Modeling of mortgage-backed securities based on stochastic processes

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Abstract:
In this paper, we first present a nonlinear structural model for pricing mortgage-backed securities. These derivatives are considered to be the primary cause of the 2008 financial crisis that was raised in the United States. We focus our work on pass-through mortgages, which pay both the principal and interest to the investors. We begin our work by introducing the factors that affect the market of mortgage-backed securities. Then, by applying some assumptions and conditions to the parameters of the initial model, and without the loss of generality, we show that this model can be greatly simplified. We focus our attention on how the change in interest rates can affect the value of mortgage-backed securities. Various numerical methods can be used to solve the reduced model that is achieved. We adapt the mesh-less method of radial basis functions to solve the reduced model. The numerical results indicate that the method that we have used can capture the market trends in a specific interval.

Keywords: Mortgage backed security, Reduced modeling, Radial Basis Functions, Prepayment, Financial crisis.
MSC2010 Classification: 91G20, 65D12, 91G30.

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Received: 2021-08-16 Approved: 2021-11-23

Introduction

Modeling complex real-world phenomenon is often very difficult or impossible due to the focus on complex or nonlinear processes with a large number of variables. This shows the great importance of reduced modeling in the applied sciences which deal with such phenomena.

One of the methods used to simplify complex models is to apply various limitations to model parameters. This is particularly favorable when applying these restrictions is feasible in the real world. This approach has been carried out successfully in analyzing partial differential equations that appear in fluid convection. [4, 15] In this paper, we focus on partial differential equations used for the pricing of mortgage-backed securities (abbreviated MBS).

An asset-backed security is a type of investment in which its cash flow is derived from a pool of assets as collateral. Sometimes mortgage-backed securities are considered as a part of asset-backed securities but they are usually considered as independent investments. These securities are debt instruments that represent the rights of the holder on the cash flow received by the issuer. They are usually formed by pooling a group of mortgages together and then selling this pool as a security to investors. It should be noted that mortgage-backed securities existed long before asset-backed securities; therefore in our work we consider them as independent investment instruments.

Institutions that issue MBS buy mortgage-backed securities documents from various banks and other loan providers. They classify these loans according to their common characteristics and then divide them into different tranches based on the risk and interest rate, and then issue securities and sell them to investors. This enables investors to choose which MBS to invest in according to their risk tolerance. Institutes use the money obtained from selling MBS in the process of making new loans. In fact, this measure provides the necessary capital to provide loans to new applicants.

The cash flow of mortgage-backed securities includes principal and interest of the pool of mortgages and the value is quoted as a percentage of the underlying mortgage balance and the price observed in the market. Prepayments on these bonds are usually made on a monthly basis until the maturity.

Since the mortgagors have the option to prepay more than the monthly installments, at any time before the maturity, the monthly cash flow from the mortgage-backed securities is not guaranteed, and in particular, it may stop.

Mortgage-backed securities were considered the primary cause of the 2008 economic recession and financial crisis in the United States. Thus it is beneficial for both academics and practitioners to gain a further understanding of the dynamics of mortgage-backed securities.

One of the most notable aspects of financial mathematics is the continuous interaction between theory and practice. This has been given much importance ever since the seminal work of Black and Scholes on the pricing of derivative securities with partial differential equations. We can point out the progressive evolution and improvement of the market of mortgage-backed securities and the emphasis of researchers on the importance of various types of derivatives including MBSs as an example of such interactions.

Mortgage-backed securities, as we know them today, were first issued by the United States government in 1968, and have continued to grow and develop ever since. The main purpose of mortgage-backed securities was to provide housing funds for more people, create stability, and increase liquidity and investment in the housing market of the United States.

Over the years, there have been many changes in the structure of MBS and the underlying asset, in the markets. However, researchers have always focused on two main categories of models; structural models, and reduced models for pricing of the mortgage-backed securities. A reduced model is based on a structural model, but they differ in the economic estimates used.

