causes the problem of being stochastically unstable [16]. In this paper to design a highly accurate method for the numerical solution of the SVIE (1), a fourth-order implicit finite difference method and the explicit Milstein method are implemented for the discretization of non-stochastic and stochastic integral parts, respectively. The proposed finite difference method is of order $(4, 1)$, and specially it is very appropriate for SVIEs with small noise. Unlike the operational matrix based methods, the proposed finite difference method just needs to solve a 2×2 system at each step for one-dimensional SVIEs. Therefore, the computational cost of the new method is less than the operational matrix based methods. The convergence analysis of the proposed method is proven. The efficiency and high accuracy of the proposed method are verified throughout some examples. Moreover, the comparative numerical results show that this method is more accurate than the other existing methods.

2 Some definitions and preliminary results

Here, we provide some basic mathematical preliminaries of the stochastic calculus and numerical integration methods. At first, we consider the equidistant discretization $\mathcal{I}_h = \{0 = t_0 < t_1 < ... < t_N = T\}$ of the time interval $[0, T]$ with stepsize $h = \frac{T}{N}$ and $t_j = jh$ for $j = 0, 1, ..., N$ and time discrete approximation $Y(t), t \in \mathcal{I}_h$. Also, in this paper for more simplicity, we use Y_n instead of $Y(t_n)$.

Definition 2.1. A stochastic process $B(t), t \in [0, T]$ is called Brownian motion, if the following properties are satisfied [25]

(i) For $0 \leq t_1 \leq \cdots \leq t_n = T$, the increments of the process $B(t)$ is independent.

(ii) $\forall t \geq 0$, $B(t+h) - B(t) = \sqrt{h}\mathcal{N}(0,1)$ where $\mathcal{N}(0,1)$ is the standard normal distribution.

(iii) $\forall t \geq 0$, the paths of $B(t)$ is continuous.

It should be pointed out that in this paper for numerical computational, the assumption $B(0) = 0$ with probability one is considered.
Consider $\mathcal{V} = \mathcal{V}(T, S)$ as the class of functions $f(t, w) : [0, \infty) \times \Omega \to \mathbb{R}$ such that

(i) The function $f(t, w)$ is $\mathcal{B} \times \mathcal{F}-$ measurable, where $\mathcal{F}$ is the $\sigma-$algebra on Ω and $\mathcal{B}$ signifies the Borel algebra on $[0, \infty)$.

(ii) f is adapted to $\mathcal{F}_t$, where $\mathcal{F}_t$ denotes the $\sigma-$algebra generated by the random variables $B(s)$ for $s \leq t$.

(iii) $E\left[\int_S^T f^2(t, w)dt\right] < \infty$.

Definition 2.2. If the function $\varphi \in \mathcal{V}$ has a representation in form $\varphi(t, w) = \sum_j e_j(w)\chi[t_j, t_{j+1})(t)$, then it calls as a elementary function. The symbol χ signifies the characteristic function, and each function e_j is $\mathcal{F}_t-$measurable.

Definition 2.3. [25] Let $f \in \mathcal{V}(T, S)$, then the Itô integral of f is defined by

$$\int_S^T f(t, w)dB_t(w) = \lim_{n\to\infty} \int_S^T \varphi_n(t, w)dB_t(w)$$

where, φ_n is a sequence of elementary functions such that $E\left[\int_S^T (f(t, w) - \varphi_n(t, w))^2 dt\right]$ $\to 0$ as $n \to \infty$.

The stochastic linear multi-step method can be consider as follows

$$Y_n = f_n + h\sum_{j=0}^{n} \alpha_j k^1(t_j, t_n)u^1(Y_j) + \sum_{j=0}^{n-1} \Phi(t_j, t_{j-1}, \vartheta^\nu(h), Y_j), \quad n = 1, 2, \cdots, N, \nu \in \mathcal{M}, \tag{2}$$

in which $h\sum_{j=0}^{n}\alpha_j k^1(t_j,t_n)u^1(Y_j)$ is a numerical integration method that approximate the non-stochastic integral $\int_{t_0}^{t_n} k^1(s,t_n)u^1(y(s))ds$ and $\sum_{j=0}^{n-1}\Phi(t_j,t_{j-1},\vartheta^{\nu}(h),Y_j)$ is an stochastic numerical integration method that approximate the stochastic integral $\int_{t_0}^{t_n} k^2(s,t_n)u^2(y(s))dB(s)$ where $\mathcal{M}$ is a finite set of multi-indices with cardinality p and $\vartheta^{\nu}(h)$ is a random vector that satisfies the moment condition [2, 27]

$$E(\vartheta_{\nu_1}^{q_1}(h)\cdot\ldots\cdot\vartheta_{\nu_p}^{q_p}(h)) = O(h^{(q_1+\cdots+q_p)/2}) \tag{3}$$

for all $q_i \in \mathbb{N}_0, \nu_i \in \mathcal{M}$, where $\mathbb{N}_0$ is the set of nonnegative integers. For example, if we use trapezoidal rule and Euler-Maruyama method for non-stochastic and stochastic integral parts, respectively, then we obtain the following numerical method [16],

$$\begin{aligned} Y_n =& f_n + \frac{h}{2}\left(k^1(t_0,t_n)u_0^1 + k^1(t_n,t_n)u_n^1\right) + h\sum_{j=1}^{n-1}k^1(t_j,t_n)u_j^1 \\ &+ \sum_{j=0}^{n-1}k^2(t_j,t_n)u_j^2\Delta B_j, \quad n = 1,2,\cdots,N, \end{aligned}$$

where $\Delta B_j = B(t_{j+1}) - B(t_j)$.

Definition 2.4. We say the stochastic linear multi-step (2) for the approximation of the solution of the SVIE (1) mean-square converges if the following property for global error $y(t_n) - Y_n$ is established

$$\max_{n=1,\cdots,N} y(t_n) - Y_{n2} \longrightarrow 0 \quad as \quad h \longrightarrow 0,$$

and also the discrete time approximation $Y(t)$ mean-square convergent with order $\delta > 0$, if there exist constants $C < \infty$, not depending on h, such that the global error satisfies

$$\max_{n=1,\cdots,N} y(t_n) - Y_{n2} \le C \times h^{\delta}.$$

In the following we denote the stochastic strong order of convergence of the scheme by p_S and in the deterministic case ($u^2 \equiv 0$) by p_D and for their pair by (p_D, p_S).

