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Research paper

Tau method for pricing American options under complex models

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Abstract:
In this paper, we will study the numerical solutions of a class of complex partial differential equations (PDE) systems with free boundary conditions. This kind of problems arise naturally in pricing (finite-maturity) American options, which is applies to a wide variety of asset price models including the constant elasticity of variance (CEV), hyper-exponential jump-diffusion (HEJD) and the finite moment log stable (FMLS) models. Developing efficient numerical schemes will have significant applications in finance computation. These equations have already been solved by the Hybrid Laplace transform-finite difference methods and the Laplace transform method(LTM). In this paper we will introduce a method to solve these equations by Tau method. Also, we will show that using this method will end up to a faster convergence. Numerical examples demonstrate the accuracy and velocity of the method in CEV models.

Keywords: American option pricing, CEV model, Fractional partial differential equations, Tau method
MSC2010 Classifications: 65Nxx, 60H15, 91B28.

1 Introduction

In finance, an option is a contract which gives the buyer (the owner or holder of the option) the right, but not the obligation, to buy or sell an underlying asset or instrument at a specified strike price before to or on a specified date. The vast majority of options are either European or American (style) options. A European option can only exercise at the expiration date, i.e. at a single pre-defined point in time. An American option, on the other hand, maybe exercised at any time before the expiration date. The pricing of options has its origins in work of Black and Scholes [1], which assume that the price of an underlying asset follows a geometric Brownian motion with constant volatility. However, there are sufficient empirical

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evidences to suggest that in many cases, the assumption of constant volatility does not match well to the observed market data. As a result, there are various ideas to how to modify and extend the basic the Black-Scholes framework, to account for this phenomenon. One of these idea is the constant elasticity of variance (CEV) diffusion model, which was introduced by Cox and Ross [4] in the context of European options. Studies using data from real markets [5], which include both equity and index options, suggest that the CEV models will better fit than the Black-Scholes as they lead to smaller smiles/frowns. Mathematically the American options lead to partial differential equations (PDE)with free boundaries, which can only rarely be solved explicitly. However, there exists little or no analytical work for the valuation of American options under a CEV process. The analysis of these options is more difficult than the corresponding European options.

Pricing these derivatives has attracted a lot of interests in academia in recent years [6]. Because there is no known closed-form solution, many researchers have studied with various numerical methods. Among the studies, the Laplace transform method [11], a simple least-squares approach [14], the Monte Carlo [20], the Penalty method [16] and the $\theta-$method [10], are used to study the valuation of American options.

The numerical methods for solving the PDEs system for pricing American options with regime-switching models have studied in papers [8, 9]. The option price under this kind of model is governed by fractional partial differential equations. This kind of fractional partial differential equation is challenging to solve. Recently, Laplace transform methods have developed to solve the free boundary problem arising in American option pricing under the geometric Brownian motion (GBM) model [23]. The constant elasticity of variance (CEV) model [21] and the hyper-exponential jump diffusions (HEJD) model [12]. Recently J. Ma et al in [15] proposed a hybrid Laplace transform and finite difference method (hybrid LT-FDM). They use finite difference methods to discretize the ODEs coupled with the approximation of the free boundary conditions. The approximate free boundary value in the Laplace space has obtained from an iterative algorithm based on a discrete version of the smooth pasting condition.

The Tau method is a spectral method, originally developed by Lanczos in the 30s [17] which delivers polynomial approximations to the solution of differential problems. The method tackles both initial and boundary value problem with ease. It is a spectral method that ensuring excellent error properties, whenever the solution is smooth. Initially developed for linear differential problems with polynomial coefficients, it has used to solve broader mathematical formulations: functional coefficients, non-linear differential and integro-differential equations. Several studies applying the Tau method have been performed to approximate the solution of differential linear and non-linear equations, partial differential equations and integro-differential equations, among others. In [18, 19], we solved the European option pricing problem with the jump and the delay model with Tau method, respectively, and we arrived at the proper precision of the solution. Therefore, we decided to solve the pricing problem of the American options under complex models with Tau method.