Historically the focus of studies on mortgage-backed securities initially focused on the structural models. The first structural model for analyzing GNMA MBS was performed by Dunn and McConnell (1981) in which they valued mortgages as individual loans. [12] In 1985, Brennan and Schwartz proposed a model based on long-term and short-term interest rates. [12] They were among the first to encourage their fellow researchers to develop reduced models.

Schwartz and Torous first proposed a reduced model for mortgage-backed securities in 1989, in which the prepayment function was estimated empirically using the previously proposed model. [12, 20] McConnell and Singh (1993) then presented a redused model using the Black

Government National Mortgage Association and Scholes model, along with the prepayment function used by Schwartz and Torous, and derived a reduced model. [11, 12]

In 2004, Marco Papi and Maya Briani introduced a new model based on partial differential equations for mortgage-backed securities. They proved that the price of mortgage-backed securities could be represented as the solution to a parabolic semi-linear equation. They also proved the existence, uniqueness, and regularity results in the framework of viscosity solutions. [14]

In 2008 house prices in the United States increased, loan defaults escalated and the institutions that had the support of the government lost billions of dollars. Many attributed the problems that led to the 2008 financial crisis to the complexity of the financial models of mortgage-backed securities. Critics hold the inadequacy of models accountable as the main factor that led banks and other investors to take risks that they could not manage. This led financial institutions to pay more attention to the use of models provided by the researchers in the field of financial mathematics, which resulted in the revival of mortgage-backed securities to the market in 2011.

The process of real estate pricing in Iran is such that over time, we usually witness price increases, sometimes with a steep and sometimes with a slow slope. This increase in prices leads to serious problems for the majority of the population. On the other hand, due to the limitations in the financial resources of banks and financial institutions, the loans they can provide or offer are also limited. These conditions make it almost impossible for many people to own a house. The definition of mortgage-backed securities allows banks and financial institutions to provide housing loans to more people.

In this paper, we intend to first apply the assumptions suggested by Parshad (2010) and implement them on the model proposed by Brianni and Papi (2004) and without loss of generality, derive a reduced model for the pricing of mortgage-backed securities. [14, 15] We then solve the model using the numerical method of radial basis functions. (abbreviated RBF)

The organization of this paper is as follows. In section 2 we first derive a partial differential equation for pricing of mortgage-backed securities. Then by considering some assumptions on the parameters, we simplify the PDE to a reduced diffusion equation. In section 3 we introduce the radial basis functions method and employ it to solve the reduced model.

In section 4 we solve an example and finally in section 5 the conclusion is presented.

Mortgage-Backed Securities Modelling

As mentioned in the introduction, in this section we intend to provide a model for mortgage-backed securities. To proceed we consider a market of k mortgage-backed securities with the same maturity T. The asset price is presented by V_t. Investors can invest in the riskless asset V_t^* or a finite set of mortgage-backed securities. Therefore, the asset price vector is as follows:

$$V_t = (V_t^*, V_t^{\text{MBS}}) \equiv (V_t^*, V_t^{\text{MBS},1}, ..., V_t^{\text{MBS},k}), \quad t \in [0, T].$$

The price of mortgage-backed securities depend on various economic factors including interest rate level, the value of the houses, and possible prepayment specific variables such as divorce, death of a spouse, etc. As the first step, we introduce the model assumptions and the assumptions on the dynamics of the stochastic processes affecting mortgage-backed securities prices.

We assume that $x \in \mathbb{R}^N$ indicates all the various economic factors, such as interest rates, that affect the price of mortgage-backed securities. There exists a collection of stochastic processes $\{X_t^x : t \in [0, T]\}_x$ satisfying the following model of Brownian motion:

$$dX_t^x = \mu(X_t^x, T - t)dt + \sigma(X_t^x, T - t)dW_t, \tag{1}$$

where $X_0^x = 0$ and the coefficients μ and σ are mean and volatility which are continuous in $\mathbb{R}^N \times [0, T]$ and we consider them to be constant.