3 Theory of finite difference method for nonlinear SVIEs

In this section, we try to construct some numerical integration techniques to achieve a fourth-order finite difference method for the non-stochastic integral part of SVIEs (1) and also apply the Milstein method for corresponding its stochastic integral part in such a way that the numerical procedure not only does not face stochastically unstable phenomena but also preserves the order of the convergence. It should be noted out that, if we directly employ Simpson's rule of integral approximation, then we must have $N = 2m$, i.e. the number of nodes must be odd, and so we obtain:

$$\begin{aligned}Y_{2n} =& f_{2n} + \sum_{j=1}^{n} \int_{t_{2(j-1)}}^{t_{2j}} k^1(s, t_{2n}) u^1(y(s)) ds \\ &+ \sum_{j=1}^{n} \int_{t_{2(j-1)}}^{t_{2j}} k^2(s, t_{2n}) u^2(y(s)) dB(s), \quad n = 1, 2, \cdots, m,\end{aligned}$$

that results in

$$\begin{aligned}Y_{2n} =& f_{2n} + \frac{h}{3} \sum_{j=1}^{n} \left(k^1_{2(j-1),2n} u^1_{2(j-1)} + 4k^1_{2j-1,2n} u^1_{2j-1} + k^1_{2j,2n} u^1_{2j} \right) \\ &+ \sum_{j=1}^{n} \int_{t_{2(j-1)}}^{t_{2j}} k^2(s, t_{2n}) u^2(y(s)) dB(s), \quad n = 1, 2, \cdots, m,\end{aligned}$$

now, if we apply Milstein method [16] for stochastic integral parts then we obtain

$$\begin{aligned}Y_{2n} =& f_{2n} + \frac{h}{3} \sum_{j=1}^{n} \left(k^1_{2(j-1),2n} u^1_{2(j-1)} + 4k^1_{2j-1,2n} u^1_{2j-1} + k^1_{2j,2n} u^1_{2j} \right) \\ &+ \sum_{j=1}^{n} \Big(k^2_{2(j-1),2n} u^2_{2(j-1)} \left(I^0_{2(j-1)} + I^1_{2(j-1)} Du^2_{2(j-1)} \right) \\ &\qquad + k^2_{2j-1,2n} u^2_{2j-1} (I^0_{2j-1} + I^1_{2j-1} Du^2_{2j-1}) \Big),\end{aligned}$$

in which $Du^2 = \frac{du^2}{dy}$ and I^0 and I^1 are distributed random variables with $I^0_l = \int_{t_l}^{t_{l+1}} dB(s) = \sqrt{h}\mathcal{N}(0,1)$ and $I^1_l = \frac{1}{2}\left(\left(I^0_l\right)^2 - h \right)$ where $\mathcal{N}(0,1)$ is the standard normal distribution. It should be mentioned that with the above numerical procedure, the corresponding system of nonlinear equations will be implicit with respect to the stochastic terms that cause the finite difference to be stochastically unstable. For more details about this phenomena see [1,16,21]. In the following, with the help of some useful finite difference methods, we try to overcome this drawback. So, at first, we add the auxiliary nodes $\mathcal{I}_{h/2} = \{t_{j+\frac{1}{2}} = t_j + \frac{h}{2}, j = 0, 1, \cdots, N-1\}$ to the $\mathcal{I}_h$ and then we take $\bar{\mathcal{I}}_h = \mathcal{I}_h \cup \mathcal{I}_{h/2}$. Therefore, if we take $t = t_j, j = 1, 2, \cdots, N$ in (1) then we have

$$\begin{aligned}Y_j =& f_j + \frac{h}{6} \sum_{i=0}^{j-1} \left(k^1_{i,j} u^1_i + 4k^1_{i+\frac{1}{2},j} u^1_{i+\frac{1}{2}} + k^1_{i+1,j} u^1_{i+1} \right) \\ &+ \sum_{i=0}^{j-1} \left(k^2_{i,j} u^2_i \left(I^0_i + I^1_i Du^2_i\right) + k^2_{i+\frac{1}{2},j} u^2_{i+\frac{1}{2}} \left(I^0_{i+\frac{1}{2}} + I^1_{i+\frac{1}{2}} Du^2_{i+\frac{1}{2}} \right) \right). \qquad (4)\end{aligned}$$

On the other hand, taking $t = t_{j+\frac{1}{2}}, j = 0, 1, 2, \cdots, N-1$, concluding

$$\begin{aligned} y_{j+\frac{1}{2}} =& f_{j+\frac{1}{2}} + H_j + \frac{h}{6}\sum_{i=0}^{j-1}\left(k^1_{i+\frac{1}{2},j+\frac{1}{2}}u^1_{i+\frac{1}{2}} + 4k^1_{i+1,j+\frac{1}{2}}u^1_{i+1} + k^1_{i+\frac{3}{2},j+\frac{1}{2}}u^1_{i+\frac{3}{2}}\right) \\ &+ \sum_{i=0}^{j-1}\left(k^2_{i,j+\frac{1}{2}}u^2_i\left(I^0_i + I^1_i Du^2_i\right) + k^2_{i+\frac{1}{2},j+\frac{1}{2}}u^2_{i+\frac{1}{2}}\left(I^0_{i+\frac{1}{2}} + I^1_{i+\frac{1}{2}}Du^2_{i+\frac{1}{2}}\right)\right) \\ &+ k^2_{j,j+\frac{1}{2}}u^2_j\left(I^0_j + I^1_j Du^2_j\right), \end{aligned} \tag{5}$$

in which H_j refers to a suitable method of numerical integration for approximating

$$\int_0^{t_{\frac{1}{2}}} k^1(s, t_{j+\frac{1}{2}})u^1(y(s))ds,$$

in half domain $[0, t_{\frac{1}{2}}]$. Since the Simpson integral approximation method guarantees fourth order for the corresponding non-stochastic integral part of SVIEs (1), we must construct a method with the same order of convergence for H_j to preserve fourth order convergence of the method. Hence, the following tree point approximating procedure is proposed for the above-mentioned integral term,

$$\int_0^{t_{\frac{1}{2}}} k^1(s, t_{j+\frac{1}{2}})u^1(y(s))ds \approx h\left(\alpha_1 k^1_{0,j+\frac{1}{2}}u^1_0 + \alpha_2 k^1_{\frac{1}{2},j+\frac{1}{2}}u^1_{\frac{1}{2}} + \alpha_3 k^1_{1,j+\frac{1}{2}}u^1_1\right). \tag{6}$$