In this work, we present and test the Tau method to discretize the PDEs coupled with the approximation of the free boundary conditions. Then to obtain a numerical solution of the American option pricing problem, we solve the matrix equation $AX = B$.

We will present some numerical example and compare the Tau method with the Laplace transforms method derived in [15]. The numerical results showed that the accuracy of our solutions are better when we use the Tau method.

This paper is organized as follows: In Section 2, we introduce The CEV model. In Section 3, we present some theoretical results of simplifying the application of the Tau method. In Section 4, we solve the hybrid LT-price (finite-maturity) American options with the Tau method. In Section 5, the accuracy and velocity of the method are checked by solving some problems. Section 6 concludes the paper.

2 The CEV model

Let $V(S,\tau)$ denotes the price of an American put option for an asset with price S at $\tau = T - t$ (remaining time to maturity T) with strike K. We assume that the underlying asset price S satisfies the constant elasticity of variance (CEV) model (see e.g., [3])

$$dS_t = (r-q)S_t dt + \delta S_t^{\beta+1} dW_t, \tag{1}$$

where r is the risk-free interest rate, q is the dividend yield, and W_t is a standard Brownian motion. The term $\sigma(S) = \delta S^{\beta}$ represents the local volatility function and β can be interpreted as the elasticity of $\sigma(S)$, i.e., $\frac{d\sigma/\sigma}{dS/S} = \beta$. Here, δ is the scale parameter fixing the initial instantaneous volatility at time $t = 0$, and $\sigma_0 = \sigma(S_0) = \delta S_0^{\beta}$. If $\beta = 0$, then the SDE (1) becomes a geometric Brownian motion with the constant volatility rate $\sigma_0 = \delta$. Hense, $V(S,\tau)$ satisfies the following PDE with a free boundary:

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\delta^2 S^{2\beta+2}\frac{\partial^2 V}{\delta S^2} + (r-q)S\frac{\partial V}{\partial S} - rV, \quad S > S_f(\tau), \quad \tau > 0 \tag{2}$$

$$V(S,\tau) = K - S, \quad 0 < S < S_f(\tau), \tag{3}$$

$$V(S,0) = max(K-S,0), \tag{4}$$

$$V(S_f(\tau),\tau) = K - S_f(\tau), \tag{5}$$

$$\frac{\partial V(S_f(\tau),\tau)}{\partial S} = -1, \tag{6}$$

$$\lim_{S\to\infty} V(S,\tau) = 0, \tag{7}$$

where $S_f(\tau)$ is the corresponding optimal exercise boundary.

3 Some preliminary results of the Tau method

We recall from [17] that the Tau method is based on the following simple matrices

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & \dots \\ 0 & 0 & 3 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mu = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

And we assume

$$\underline{X} = (1, x, x^2, \dots, x^n, \dots)^T, \tag{8}$$

$$\underline{T} = (1, t, t^2, \dots, t^n, \dots)^T, \tag{9}$$

$$C = (c_{ij})_{n\times n} = \begin{pmatrix} c_{00} & c_{01} & \cdots & c_{0n} \\ c_{10} & c_{11} & \cdots & c_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n0} & c_{n1} & \cdots & c_{nn} \end{pmatrix}.$$

Also we need to state the following properties of the Tau method:

Lemma 3.1. *If* $V(x,t) = \underline{X}^T C \underline{T}$ *then*

a. $V_x(x,t) = \underline{X}^T \eta^T C \underline{T}$,

b. $V_{xx}(x,t) = \underline{X}^T \eta^{T^2} C \underline{T}$,

c. $V_t(x,t) = \underline{X}^T C \eta \underline{T}$,

d. $x^m V(x,t) = \underline{X}^T \mu^m C \underline{T}$.