$S(x, t)$ is the prepayment function, where $S \in C^{1,2}(\mathbb{R}^N \times [0, T])$, and also $S \geq 0$. Generally, there is no closed-form for this function and it is often constructed using empirical data. However, one of the popular approaches for the form of this prepayment function is: [20]

$$S(x, t) = g(t) \exp\big(\sum_{j=1}^{n} \beta_j x_j\big), \tag{2}$$

here g is given by

$$g(t) = \frac{p\gamma(\gamma t)^{p-1}}{1 + (\gamma t)^p}, \tag{3}$$

where γ and p are appropriately chosen parameter values.

We assume that $h(x,t)$ represents the mortgage's principal at time t. We define $S(x,t)$ the $\mathcal{F}_t$-adopted prepayment process so that $h(x,t)$ is

$$h(x,t) = MB(t)e^{-S(x,t)}, \tag{4}$$

here $MB(t)$ is the remaining principal at time t, if there are no prepayments, we have

$$MB(t) = MB(0)\frac{e^{\tau' T} - e^{\tau' t}}{e^{\tau' T} - 1}, \quad t \in [0,T] \tag{5}$$

where τ' is the fixed rate paid by the mortgagor, whereas τ is the fixed rate coupon that the investor recieves, and also we have $\tau < \tau'$.

G_t^i is the gain process associated with the i-th asset in the mortgage-backed securities market, and has the following form

$$G_t^i = V_t^i + \int_0^t dc_s^i, \qquad \forall i = 1,...,k \tag{6}$$

where dc_t^i is defined as the i-th cash flow process of the securities. The process of securities cash flow for the pass-through securities is $dc_t = \tau h_t dt - dh_t$. For any $t \in [0,T]$ and $i = 1,...,k$; we have the following representation

$$dG_t^i = \delta(t)V_t^i dt + D_t G_t^i d\widehat{W}_t, \tag{7}$$

where $\delta(t)$ is the continuous risk-less discount rate and $D_t G_t^i$ for the pass-through securities, that pay both the principal and interets, is obtained by the following equation:

$$D_t G_t^i = -\mathbb{E}^{\mathbb{Q}}[\int_t^T \big(\tau_i - \delta(s)\big) e^{\int_t^s \delta(u)du} h_s^i \Big(D_t S_s^i + \int_t^s D_t \gamma_u d\widehat{W}_u\Big) ds|\mathcal{F}_t]. \tag{8}$$

For $i = 1,...,k$, there exist functions $u_i \in C^{1,2}(\mathbb{R}^N \times [0,T])$ which are the price functions of mortgage-backed securities, such that $u_i \geq -h_{0,i}$ and we have:

$$V_t^i(x) = u_i(x,t) + h(x,t), \quad \forall t \in [0,T], \quad x \in \mathbb{R}^N. \tag{9}$$

The price of mortgage-backed securities as mentioned earlier is affected by a variety of economic factors, including interest rates, value of the house, inflation rates, and so on. In this study, we assume that all factors except interest rates are fixed.

An equilibrium for the mortgage-backed securities market is a d-dimensional process $\gamma(x)$, adapted to the filtration of W. If we define $\mathbb{Q}$ as the risk neural measure associated to $\gamma(x)$; then for each $t \in [0,T]$, relation (7) holds true, and therefore we have:

$$\gamma_t(x) = \rho \frac{\sum_{i=1}^k D_t G_t^i(x)}{\sum_{i=1}^k V_t^i(x) + V_t^*}, \quad \forall t \in [0,T]. \tag{10}$$

By using relations (10) and (7), for every $s \in [0,T]$ and $i = 1, ..., k$, we have

$$\langle \sigma_{G,i}(s), \gamma_s(x) \rangle = \mu_{G,i}(s) - \delta(s) V_s^i \tag{11}$$