Expanding the right hand side of (6) in terms of $k^1_{0,j+\frac{1}{2}}u^1_0$ yields that with parameters

$$\alpha_1 = \frac{5}{24}, \alpha_2 = \frac{8}{24}, \alpha_3 = -\frac{1}{24}$$

the numerical integration method for corresponding integral will be as follow

$$H_j = \frac{h}{24}\left(5k^1_{0,j+\frac{1}{2}}u^1_0 + 8k^1_{\frac{1}{2},j+\frac{1}{2}}u^1_{\frac{1}{2}} - k^1_{1,j+\frac{1}{2}}u^1_1\right), \tag{7}$$

and Also, we are able to conclude from the Taylor expansion that the numerical integration technique (7) guarantees the fourth order of convergence. Therefore, putting (7) in to the relation (5) we obtain

$$\begin{aligned} Y_{j+\frac{1}{2}} =& f_{j+\frac{1}{2}} + \frac{h}{24}\left(5k^1_{0,j+\frac{1}{2}}u^1_0 + 8k^1_{\frac{1}{2},j+\frac{1}{2}}u^1_{\frac{1}{2}} - k^1_{1,j+\frac{1}{2}}u^1_1\right) \\ &+ \frac{h}{6}\sum_{i=0}^{j-1}\left(k^1_{i+\frac{1}{2},j+\frac{1}{2}}u^1_{i+\frac{1}{2}} + 4k^1_{i+1,j+\frac{1}{2}}u^1_{i+1} + k^1_{i+\frac{3}{2},j+\frac{1}{2}}u^1_{i+\frac{3}{2}}\right) \\ &+ \sum_{i=0}^{j-1}\left(k^2_{i,j+\frac{1}{2}}u^2_i\left(I^0_i + I^1_i Du^2_i\right) + k^2_{i+\frac{1}{2},j+\frac{1}{2}}u^2_{i+\frac{1}{2}}\left(I^0_{i+\frac{1}{2}} + I^1_{i+\frac{1}{2}}Du^2_{i+\frac{1}{2}}\right)\right) \\ &+ k^2_{j,j+\frac{1}{2}}u^2_j\left(I^0_j + I^1_j Du^2_j\right). \end{aligned} \tag{8}$$

If we solve the overall system of the equations with respect to the unknowns Y_j and $Y_{j+\frac{1}{2}}$ then the resulting nonlinear system will be implicit with respect to the stochastic terms which causes the method to be stochastically unstable that mentioned in the previous section. Therefore, at first, using relations (4) and (8) for $j = 1$ and $j = 0$ respectively, we get

$$Y_1 = f_1 + \frac{h}{6}\left(k^1_{0,1}u^1_0 + 4k^1_{\frac{1}{2},1}u^1_{\frac{1}{2}} + k^1_{1,1}u^1_1\right) + \left(k^2_{0,1}u^2_0\left(I^0_0 + I^1_0 Du^2_0\right) + k^2_{\frac{1}{2},1}u^2_{\frac{1}{2}}\left(I^0_{\frac{1}{2}} + I^1_{\frac{1}{2}} Du^2_{\frac{1}{2}}\right)\right), \tag{9}$$

$$Y_{\frac{1}{2}} = f_{\frac{1}{2}} + \frac{h}{24}\left(5k^1_{0,\frac{1}{2}}u^1_0 + 8k^1_{\frac{1}{2},\frac{1}{2}}u^1_{\frac{1}{2}} - k^1_{1,\frac{1}{2}}u^1_1\right) + k^2_{0,\frac{1}{2}}u^2_0\left(I^0_0 + I^1_0 Du^2_0\right). \tag{10}$$

It should be pointed out that the above 2×2 system is implicit w.r.t the stochastic terms, so, for the terms in relation (9) that include $Y_{\frac{1}{2}}$ replace $Y_{\frac{1}{2}}$ with y_0 and then solving resulting 2×2 system obtains $Y_{\frac{1}{2}}$ and Y_1. Therefore the other equations solving in the manner that we don't face with any problem that causes the method to be stochastically unstable.

4 Convergence analysis

In this section it is proved that the presented finite difference method to solve the SVIEs (1) is of order $p_D = 4$ and $p_S = 1$ in the Itô sense. The following theorem shows that the proposed method has the fourth-order of accuracy for the SVIEs (1) with $u^2 \equiv 0$.

Theorem 4.1. *For integral equation (1), let $u^2 \equiv 0$, functions $k^1(s,t)$ and $u^1(y(s))$ are differentiable, and their fourth-order derivatives are piecewise continuous. Then there exists positive constant C such that the inequality*

$$LTE_l(h) := y(t_l) - Y_l \le Ch^4.$$

holds for all $t_l \in \bar{\mathcal{I}}_h$, where $y(t_l)$ and Y_l are the solutions of Eqs. (1) and (4)-(5), respectively.

Proof. Using Eqs. (1) and (4), we have

$$\begin{aligned}
y(t_j) - Y_j &= f(t_j) - f_j + \int_0^{t_j} k^1(s,t_j)u^1(y(s))ds - \frac{h}{6}\sum_{i=0}^{j-1}\left(k^1_{i,j}u^1_i + 4k^1_{i+\frac{1}{2},j}u^1_{i+\frac{1}{2}} + k^1_{i+1,j}u^1_{i+1}\right)\\
&= \sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}} k^1(s,t_j)u^1(y(s))ds - \frac{h}{6}\sum_{i=0}^{j-1}\left(k^1_{i,j}u^1_i + 4k^1_{i+\frac{1}{2},j}u^1_{i+\frac{1}{2}} + k^1_{i+1,j}u^1_{i+1}\right)\\
&= \sum_{i=0}^{j-1}\left(\int_{t_i}^{t_{i+1}} k^1(s,t_j)u^1(y(s))ds - \frac{h}{6}\left(k^1_{i,j}u^1_i + 4k^1_{i+\frac{1}{2},j}u^1_{i+\frac{1}{2}} + k^1_{i+1,j}u^1_{i+1}\right)\right).
\end{aligned} \tag{11}$$

According to the bound of error of the Simpson rule of the integral approximation, we can derive

$$\begin{aligned}
&\int_{t_i}^{t_{i+1}} k^1(s,t_j)u^1(y(s))ds - \frac{h}{6}\left(k^1_{i,j}u^1_i + 4k^1_{i+\frac{1}{2},j}u^1_{i+\frac{1}{2}} + k^1_{i+1,j}u^1_{i+1}\right)\\
&= C_1 h^5 \frac{\partial^4}{\partial s^4} k^1(s,t_j)u^1(y(s))|_{s=\eta_i},
\end{aligned} \tag{12}$$

where $\eta_i \in [t_i, t_{i+1}]$ and C_1 is a constant. Taking (12) in to (11) and using continuity of fourth order derivatives of functions, we conclude that

$$y(t_j) - Y_j = C_1 h^5 \sum_{i=0}^{j-1} \left(\frac{\partial^4}{\partial s^4} k^1(s,t_j) u^1(y(s))|_{s=\eta_i} \right) = C_1 h^5 j \frac{\partial^4}{\partial s^4} k^1(s,t_j) u^1(y(s))|_{s=\eta_j}, \quad (13)$$

in which $\eta_j \in [0, t_j]$. It should be noted that, from the continuity of fourth order derivatives of functions and utilizing the intermediate value theorem, the last one equality (13) is fulfilled. Therefore, we obtain

$$y(t_j) - Y_j \le C_2 h^4, \quad \text{where} \quad C_2 = C_1 T \frac{\partial^4}{\partial s^4} k^1(s,t_j) u^1(y(s))|_{s=\eta_j}.$$