Proof. Proof is clear. □

4 Transformation of the CEV model to a system of linear algebraic equations

To transform Eq. (2) and free boundary conditions (3) – (9) to a system of linear algebraic equations by using the operational approach of the Tau method, we consider the solution of the Eq. (2) in the following form:

$$V(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} c_{ij} x^i t^j = \underline{X}^T C \underline{T}, \tag{10}$$

where

$$\underset{\sim}{X} = (1, x, x^2, \ldots, x^n)^T, \tag{11}$$

and

$$\underset{\sim}{T} = (1, t, t^2, \ldots, t^n)^T, \tag{12}$$

are base vectors and C is an $(N+1)\times(N+1)$ matrix containing the unknown values c_{ij}:

$$C = (c_{ij})_{n\times n} = \begin{pmatrix} c_{00} & c_{01} & \cdots & c_{0n} \\ c_{10} & c_{11} & \cdots & c_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n0} & c_{n1} & \cdots & c_{nn} \end{pmatrix}.$$

By substituting the relations (a), (b), (c) and (d) from lemma 3.1 into Eq. (2) we obtain:

$$\underset{\sim}{X}^T \eta C \underset{\sim}{T} = \frac{\delta^2}{2} \underset{\sim}{X}^T \mu^{2\beta+2}\eta^{T^2} C \underset{\sim}{T} + (r-q) \underset{\sim}{X}^T \mu\eta^T C \underset{\sim}{T} - r \underset{\sim}{X}^T C \underset{\sim}{T}. \tag{13}$$

Since $\underset{\sim}{X}$ and $\underset{\sim}{T}$ are bases vectors, we have:

$$(\frac{\delta^2}{2}\mu^{2\beta+2}\eta^{T^2} + (r-q)\mu\eta^T - rI)C - C\eta = 0.$$

Therefore, Eq. (2) will be transformed to the following system of linear algebraic equations

$$LC - C\eta = 0, \tag{14}$$

where

$$L = \frac{\delta^2}{2}\mu^{2\beta+2}\eta^{T^2} + (r-q)\mu\eta^T - rI.$$

To transform the supplementary conditions (4), (6), suppose $V(S,0)$ and $V(S_f(\tau),\tau)$ are in the forms

$$V(S,0) = (K-S)^+ = \underset{\sim}{X}^T a, \tag{15}$$

$$V(S_f(\tau),\tau) = K - S_f(\tau) = b^T \underset{\sim}{T}, \tag{16}$$

where $a = (a_0, \cdots, a_n)^T$, and $b = (b_0, \cdots, b_n)^T$.

If we define

$$T_1 = (1, 0, 0, ..., 0)^T,$$

and

$$G_1 = (1, s_f, s_f^2, \cdots, s_f^n)^T,$$

then from (15), (16) we have

$$CT_1 = a, \tag{17}$$

and

$$G_1 C = b^T. \tag{18}$$

Now, we use the Kronecker product, then 15, 17 and 18 can be expressed as the following form:

$$AX = B. \tag{19}$$

Note that the system 16 cannot be solved directly, since it involves a free boundary $S_f(\lambda)$ which needs to be simultaneously solved.
We summarize the above steps in the following algorithm:

Algorithm: (Solving the Free Boundary Problem (2)-(9)).

(i) let $L_f = \frac{rk}{\delta}, \quad R_f = S.$

(ii) for $k = 1 : n$

(a) let $S_f^{(k)} = \frac{(R_f + L_f)}{2}$.

(b) calculate A and b with replacement $S_f^{(k)}$.

(c) solve the linear system 16 and get the solution vector V.

(d) if

$$\frac{\partial V(S_f^{(k)}(\tau), \tau)}{\partial S} < -1,$$

then we let $L_f = S_f^{(k)}$;

if

$$\frac{\partial V(S_f^{(k)}(\tau), \tau)}{\partial S} > -1,$$

then we let $R_f = S_f^{(k)}$.

Note: we can use the above method for hyper-exponential jump-diffusion model [13], Markov regime switching model [7] and The finite moment log stable (FMLS) model [2].
In the following, we will check the accuracy of the algorithm by mentioning some examples.