We suppose ∇f to be the gradient of f with respect to x, and $\nabla^2 f$ to be the Hessian matrix of f. By applying Ito's lemma to (6) the differential of the gain process is obtained from the following relation:

$$\begin{aligned} dG_s^i &= d\big(u_i(X_s,s) + h_{0,i}(X_s,s)\big) + \tau_i h_{0,i}(X_s,s)ds - dh_{0,i}(X_s,s) \\ &= \Big[\langle \nabla u_i(X_s,s), \mu_0(X_s,s)\rangle + \frac{\partial u_i}{\partial s}(X_s,s) + \frac{1}{2} tr\big(\sigma_0 \sigma_0^\top \nabla^2 u_i(X_s,s)\big) \\ &\quad + \tau_i h_{0,i}(X_s,s)\Big] ds + \nabla^\top u_i(X_s,s)\sigma_0 dW_s \end{aligned} \tag{12}$$

therefore

$$\begin{aligned} \sigma_{G,i}(s) &= \nabla^\top u_i(X_s,s)\sigma_0(X_s,s) \\ \mu_{G,i}(s) &= \langle \nabla u_i, \mu_0 \rangle + \frac{\partial u_i}{\partial s} + \frac{1}{2} tr(\sigma_0 \sigma_0^\top \nabla^2 u_i) + \tau_i h_{0,i} \end{aligned}$$

Now By using relations (9) and (10), we have:

$$\gamma_s(x) = \rho \frac{\sum_{j=1}^k \sigma_0^\top(X_s,s) \nabla u_j(X_s,s)}{\sum_{j=1}^k \big[u_j(X_s,s) + h_{0,j}(X_s,s)\big] + V_s^*}. \tag{13}$$

By substituting the relations we had so far to (11), for each $s \in [0,T]$, we obtain

$$\begin{aligned} &\rho \Big\langle \sigma_0^\top(X_s,s)\nabla u_i(X_s,s), \frac{\sum_{j=1}^k \sigma_0^\top(X_s,s) \nabla u_j(X_s,s)}{\sum_{j=1}^k \big[u_j(X_s,s) + h_{0,j}(X_s,s)\big] + V_s^*} \Big\rangle \\ &= \langle \nabla u_i(X_s,s), \mu_0(X_s,s)\rangle + \frac{1}{2} tr\big(\sigma_0 \sigma_0^\top \nabla^2 u_i(X_s,s)\big) + \tau_i h_{0,i}(X_s,s) \\ &\quad + \frac{\partial u_i}{\partial s}(X_s,s) - \delta(s)\big(u_i(X_s,s) + h_{0,i}(X_s,s)\big) \end{aligned} \tag{14}$$

Let us make the following change of variable $s = T - t$ so that we have:

$$h(x,t) = h_{0,1}(x, T-t), \quad r(t) = \delta(T-t), \quad U(x,t) = u_1(x, T-t),$$

and

$$\xi(t) = A_0 e^{\int_0^{T-t} \delta(s) ds}.$$

By applying these changes the relation (14) reduces to the following equation:

$$\frac{\partial U}{\partial t} - \frac{1}{2} tr(\sigma(x,t)\sigma^\top(x,t)\nabla^2 U) - \langle \mu(x,t), \nabla U \rangle + \rho \frac{|\sigma^\top(x,t)\nabla U|^2}{U + h(x,t) + \xi(t)} + \tau h(x,t) - r(t)\big(U + h(x,t)\big) = 0 \tag{15}$$

With $U(x,0) = 0$, everywhere in $\mathbb{R}^N \times [0,T]$. Obviously to be consistent with the financial market, we require that $U + h$ be nonnegative (i.e. $V \geq 0$), also U needs to satisfy the stochastic representation

$$U(X_t^x, T-t) = \mathbb{E}^{\mathbb{Q}}\Big[\int_t^T \big(\tau - r(T-s)\big) e^{-\int_t^s r(T-\kappa) d\kappa} h(X_s^x, T-s) ds \Big| \mathcal{F}_t\Big] \tag{16}$$

Now using the partial differential equation obtained in (15) we can derive our reduced diffusion equation. By the definition of reduced modeling and considering that the price of mortgage securities depend on various factors, we first consider some assumptions on the model.