On the other hand for nodes $t_{j+\frac{1}{2}}$ we get

$$\begin{aligned}
&y(t_{j+\frac{1}{2}}) - Y_{j+\frac{1}{2}} = f(t_{j+\frac{1}{2}}) - f_{j+\frac{1}{2}} + \int_0^{t_{j+\frac{1}{2}}} k^1(s, t_{j+\frac{1}{2}}) u^1(y(s)) ds \\
&- \frac{h}{24} \left(5k^1_{0,j+\frac{1}{2}} u^1_0 + 8k^1_{\frac{1}{2},j+\frac{1}{2}} u^1_{\frac{1}{2}} - k^1_{1,j+\frac{1}{2}} u^1_1 \right) \\
&- \frac{h}{6} \sum_{i=0}^{j-1} \left(k^1_{i+\frac{1}{2},j+\frac{1}{2}} u^1_{i+\frac{1}{2}} + 4k^1_{i+1,j+\frac{1}{2}} u^1_{i+1} + k^1_{i+\frac{3}{2},j+\frac{1}{2}} u^1_{i+\frac{3}{2}} \right) \\
&= \int_0^{t_{\frac{1}{2}}} k^1(s, t_{j+\frac{1}{2}}) u^1(y(s)) ds - \frac{h}{24} \left(5k^1_{0,j+\frac{1}{2}} u^1_0 + 8k^1_{\frac{1}{2},j+\frac{1}{2}} u^1_{\frac{1}{2}} - k^1_{1,j+\frac{1}{2}} u^1_1 \right) \\
&+ \sum_{i=0}^{j-1} \Bigg(\int_{t_{i+\frac{1}{2}}}^{t_{i+\frac{3}{2}}} k^1(s, t_{j+\frac{1}{2}}) u^1(y(s)) ds \\
&\qquad - \frac{h}{6} \left(k^1_{i+\frac{1}{2},j+\frac{1}{2}} u^1_{i+\frac{1}{2}} + 4k^1_{i+1,j+\frac{1}{2}} u^1_{i+1} + k^1_{i+\frac{3}{2},j+\frac{1}{2}} u^1_{i+\frac{3}{2}} \right) \Bigg) \\
&= C_3 h^4 \frac{\partial^3}{\partial s^3} k^1(s, t_{j+\frac{1}{2}}) u^1(y(s))|_{s=\eta_{\frac{1}{2}}} + C_4 h^4 T \frac{\partial^4}{\partial s^4} k^1(s, t_{j+\frac{1}{2}}) u^1(y(s))|_{s=\eta_{j+\frac{1}{2}}},
\end{aligned}$$

in which $\eta_{\frac{1}{2}} \in [0, t_{\frac{1}{2}}], \eta_{j+\frac{1}{2}} \in [0, t_{j+\frac{1}{2}}]$ and C_1, C_2 are some constants. Using some calculations, the proof can be easily completed. $\square$

The following theorem demonstrates that the proposed method (4) is the mean-square convergent to the SVIEs (1).

Theorem 4.2. *For the SVIEs (1), let that functions $k^1(s,t)$ and $u^1(y(s))$ are differentiable, and their fourth-order derivatives are piecewise continuous. Furthermore suppose that functions $k^2(s,t)$ and $u^2(y(s))$ are differentiable, and their second-order derivatives are piecewise continuous. Then there exists some constants C_1, C_2 such that the inequality*

$$E[LTE_l(h)^2] = E[y(t_l) - Y_l^2] \le C_1 \times h^8 + C_2 \times h^2,$$

holds for all $t_l \in \bar{\mathcal{I}}_h$.

Proof. For $l = 1, \ldots, N$ and from (4), we have

$$y(t_j) - Y_j = \sum_{i=0}^{j-1} \left(\int_{t_i}^{t_{i+1}} k^1(s,t_j)u^1(y(s))ds - \frac{h}{6}\left(k^1_{i,j}u^1_i + 4k^1_{i+\frac{1}{2},j}u^1_{i+\frac{1}{2}} + k^1_{i+1,j}u^1_{i+1}\right)\right)$$
$$+ \sum_{i=0}^{j-1} \left(\int_{t_i}^{t_{i+1}} k^2(s,t_j)u^2(y(s))dBs - k^2_{i,j}u^2_i\left(I^0_i + I^1_i Du^2_i\right) - k^2_{i+\frac{1}{2},j}u^2_{i+\frac{1}{2}}\left(I^0_{i+\frac{1}{2}} + I^1_{i+\frac{1}{2}}Du^2_{i+\frac{1}{2}}\right)\right),$$

and therefore because of the properties of stochastic integrals I^0 and I^1 we conclude that

$$E[(y(t_j) - Y_j)^2] = \left(\sum_{i=0}^{j-1}\left(\int_{t_i}^{t_{i+1}} k^1(s,t_j)u^1(y(s))ds - \frac{h}{6}(k^1_{i,j}u^1_i + 4k^1_{i+\frac{1}{2},j}u^1_{i+\frac{1}{2}} + k^1_{i+1,j}u^1_{i+1})\right)\right)^2$$
$$+ E\left[\left(\sum_{i=0}^{j-1}\left(\int_{t_i}^{t_{i+1}} k^2(s,t_j)u^2(y(s))dBs - k^2_{i,j}u^2_i(I^0_i + I^1_i Du^2_i) - k^2_{i+\frac{1}{2},j}u^2_{i+\frac{1}{2}}(I^0_{i+\frac{1}{2}} + I^1_{i+\frac{1}{2}}Du^2_{i+\frac{1}{2}})\right)\right)^2\right].$$

From Theorem 4.1, we can derive

$$\left(\sum_{i=0}^{j-1}\left(\int_{t_i}^{t_{i+1}} k^1(s,t_j)u^1(y(s))ds - \frac{h}{6}\left(k^1_{i,j}u^1_i + 4k^1_{i+\frac{1}{2},j}u^1_{i+\frac{1}{2}} + k^1_{i+1,j}u^1_{i+1}\right)\right)\right)^2 \leq C_1h^8.$$

Cosequently, we get

$$E\left[\left(\sum_{i=0}^{j-1}\left(\int_{t_i}^{t_{i+1}} k^2(s,t_j)u^2(y(s))dBs - k^2_{i,j}u^2_i(I^0_i + I^1_i Du^2_i) - k^2_{i+\frac{1}{2},j}u^2_{i+\frac{1}{2}}(I^0_{i+\frac{1}{2}} + I^1_{i+\frac{1}{2}}Du^2_{i+\frac{1}{2}})\right)\right)^2\right]$$
$$= \sum_{i=0}^{j-1} E\left[\int_{t_i}^{t_{i+1}} k^2(s,t_j)u^2(y(s))dBs - k^2_{i,j}u^2_i(I^0_i + I^1_i Du^2_i) - k^2_{i+\frac{1}{2},j}u^2_{i+\frac{1}{2}}\left(I^0_{i+\frac{1}{2}} + I^1_{i+\frac{1}{2}}Du^2_{i+\frac{1}{2}}\right)\right]^2.$$