5 Numerical examples

In this section, we compare the Tau method with two methods namely, the Hybrid LT-FDM and the Laplace transform method(LTM). The computations are run by $MATLABR2014a$ on a PC with the configuration: Intel(R) Core(TM) i5.

Hybrid Laplace transform and finite difference methods

Example 5.1. Solve the following system of equations [15]:

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\delta^2 S^{2\beta+2}\frac{\partial^2 V}{\delta S^2} + (r-q)S\frac{\partial V}{\partial S} - rV, \quad S > S_f(\tau), \quad \tau > 0 \tag{20}$$

$$V(S,\tau) = K - S, \quad 0 < S < S_f(\tau),$$

$$V(S,0) = max(K-S,0),$$

$$V(S_f(\tau),\tau) = K - S_f(\tau),$$

$$\frac{\partial V(S_f(\tau),\tau)}{\partial S} = -1,$$

$$\lim_{S\to\infty} V(S,\tau) = 0,$$

with the parameter values

$$S_0 = 40, \quad r = 0.05, \quad q = 0, \quad k = 40, \qquad T = 3.$$

Table 1 summarizes the computational results of this example. In this table, the label "HLD" represents the hybrid LT-FDM and the label "Tau" represents the Tau method.

The results in this table show that for the initial instantaneous volatility $\sigma_0 = 0.2$, the error between HLD and Tau method is about 10^{-3} while the average computational time for HLD is about 4.3868s and for Tau method is about 0.071754s.

Example 5.2. Solve the equation 20 with the parameter values:

$$T = 1, \quad S_0 = 100 \quad , r = 0.04, \quad q = 0.07, \quad k = 95.$$

Table 2 summarizes the computational results. Columns entitled "LTM" and "Tau" represent the Laplace transform method and the Tau method, respectively.

From Tables 1, 2 we realize that the results are consistent with the prices listed in [15, 22] while the Tau method takes much less CPU time than the HLD and LTM.

6 Conclusions

In this work, we proposed the Tau method for the solution of American option pricing under the constant elasticity of variance (CEV) model. We compared our numerical results with Hybrid Laplace transform and finite difference methods (HLD) and the Laplace transform method(LTM). We showed that the Tau method is more efficient in both accuracy and CPU time. Also, we can apply the method used in this paper to various problems arising in finance (Hyper-exponential jump-diffusion model, Markov regime switching model and the finite moment log stable (FMLS) model).

Table 1: Prices of American put option under CEV models. The parameters are set as follows: $T = 3, S_0 = 40, k = 40, r = 0.05, q = 0$ and other parameters are listed in the table.

		$\sigma_0 = 0.2$			$\sigma_0 = 0.4$		
β	k	HLD	Tau	sec	HLD	Tau	sec
0	35	1.6762	1.6762	0.073	5.8978	5.8978	0.0736
	40	3.4655	3.4655	0.0873	8.3750	8.3750	0.0894
	45	6.1976	6.1980	0.0901	11.2352	11.2351	0.0941
-1	35	1.8820	1.8821	0.0740	6.5013	6.5014	0.0741
	40	3.3783	3.3784	0.0879	8.3018	8.3018	0.0879
	45	5.8522	5.8521	0.0931	10.4950	10.4951	0.0955
-2	35	2.14652	2.1466	0.0771	7.3065	7.3066	0.0785
	40	3.3217	3.3217	0.0885	8.6540	8.6541	0.0895
	45	5.5632	5.5632	0.0941	10.2660	10.2660	0.0961

Table 2: Prices of American put option under CEV models. The parameters are set as follows: $T = 1, S_0 = 100, k = 95, r = 0.04, q = 0.7$ and other parameters are listed in the table.

	$\sigma_0 = 0.1$		$\sigma_0 = 0.2$	
	LTM	Tau	LTM	Tau
β =-1	1.1877	1.2026	3.6108	3.5921
CPU time (s)	1.53	0.0455	1.98	0.0323
β =0	1.2189	1.2013	3.4116	3.4025
CPU time (s)	1.65	0.04621	1.73	0.03801

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