We consider an economic situation of decreasing interest rates. A decrease in interest rates affects the mortgagor's behavior and consequently the MBS price. We assume x approaches to zero. We know that the interest rate cannot hit zero in reality, but this assumption is merely a theoretical construction to gain some insight into the behavior of mortgage-backed securities prices in a situation of sharp fall in interest rates.

We also consider a limiting situation for the prepayment function (2) and we have:

$$S(x,t) = \lim_{x \to 0} g(t) e^{\beta x} = g(t) = S(t).$$

We assume the function $u(x,0) \in L^2(U)$ is locally bounded and takes zero as the boundary value. For convenience, we further assume the problem on some bounded domain Ω and not the whole domain ($\Omega \in \mathbb{R}^3$). Also, we assume $\sigma\sigma^T = I$ and $r(t) = r$ be a fixed discount rate.

Once the above assumptions are applied to (15) we get the following equation which is greatly simplified

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \mu \nabla u - \rho \frac{|\sigma^T \nabla u|^2}{u + h(t) + \xi(t)} - (r - \tau)h(t) - ru, \qquad \text{in } U \tag{17}$$

$$u = 0 \qquad \text{on } \partial U$$

$$u(x,t) = u_0(x)$$

We first begin by making the substitution

$$\upsilon(x,t) = u(x,t) + h(t) + \xi(t) \tag{18}$$

by inserting $\upsilon(x,t)$ in (17) yields

$$\frac{\partial \upsilon}{\partial t} - \frac{1}{2}\frac{\partial^2 \upsilon}{\partial x^2} - \mu \frac{\partial \upsilon}{\partial x} + \rho \frac{|\frac{\partial \upsilon}{\partial x}|^2}{\upsilon} + r\upsilon = F(t) \tag{19}$$

here

$$F(t) = \tau h(t) + r\xi(t) + h'(t) + \xi'(t) \tag{20}$$

Next, by making a set of substitutions, and inserting them into the equivalent equations, we obtain the following equation:

$$\frac{\partial k}{\partial t} - \frac{1}{2}\frac{\partial^2 k}{\partial x^2} = \left[(1-2\rho)F(t)\left(e^{(\frac{1}{2\rho-1})\left(((2\rho-1)r - \frac{\mu^2}{2})t + \mu x\right)}\right)k^{(\frac{2\rho}{2\rho-1})}\right]. \tag{21}$$

Note that via the above transforms for $u(x,t)$ we have

$$u(x,t) = (e^{(\frac{1}{2\rho-1})\left(((2\rho-1)r - \frac{\mu^2}{2})t + \mu x\right)})(k(x,t))^{\frac{1}{1-2\rho}} - h(t) - \xi(t) \tag{22}$$

Hence on ∂U we have

$$u(x,t) = 0 = (e^{(\frac{1}{2\rho-1})\left(((2\rho-1)r - \frac{\mu^2}{2})t + \mu x\right)})(k(x,t))^{\frac{1}{1-2\rho}} - h(t) - \xi(t) \tag{23}$$

This implies that on the boundary of U:

$$k(x,t) = (h(t) + \xi(t))^{1-2\rho} e^{-(((2\rho-1)r - \frac{\mu^2}{2})t + \mu x)} \tag{24}$$