From [16,21] and using the local error bound of the Milstein method, we have

$$E\left[\int_{t_i}^{t_{i+1}} k^2(s,t_j)u^2(y(s))dBs - k^2_{i,j}u^2_i(I^0_i + I^1_i Du^2_i) - k^2_{i+\frac{1}{2},j}u^2_{i+\frac{1}{2}}\left(I^0_{i+\frac{1}{2}} + I^1_{i+\frac{1}{2}}Du^2_{i+\frac{1}{2}}\right)\right]^2 \leq C_2h^3.$$

This leads to

$$E\left[\left(\sum_{i=0}^{j-1}\left(\int_{t_i}^{t_{i+1}} k^2(s,t_j)u^2(y(s))dBs - k^2_{i,j}u^2_i(I^0_i + I^1_i Du^2_i) - k^2_{i+\frac{1}{2},j}u^2_{i+\frac{1}{2}}(I^0_{i+\frac{1}{2}} + I^1_{i+\frac{1}{2}}Du^2_{i+\frac{1}{2}})\right)\right)^2\right] \leq C_2h^2.$$

Finally, with the above calculations, we obtain

$$E[(y(t_j) - Y_j)^2] \leq C_1h^8 + C_2h^2,$$

where C_1 and C_2 are constants. In the same way we can attain similar relation for $t_l = t_{j+\frac{1}{2}}$, and thus the proof of the theorem will be completed. □

So, according to the theorem 4.2 the presented finite difference method has pair order $(4, 1)$ when applied SVIEs (1).

Remark 4.3. In a special case, if we consider SVIEs (1) with a small noise term, it may be observed that the impact of the noise is not dominant, leading to notable enhancements in the characteristics of the proposed method within a deterministic framework.

5 Application to a system of Itô SVIEs with two dimensional noise

In this section we consider the following two-dimensional Itô SVIEs with two-dimensional noise

$$\begin{aligned} y^1(t) =& f^1(t) + \int_0^t k^1(s,t)u^1(y^1(s),y^2(s))ds + \int_0^t g^{1,1}(s,t)q^{1,1}(y^1(s),y^2(s))dB^1(s) \\ &+ \int_0^t g^{1,2}(s,t)q^{1,2}(y^1(s),y^2(s))dB^2(s), \qquad (14) \\ y^2(t) =& f^2(t) + \int_0^t k^2(s,t)u^2(y^1(s),y^2(s))ds + \int_0^t g^{2,1}(s,t)q^{2,1}(y^1(s),y^2(s))dB^1(s) \\ &+ \int_0^t g^{2,2}(s,t)q^{2,2}(y^1(s),y^2(s))dB^2(s), \qquad (15) \end{aligned}$$

in which $f^i(t)$ and kernels $k^i, u^i, g^{i,j}, q^{i,j}, i, j = 1, 2$, are known L_2 functions, while $y^i(t), i = 1, 2$, is the unknown L_2 functions and also $B(t) = (B^1(t), B^2(t))$ is a two-dimensional Brownian motion process. The numerical procedure presented in the previous section can be utilized to solve the two-dimensional problem Itô SVIEs (14)-(15). Therefore, setting $t = t_j, t_{j+\frac{1}{2}}$ results in

$$\begin{aligned} Y_j^\nu =& f_j^\nu + \frac{h}{6}\sum_{i=0}^{j-1}\left(k_{i,j}^\nu u_i^\nu + 4k_{i+\frac{1}{2},j}^\nu u_{i+\frac{1}{2}}^\nu + k_{i+1,j}^\nu u_{i+1}^\nu\right) \\ &+ \sum_{\ell=1}^{2}\sum_{i=0}^{j-1}\left(g_{i,j}^{\nu,\ell} q_i^{\nu,\ell} I_i^{(\ell)} + g_{i+\frac{1}{2},j}^{\nu,\ell} q_{i+\frac{1}{2}}^{\nu,\ell} I_{i+\frac{1}{2}}^{(\ell)}\right) \\ &+ \sum_{\ell_1,\ell_2=1}^{2}\sum_{i=0}^{j-1}\left(L_{i,j}^{\ell_1}\left(g^{\nu,\ell_2}q^{\nu,\ell_2}\right) I_i^{(\ell_1,\ell_2)} + L_{i+\frac{1}{2},j}^{\ell_1}\left(g^{\nu,\ell_2}q^{\nu,\ell_2}\right) I_{i+\frac{1}{2}}^{(\ell_1,\ell_2)}\right), \nu = 1, 2, \end{aligned} \qquad (16)$$

and also taking $t = t_{j+\frac{1}{2}}, j = 0, 1, 2, \cdots, N-1$, concluding

$$
\begin{aligned}
Y^{\nu}_{j+\frac{1}{2}} =& f^{\nu}_{j+\frac{1}{2}} + \frac{h}{24}\left(5k^{\nu}_{0,j+\frac{1}{2}}u^{\nu}_0 + 8k^{\nu}_{\frac{1}{2},j+\frac{1}{2}}u^{\nu}_{\frac{1}{2}} - k^{\nu}_{1,j+\frac{1}{2}}u^{\nu}_1\right) \\
&+ \frac{h}{6}\sum_{i=0}^{j-1}\left(k^{\nu}_{i+\frac{1}{2},j+\frac{1}{2}}u^{\nu}_{i+\frac{1}{2}} + 4k^{\nu}_{i+1,j+\frac{1}{2}}u^{\nu}_{i+1} + k^{\nu}_{i+\frac{3}{2},j+\frac{1}{2}}u^{\nu}_{i+\frac{3}{2}}\right) \\
&+ \sum_{\ell=1}^{2}\sum_{i=0}^{j-1}\left(g^{^{\nu},\ell}_{i,j+\frac{1}{2}}q^{^{\nu},\ell}_i I^{(\ell)}_i + g^{^{\nu},\ell}_{i+\frac{1}{2},j+\frac{1}{2}}q^{^{\nu},\ell}_{i+\frac{1}{2}}I^{(\ell)}_{i+\frac{1}{2}}\right) + \sum_{\ell=1}^{2} g^{^{\nu},\ell}_{j,j+\frac{1}{2}}q^{^{\nu},\ell}_j I^{(\ell)}_j \\
&+ \sum_{\ell_1,\ell_2=1}^{2}\sum_{i=0}^{j-1}\left(L^{\ell_1}_{i,j+\frac{1}{2}}\left(g^{^{\nu},\ell_2}q^{^{\nu},\ell_2}\right)I^{(\ell_1,\ell_2)}_i + L^{\ell_1}_{i+\frac{1}{2},j+\frac{1}{2}}\left(g^{^{\nu},\ell_2}q^{^{\nu},\ell_2}\right)I^{(\ell_1,\ell_2)}_{i+\frac{1}{2}}\right) \\
&+ \sum_{\ell_1,\ell_2=1}^{2} L^{\ell_1}_{j,j+\frac{1}{2}}\left(g^{^{\nu},\ell_2}q^{^{\nu},\ell_2}\right)I^{(\ell_1,\ell_2)}_j,
\end{aligned}
\tag{17}
$$