Then we homogenize it by setting

$$l(x,t) = (h(t) + \xi(t))^{1-2\rho} e^{-(((2\rho-1)r - \frac{\mu^2}{2})t + \mu x)} \tag{25}$$

and considering the function $V(x,t)$ to be

$$V(x,t) = k(x,t) - l(x,t) \tag{26}$$

finally by inserting this into (21) yields

$$\frac{\partial V}{\partial t} = \frac{1}{2}\frac{\partial^2 V}{\partial x^2} + \tilde{K}(V,t), \tag{27}$$
$$V = 0 \quad \text{on } \partial U \tag{28}$$
$$V(x,0) = V_0(x) \tag{29}$$

here

$$\begin{aligned}\tilde{K}(V,t) = (1-2\rho)F(t)(e^{(\frac{1}{2\rho-1})\left(\left((2\rho-1)r-\frac{\mu^2}{2}\right)t+\mu x\right)})(V+l)^{(\frac{2\rho}{2\rho-1})} \\ -\frac{\partial l}{\partial t} + \frac{1}{2}\frac{\partial^2 l}{\partial x^2}\end{aligned} \tag{30}$$

We call equations (27) to (29) a reduced model for mortgage-backed securities. Many methods exist for solving this model; in the next section, we chose the radial basis method.

Radial Basis Functions Method

In recent years, the radial basis functions methods have been used to solve partial differential equations, due to the flexibility while maintaining the geometry of the computational domain. RBF is a meshless method and a good alternative for the classical interpolation methods such as polynomial or spline interpolation, especially in higher orders, where the traditional methods do not work or are difficult to use. The main idea of the RBF method is to approximate the solution of a equation using a linear combination of infinitely differentiable radial basis functions ϕ. One of the characteristics of this method, in addition to not requiring the generation of a grid, is obtaining a high order of convergence, and that the computational domain is composed of scattered collocation points.

The radial basis functions method was first introduced by Hardy (1971) to approximate a two-dimensional surface in geometry. Hardy employed MQ functions to approximate 2-dimensional geographical surfaces. The MQ function was stated to be one of the best functions among all the other interpolation schemes based on accuracy, stability, efficiency, required memory, and ease of interpolation. [8]

Kansa (1990) was able to successfully modify the MQ to solve partial differential equations of elliptic, parabolic, and hyperbolic types. Thus he became a pioneer in solving partial differential equations by using the RBF method. [9]

However, it should be noted that the efficiency of the MQ is greatly affected by the choice of ϵ, which is a positive parameter and in the case of MQ is commonly called the shape parameter.

Table 1: Radia Basis Functions

RBF	$\phi_j(y)$
Gausian	$\exp(-\epsilon^2(y-y_j)^2)$
Multi-quadratic	$\sqrt{(y-y_j)^2+\epsilon^2}$
Invese multi-quadratic	$\frac{1}{\sqrt{(y-y_j)^2+\epsilon^2}}$
Thin plate spline	$(y-y_j)^2\log(\lvert y-y_j\rvert)$

To solve the reduced mortgage-backed securities model derived in the previous section we use the radial basis functions scheme.

We first interpolate the function V by using the following radial basis function

$$V(x,t) \simeq \sum_{j=1}^{N} \alpha_j(t)\phi(|x-x_j|) \tag{31}$$

here α_j are unknown coefficients that depend on time, and $\phi_j(x) = \phi(|x-x_j|)$ are radial basis functions, that $|x-x_j|$ denotes the radial distance of the N scattered data points x_j. By substituting (31) into (27) and (30), for $i = 1, ..., N$, we obtain the following system of linear equations

$$\frac{\partial V(x_i,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 V(x_i,t)}{\partial x^2} + \tilde{K}(V,t) \tag{32}$$

$$\tilde{K}(V,t) = (1-2\rho)F(t)e^n(V(x_i,t)+l)^m - \frac{\partial l}{\partial t} + \frac{1}{2}\frac{\partial^2 l}{\partial x^2}$$

where

$$n = (\frac{1}{2\rho-1})\Big(((2\rho-1)r - \frac{\mu^2}{2})t + \mu x\Big), \quad \text{and} \quad m = \frac{2\rho}{2\rho-1}.$$

Also here $l = l(x,t)$ is a known function, obtained from the computations in the previous section to achieve the reduced model, and is introduced in equation (25).