where operator L^{ϑ} is defined as [16,21]

$$
L^{\vartheta}_{\rho,\sigma}\left(g^{^{\nu},\ell}q^{^{\nu},\ell}\right) = GQ^{\vartheta}\cdot\nabla\left(g^{^{\nu},\ell}q^{^{\nu},\ell}\right)|_{t=t_\rho,t=t_\sigma}, \quad \nu,\ell,\vartheta = 1,2
$$

such that $GQ^{\vartheta} = \left(g^{1,\vartheta}q^{1,\vartheta}, g^{2,\vartheta}q^{2,\vartheta}\right)$ and $\nabla = (\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2})$ is operator of the gradient vector and the multiple Itô stochastic integrals $I^{(\ell)}_{\lambda}, I^{(\ell_1,\ell_2)}_{\lambda}$ is defined by

$$
I^{(\ell)}_{\lambda} = \int_{t_\lambda}^{t_{\lambda+\frac{1}{2}}} dB^{\ell}(s), \quad I^{(\ell_1,\ell_2)}_{\lambda} = \int_{t_\lambda}^{t_{\lambda+\frac{1}{2}}}\int_{t_\lambda}^{s} dB^{\ell_1}(t)dB^{\ell_2}(s). \tag{18}
$$

In order to proceed, we need to establish numerical approximations $Y^{\nu}_{\frac{1}{2}}, Y^{\nu}_1, \nu = 1,2$, first. So, from (16)-(17) we can solve a system of 4×4 nonlinear equations to find numerical estimations of $y^{\nu}(t_{\frac{1}{2}}), y^{\nu}(t_1), \nu = 1,2$.

Remark 5.1. According to the definition 2.1 and (18), we can put for the Itô stochastic integrals $I^{(\ell)}_{\lambda}$ that $I^{(\ell)}_{\lambda} = \int_{t_\lambda}^{t_{\lambda+\frac{1}{2}}} dB^{\ell}(s) = B^{\ell}(t_{\lambda+\frac{1}{2}}) - B^{\ell}(t_\lambda) = \sqrt{\frac{h}{2}}\mathcal{N}(0,1)$,

Remark 5.2. Since, some multiple Itô stochastic integrals $I^{(\ell_1,\ell_2)}_{\lambda}$ are not available, we must simulate them by some appropriate random variables denoted by $\hat{I}_{(\ell_1,\ell_2)}$ that satisfied moment condition (3) and defined by [3,27]

$$
\hat{I}_{(\ell_1,\ell_2)} = \begin{cases}
\frac{1}{2}\left(\hat{I}_{(\ell_1)}\hat{I}_{(\ell_2)} - \sqrt{\frac{h}{2}}\tilde{I}_{(\ell_1)}\right) & for \;\; \ell_1 < \ell_2 \\
\frac{1}{2}\left(\hat{I}_{(\ell_1)}\hat{I}_{(\ell_2)} + \sqrt{\frac{h}{2}}\tilde{I}_{(\ell_2)}\right) & for \;\; \ell_2 < \ell_1 \\
\frac{1}{2}\left(\hat{I}^2_{(\ell_1)} - \frac{h}{2}\right) & for \;\; \ell_1 = \ell_2
\end{cases} \tag{19}
$$

in which $\tilde{I}_{(\ell_1)}$ is defined by a two point distribution with $P\left(\tilde{I}_{(\ell_1)} = \pm\sqrt{\frac{h}{2}}\right) = \frac{1}{2}$ and we also choose $\hat{I}_{(\ell_1)}$ as three-point distributed random variables with $P\Big(\hat{I}_{(\ell_1)} = \pm\sqrt{\frac{3h}{2}}\Big) = \frac{1}{6}$ and $P(\hat{I}_{(\ell_1)} = 0) = \frac{4}{6}$.

6 Numerical simulations

The performance of the proposed finite difference method to solve the SVIEs of the second kind given by Eq. (1) is illustrated in this section. For various step sizes and $M = 10^3$ simulated trajectories, the numerical results of the Newton method after three iterations are depicted in Figures 1-3. For some test examples, the stochastic integral $\int_0^t g(s, B(s))ds$ is simulated by using the following MATLAB command which is denoted by "Algorithm 1".

Algorithm 2 Simulation of stochastic integral $\int_0^t g(s, B(s))ds$ in $\mathcal{I}_h$

```
clear all
 randn('state',50);
 format long
 T=1; dt=0.1; N=T/dt; t=0:dt:T;
 rnd1=randn(1,N);
 dWinc =sqrt(dt)*rnd1;
 W=cumsum(dWinc);W=[0,W];
 pp2=interp1(t,W,'linear','pp');
 [breaks,coefs,L,order,dim] = unmkpp(pp2);
 In=zeros(1,N);
 syms s
 for i=1:length(t)-1
   P_{i}= coefs(i,1)*(s-breaks(i))+coefs(i,2);
   In(i)=int(g(s,P_{i}),t(i),t(i+1));
 end
 StochasticInt=[0,cumsum(In)];
```

For comparison of the methods we measured the mean absolute error

$$MAE_Y = E[|y(T) - Y_N|] \approx \frac{1}{M}\sum_{\ell=1}^{M} |y_{\ell,T} - Y_{\ell,N}|,$$

in which $y_{\ell,T}$ and $Y_{\ell,N}$ refer to $y(T)$ and Y_N in ℓth simulation, respectively.

Example 1. As first example we consider the following SVIE

$$y(t) = 1 + \int_0^t \lambda y(s)ds + \int_0^t \mu y(s)dB(s), \quad t \in [0, T], \tag{20}$$

with the exact solution $y(t) = \exp\left((\lambda - \frac{1}{2}\mu^2)t + \mu B(t)\right)$.

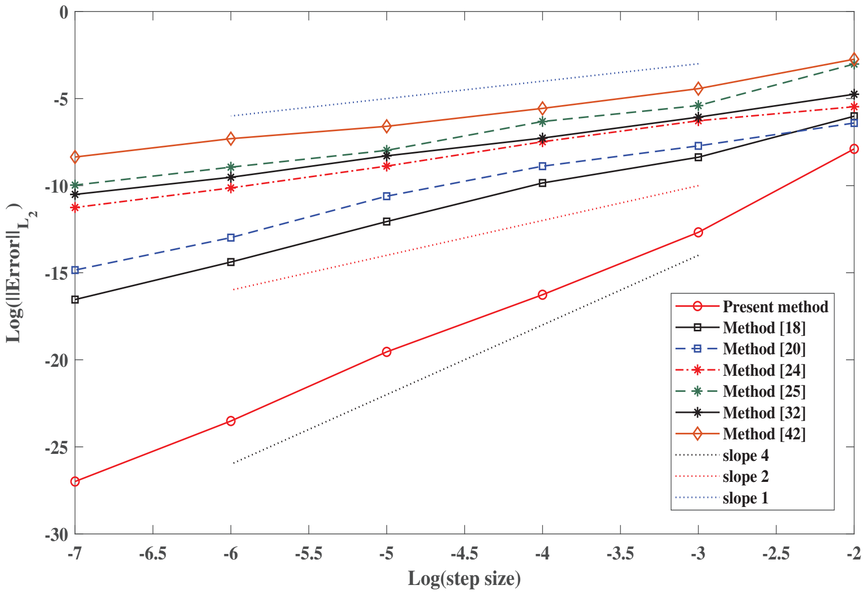


Figure 1: Numerical results of the methods for Example 1.