Since the radial basis function ϕ does not depend on time, the time derivative of V is simply given in terms of the time derivatives of the coefficients;

$$\frac{\partial V(x_i,t)}{\partial t} = \sum_{j=1}^{N} \frac{d\alpha_j(t)}{dt}\phi(x_i,x_j). \tag{33}$$

In our reduced model, we only need the second partial derivative of V with respect to x, which is given as follows:

$$\frac{\partial^2 V(x_i,t)}{\partial x^2} = \sum_{j=1}^{N} \alpha_j(t)\frac{\partial^2\phi(x_i,x_j)}{\partial x^2}. \tag{34}$$

Now among the most commonly used radial basis functions, we use the MQ interplant as the basis function. Thus when ϕ is multi-quadratic, we have

$$\phi(x_i,x_j) = \sqrt{(x_i-x_j)^2+\epsilon^2} \tag{35}$$

$$\frac{\partial^2\phi(x_i,x_j)}{\partial x^2} = \frac{1}{\sqrt{(x_i-x_j)^2+\epsilon^2}} - \frac{(x_i-x_j)^2}{((x_i-x_j)^2+\epsilon^2)^{3/2}} \tag{36}$$

here ϵ is a positive parameter, and since we are using MQ it is called the shape parameter.

The partial derivatives of any order can be computed using radial basis functions, and the results will be continuous in the computational domain.

The matrix form of the equation (32) can be expressed as follows

$$L\dot{\alpha} = \frac{1}{2}L_{xx}\alpha + \tilde{K}(L\alpha,t) \tag{37}$$

$$\tilde{K}(L\alpha,t) = (1-2\rho)F(t)e^n(L\alpha+l)^m - \frac{\partial l}{\partial t} + \frac{1}{2}\frac{\partial^2 l}{\partial x^2}$$

where α denotes the vectors containing the unknown coefficients α_j. L and L_{xx} are the $N\times N$ matrices of $\phi(x_i,x_j)$ and $\partial^2\phi(x_i,x_j)/\partial x^2$ as given by equations (35) and (36) respectively.

Powell (1992) proved that the matrix L is invertible for distinct collocation points x_j and hence its inverse L^{-1} exists. So the equation (37) can then be rewritten as

$$\dot{\alpha} = L^{-1}\Big[\frac{1}{2}L_{xx}\alpha + \tilde{K}(L\alpha, t)\Big] \equiv P\alpha \tag{38}$$

$$\tilde{K}(L\alpha, t) = (1-2\rho)F(t)e^{n}(L\alpha + l)^{m} - \frac{\partial l}{\partial t} + \frac{1}{2}\frac{\partial^2 l}{\partial x^2}$$

Where P is a $N \times N$ matrix

$$\begin{aligned} P &= \frac{1}{2}L^{-1}L_{xx} + L^{-1}\tilde{K}(L, t) \\ &= \frac{1}{2}L^{-1}L_{xx} + (1-2\rho)F(t)e^{n}L^{-1}(L + l)^{m} - \frac{\partial l}{\partial t} + \frac{1}{2}\frac{\partial^2 l}{\partial x^2} \end{aligned} \tag{39}$$

We can use any backward time integration scheme to obtain the unknown coefficients α, at each time step. We use the fourth-order Runge-Kutta scheme (RK4).

Implimentation

To facilitate the settlement of trades, securities in the United States are identified by financial codes, referred to as CUSIP . These codes consist of a combination of numbers and letters. In this section, we introduce four CUSIPs that were issued, in 2005 and 2007, by the Federal National Mortgage Association, commonly known as Fannie Mae. Specifically, we consider one of these financial codes and analyze the market during the economic crisis of 2008 and for some time after it.

Table (2) shows the characteristics of each security as of 2 September 2008. In this table, the Original Amortization is the remaining months to maturity starting from the issuance date of the mortgage, and Current Amortization represents the remaining time to maturity from 2 September 2008. The original WAC and WAM are the weighted average coupon and maturity of the securities on the issuance date, respectively. WAC is the interest rate received by the issuer from the pool of mortgages.

For example, the issuer of FNMA 2005 5.0 receives the original weighted average coupon of 5.56% (τ') while paying 5.0% (τ) of interest to investors of this particular security. The difference is kept by the issuer as service and guarantee fees.

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Table 2: MBS characteristics as of September 2, 2008

MBS	Original WAC	Original Amortization	Original WAM	Current Amortization
	(%)	(mos)	(mos)	(mos)
FNMA 2005 5.0	5.56	360	348	320
FNMA 2007 5.0	5.78	360	359	341
FNMA 2007 5.0	5.59	360	339	342
FNMA 2007 5.5	5.94	180	179	346

Table 3: MBS model parameters

MBS	p	γ	ρ	A_0
FNMA 2005 5.0	3.38	1.40	1.25	-56.80
FNMA 2007 5.0	0.70	1.18	1.25	-54.60
FNMA 2007 5.0	0.67	1.25	1.25	-54.58
FNMA 2007 5.5	2.15	2.58	1.25	-54.18

We consider the Mortgage-backed-security identified by 31402RF87. We assume that model (1) applies to the MBS. To implement our model, we consider specified apriori parameter values of $\mu = 0.5, \delta = 0.02$ and $MB(0) = 100$ arbitrary which are common to each CUSIP. Rest of the model parameters are given in table (3) and are obtained by calibration of CUSIP data.

Values of 31402RF87 since 3 September 2008, to 30 October 2010 are reported in table (4). This table shows the efficiency of our method for valuing mortgage-backed securities. It should also be noted that Q1 to Q4 represent the yearly seasonal intervals, in this table.

Figure (1) shows the process of prepayments of the remaining mortgage balance for a 30-year mortgage-backed security with a fixed rate, when the interest rate is 5%. Furthermore, we see that at the issuance (t=0) the value of the remaining payments can be considered equal to the mortgage balance because we don't have any prepayments at that

Table 4: Value of the Mortgage-backed-security

MBS	2008-Q4	2009-Q1	2009-Q2	2009-Q3	2009-Q4	2010-Q1	2010-Q2
Value of 31402RF87	6.4800	6.4215	6.3732	6.3354	6.3083	6.2920	6.2865

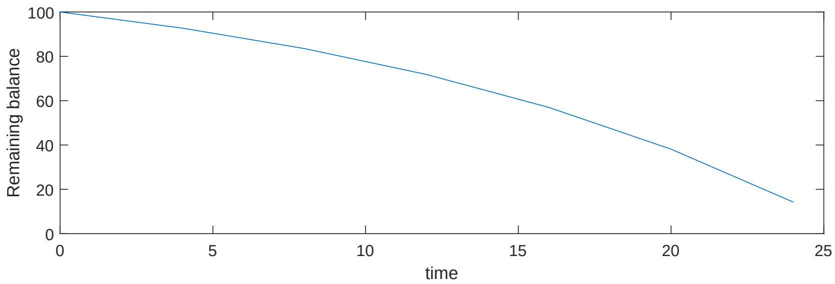

Figure 1: Remaining balance of 31402RF87

time. Calculations also show that as we approache to the maturity prepayments, including interest and principal, will decrease and eventually approach to zero at the maturity.

Concluding remarks

In conclusion, by using a structural model and by applying some assumptions to it for a specific economic situation, we derived a reduced model for predicting the price of mortgage-backed securities. Then we employed radial basis functions on our model. We also solved our model numerically for a sample of Mortgage-backed-securities pool and witnessed that our model applies to market observations immediately after the 2008 economic crisis.

As it was observed, many parameters are used in our proposed model. Obtaining these parameters requires accurate numerical methods. Therefore, we would like to suggest that Machine Learning methods be used to calibrate these parameters in later studies. In addition, using other various prepayment functions can be an interesting subject for research. We also suggest that these methods and securities be used in markets such as Iran, to diversify the markets and also create more communication between banks and financial markets.

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