Our first test problem concerned the Itô SVIEs with small noise, i.e we consider small values of μ ($0 < \mu \ll 1$). In order to investigate the influence of the deterministic order p_D on SVIEs with small noise, we consider $\mu = .005, \lambda = -2, h = 2^{-i}, i = 2, \cdots, 7$, and then we have plotted the L_2 error versus the step size in logarithmic scale with base 2. Also, lines with slopes $1, 2$ and 4 are provided in the resulting figure to enable comparisons with convergence of these orders. It can be observed that, the slope of the resulting lines corresponds to the obtained order of the methods. Computational results of the presented method with several methods which are plotted in Fig. 1 show that the proposed method has performed higher order of convergence in compared with other methods [13, 14, 17, 18, 23, 30] when applied to the Itô SVIEs (20).

Example 2. Consider the following SVIE [13, 14, 18, 23, 28, 29, 31]

$$y(t) = \frac{1}{12} + \int_0^t \cos(s)y(s)ds + \int_0^t \sin(s)y(s)dB(s), \quad t \in [0, T], \tag{21}$$

with the exact solution $y(t) = \frac{1}{12} \exp\left(-\frac{t}{4} + \sin(t) + \frac{\sin(2t)}{8} + \int_0^t \sin(s)dB(s)\right)$.

For $T = 1$ and various step sizes, numerical results of the presented finite difference method to solve the problem given by Example 2 is provided in Table 1. The approximate solution of this SVIE has also provided in [13, 14, 18, 23, 28, 29, 31]. The maximal error of the methods given in [13, 14, 18, 23, 29, 31] is of order 10^{-3}, while the mean absolute error of the proposed method is 1.65871×10^{-4} for $h = 0.01$.

By using 12221 computational knots and $h = 0.01$, the maximum bound of error of the iterative technique given in [28] with $N_1 = 20, N_2 = 100$ is 7.755×10^{-4}. While the presented new method with $h = 0.01$ uses only 201 knots to achieve the mean absolute error of order 10^{-4}. Furthermore, Table 2 displays the absolute errors obtained through both the proposed method and the method described in [31]. The results indicate that the current method yields good results when compared to the method of [31]. Consequently, the proposed method is more accurate than the methods proposed in [13, 14, 18, 23, 28, 29, 31].

Table 1: Mean absolute error (MAE_Y) for Example 2.

Step sizes	Mean absolute error MAE_Y
0.50	1.04571×10^{-1}
0.25	2.17340×10^{-2}
0.10	5.48051×10^{-3}
0.05	9.71956×10^{-4}
0.01	1.65871×10^{-4}

Table 2: Absolute errors obtained by the proposed method and method [31] for Example 2.

	Proposed method	Method of [31]
t	$h = .05$	$n = 7, \gamma = 0.5, \nu = -0.5$
0	0	2.7150e-11
0.05	1.0040e-04	5.7904e-03
0.10	9.2254e-05	1.8741e-03
0.15	1.3108e-04	3.1878e-03
0.20	2.7842e-05	6.2100e-03
0.25	8.0943e-05	1.3593e-02
0.30	5.2180e-05	6.0900e-04
0.35	6.1810e-06	1.9003e-02
0.40	2.1675e-05	9.5997e-03
0.45	3.1338e-05	4.6005e-02
0.50	3.5758e-04	6.2649e-02

Example 3. Consider the following nonlinear SVIE [28]

$$y(t) = 1 + \int_0^t (-\sin(2y(s)) - \frac{1}{4}\sin(4y(s)))ds + \int_0^t \sqrt{2}\cos^2(y(s))dB(s), \quad t \in [0, T], \tag{22}$$

in which its exact solution is $y(t) = \arctan(\tan(1)e^{-t} + \sqrt{2}\int_0^t e^{s-t}dB(s))$.

For $T = 1$ and step sizes $h = \frac{1}{30}, \frac{1}{80}, \frac{1}{100}, \frac{1}{150}$, the numerical results of the proposed method and the method of [28] are presented in Table 3. As this Table shows that the new finite difference method is more accurate than the iterative method given in [28].

Table 3: Comparison of the MAE_Y of the present method and method given in [28] to solve the Example 3.

h	Iterative technique [28] with $N_1 = 20, N_2 = 100, n = 8$	Present method
$\frac{1}{30}$	1.024×10^{-1}	2.371×10^{-2}
$\frac{1}{80}$	7.727×10^{-2}	8.004×10^{-3}
$\frac{1}{100}$	5.772×10^{-2}	2.526×10^{-3}
$\frac{1}{150}$	4.435×10^{-2}	5.174×10^{-4}

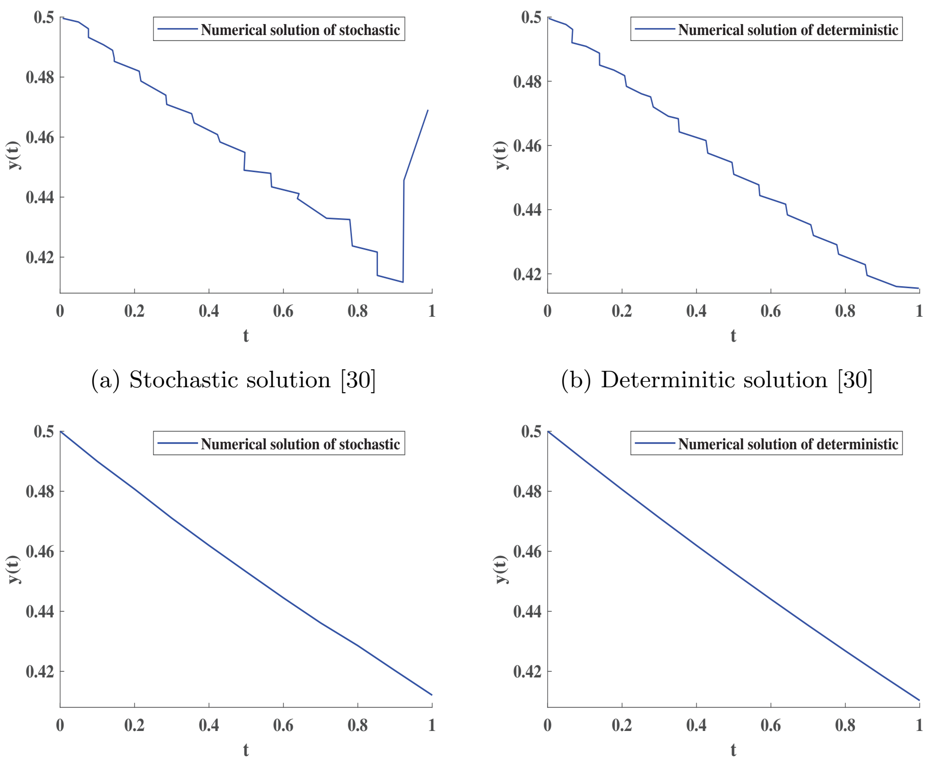
Example 4. Consider the nonlinear Cox-Ingersoll-Ross SVIE as

$$y(t) = y_0 + \int_0^t \alpha(\beta - y(s))ds + \int_0^t \sigma\sqrt{y(s)}dB(s), \quad t \in [0, T], \tag{23}$$

where $y_0 \geq 0$ and $\alpha, \sigma > 0, \beta \in \mathbb{R}$ are the initial value and parameters of model, respectively.

The Cox-Ingersoll-Ross SVIE often arises in mathematical finance and uses to describe the time evolution of interest rates, furthermore this SVIEs has a unique non-negative strong solution [8,12]. To show the efficiency of the proposed method to solve the SVIE (23) with a small noise, the initial value and parameters of this model are selected as $y_0 = .5, \alpha = .2, \beta = .005$ and $\sigma = .002$. The numerical results of the presented method and the method [30] to solve the SVIE (23) with $\sigma = .002$ (stochastic case) and $\sigma = 0$ (deterministic case) are plotted in Figure 2. In this figure we can see that the presented method is more accurate than the method given in [30] for both stochastic and deterministic cases. In Figure 3, the numerical results of the presented method to solve the Example 4 are depicted for large final time $T = 50$ and the step size $h = 0.1$. It shows that the new method preserves the positivity of the solution, experimentally.

Example 5. As last example, we consider two-dimensional Itô SVIEs with two-



(a) Stochastic solution [30]

(b) Determinitic solution [30]

(c) Stochastic solution

(d) Determinitic solution

Figure 2: (a)-(b) The plots of approximate solution of the method given in [30] and (c)-(d) the plots of approximate solution of the proposed method to solve Example 4.

dimensional noise

$$y^1(t) = \frac{1}{2} + \int_0^t y^2(s)ds + \int_0^t \theta dB^1(s), \tag{24}$$

$$y^2(t) = \frac{1}{2} - \int_0^t y^1(s)ds + \int_0^t \eta dB^2(s), \tag{25}$$

where $\theta, \eta \in \mathbb{R}$. Such problem is a model for a vibrating string subject to a stochastic force [25]. The exact solution of the above system of Itô SVIEs is

$$y^1(t) = \frac{1}{2}\left(\cos(t) + \sin(t)\right) + \int_0^t \theta \cos(t-s)dB^1(s) + \int_0^t \eta \sin(t-s)dB^2(s), \tag{26}$$

$$y^2(t) = \frac{1}{2}\left(\cos(t) - \sin(t)\right) - \int_0^t \theta \sin(t-s)dB^1(s) + \int_0^t \eta \cos(t-s)dB^2(s). \tag{27}$$

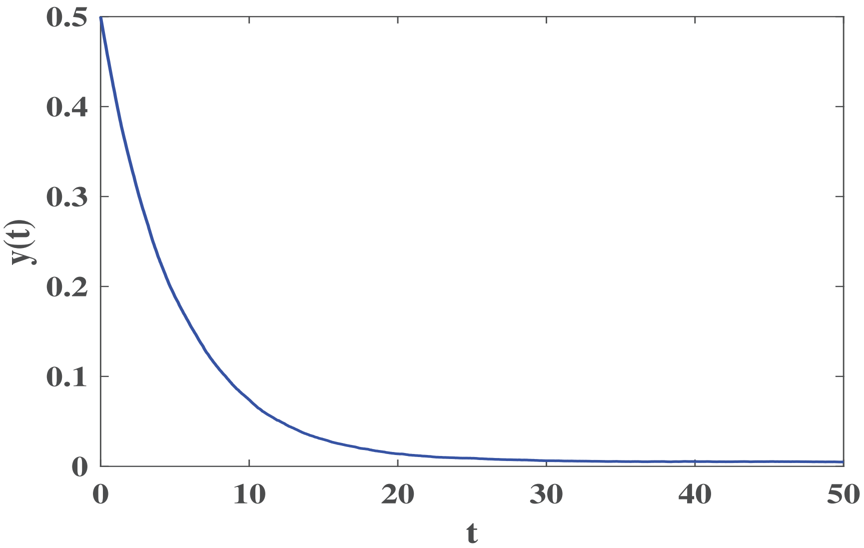


Figure 3: Numerical results of the proposed method for Example 4.

It should be mentioned that for the values of $0 < \theta, \eta \ll 1$ we face to the Itô SVIEs problem with small noise.

To solve this example, we consider $\theta = \eta = 0.01$ and use the semi-implicit Euler-Maruyama method and original Milstein method for comparing with the proposed method. It should be mentioned that the influence of the small values of θ and η is effective on the accuracy of the numerical method for this system. Also, the reported numerical results of this example in Table 4, that confirms this, exhibit that the proposed method is more accurate than the other. Furthermore, the pair order of the corresponding numerical methods semi-implicit Euler-Maruyama, Milstein and proposed method is $(1, \frac{1}{2}), (1, 1)$ and $(4, 1)$, respectively.

Table 4: Comparison of the MAE_{Y^1} of the present method with other to solve the Example 5.

h	semi-implicit Euler-Maruyama method	Milstein method	Present method
$\frac{1}{4}$	6.5661×10^{-1}	7.7278×10^{-1}	8.2028×10^{-2}
$\frac{1}{8}$	3.0428×10^{-1}	8.3014×10^{-2}	1.3946×10^{-3}
$\frac{1}{32}$	8.2374×10^{-2}	9.0779×10^{-3}	6.0638×10^{-4}
$\frac{1}{64}$	1.8978×10^{-2}	3.8225×10^{-3}	9.2874×10^{-5}

7 Conclusions

In this study we have introduced a finite difference method for the strong approximation of SVIEs in the Itô sense. The pair order of the new method is $(4, 1)$,i.e. the order of the convergence of the method is 4 when apply to corresponding Volterra integral equations (deterministic case) and also has first order of the strong convergence when applied stochastic Volterra integral equations. Also, the computational cost of the presented method is less than the operational matrix based methods, because we only need to solve a 2×2 system of equations at each step of numerical simulation for one-dimensional SVIEs. Finally, some numerical examples are prepared to exhibit the verity of the presented finite difference method. The numerical simulations demonstrates the proposed method is accurate than those methods given in [13, 14, 18, 23, 28–30].

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