Journal of Mathematics and Modeling in Finance (JMMF)
Vol. 1, No. 1, Winter & Spring 2021
Research paper

Mean-square stability and convergence of compensated split-step θ-method for nonlinear jump diffusion systems

Yasser Taherinasab[1], **Ali Reza Soheili**[2], **Mohammad Amini**[3]

[1] Department of Applied Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran.
email

[2] Department of Applied Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran.
Soheili@um.ac.ir

[3] Department of Statistics, Ferdowsi University of Mashhad, Mashhad, Iran.
email

Abstract:
In this paper, the existence and uniqueness of the numerical solution of the Stochastic Differential Equations with Jumps(SDEwJs) under the one side Lipschitz conditions and polynomial growth conditions are presented. The Compensated split step θ(CSSθ) method introduce and try to bound the moment of the numerical solutions also we analyse the strong convergence on the compact domain. We discuss the stability of SDEwJs with constant coefficient and prove some new relation between their coefficient. Finally, we present three examples to investigate the theories and methods.

Keywords: Stochastic Differential Equations with jump; Compensated Split Step θ method; Lipschitz condition; Forward-Backward Euler-Maruyama method; Mean-square stability.
MSC2010 Classifications: 65C30; 60H10; 65L20.

1 Introduction

Throughout this paper, the numerical approximation of the Stochastic Differential Equations (SDEs) with Poisson-driven jumps has been studied.

$$dx(t) = f(x(t^-))dt + g(x(t^-))dW(t) + h(x(t^-))dN(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$,$f : \mathbb{R}^n \longmapsto \mathbb{R}^n$, $g : \mathbb{R}^n \longmapsto \mathbb{R}^{nm}$, $h : \mathbb{R}^n \longmapsto \mathbb{R}^n$ for each $t \geq 0$. To make it simpler, it has been assumed that $x(0) = x_0 \in \mathbb{R}^n$ is a deterministic vector and $x(t^-) = \lim_{s \mapsto t^-} x(s)$, also $W(t)$ is an $m-$dimensional Brownian motion and $N(t)$ is a scalar Poisson process with intensity λ [2].
Recently, stochastic differential equations with jumps have been used to model real-world phenomena such as weather, economics, biology (cancer) and physics.

[2]Corresponding author
Received: 2020-08-13 Approved: 2020-11-26
http://dx.doi.org/10.22054/jmmf.2020.54500.1011

In particular, they have been utilized in mathematical finance to simulate asset prices, interest rates and volatilities [1]. Most of the existing convergence theory requires the coefficients of SDEwJs to satisfy a linear growth condition [2–4].
In the past decades, many authors devoted themselves to finding other conditions to replace the linear growth condition by employing the Lyapunov-type functions [8]. To the best of the authors' knowledge, there has been no literature concerned with the related results on the numerical solution of SDEwJs under non-linear growth conditions. While, this condition has often been met by many systems in practice and the existing results of convergence are somewhat restrictive for the purpose of practical applications.
Therefore, it is very important to establish the convergence theory of SDEwJs under some weak conditions. In [9], Wei Mao et al. have presented hard condition for a nonlinear growth condition for Euler-Maruyama that $h(x)$ must satisfy the local Lipschitz condition, whilst in this paper, we present weaker condition as follow:
If the constants $\alpha, \beta > 0$ exist, such that the coefficients of equation (1) satisfy

$$2 < x, f(x) > + \|g(x)\|^2 + \|h(x)\|^2 \leqslant \alpha + \beta\|x\|^2 + \gamma(t)e^{-\xi t}, \tag{2}$$

then

$$supE(\|x(t)\|^2) < \infty, \quad \forall T > 0, \tag{3}$$

where $0 \leq t \leq T$, $\|x\|$ denotes both the Euclidean vector norm and the Frobenius matrix norm, also $< x, y >$ denotes the scalar product of vectors $x, y \in \mathbb{R}^n$.
Mao et. al. in [8] and Hutzenthaler et al. in [5,6] used this condition in the case of super-linearly growing coefficients in without jump case. the convergence theorems and the results that presented in this paper are different with convergence results that presented in papers, [2–4, 11, 13].
However, as mentioned before, there has been no reported result bounding the numerical solutions of SDEwJs under the coercivity condition (1.2). The EM approximation may not converge in the strong L_p-sense nor in the weak sense, to the exact solution. For example, consider the following non-linear SDEs with Poisson jump

$$dx(t) = (-2x - \frac{5}{2}x^3(t))dt + x^2(t)dW(t) + (1 + x^2(t))dN(t). \tag{4}$$

The Brownian process $W(t)$ whose almost all sample paths are continuous, the Poisson random process $N(t)$ is a jump process and has the sample paths which are continuous.
In order to approximate SDEwJs (4) numerically, for any t, the partition $P = P_{\Delta t} \bigcup J$ has been defined, where $P_{\Delta t} = \{t_k = kt : k = 0, 1, 2, ..., N\}$, $N\Delta t = T$ and $J = \{t_i : jump\ node\}$ of the time interval $[0, T]$. Subsequently, we define the EM approximation $Y_{t_k} \approx x(t_k)$ of (4) by where $\Delta W_{t_k} = W(t_{k+1}) - W(t_k)$. M. Hutzenthaler et al. in [5] has shown that

$$\lim_{\Delta t \to 0} E(\|y_{t_n}\|)^2 = \infty, \tag{5}$$

for no-jump case. In a similar manner from [5], it can be shown that (5) satisfies the jump equations (4). In the other words, it is necessary to modify the EM scheme so as to prove the strong convergence theorem under condition (2). Thus, the Compensated Split-Step θ (CSSθ) method for SDEwJs has been investigated. In this paper, we will prove the convergence of the SDEwJs under the one-side Lipschitz conditions and non-linear growth conditions and we try to bound the moment of the numerical solutions.
The rest of the paper has been organized as follows. In Section 2, the existence and uniqueness of solution concerning the equation (1) will be analysed. Subsequently, the possible CSSθ method will be introduced in Section 3. In Section 4, the Forward-Backward Euler-Maruyama (FBEM) Scheme is presented. In section 5, we discuss the stability of SDEwJs with constant coefficient and prove some new relation between their coefficient. Finally, we present three examples to investigate the theories and methods.

2 Existence and Uniqueness of the Solution

For the existence and uniqueness of the solution for (1), we assume that all the coefficient f, g and h satisfy the following assumptions:

The local Lipschitz condition: There is an integer constant $m \geq 1$ and positive constant $C(m)$ such that

$$\|f(x) - f(y)\| + \|g(x) - g(y)\| + \|h(x) - h(y)\| \leqslant C(m)\|x - y\|, \qquad (6)$$

for all $x, y \in R^n$, with $|x| \vee |y| \leq m$.
Coercivity condition: There exists constants $\alpha, \beta \in \mathcal{R}$, $\xi > 0$ and non-negative continuous function $\gamma(t)$, $t \in \mathcal{R}^+$, such that

$$2 < x, f_\lambda(x) > +\|g(x)\|^2 + \|h(x)\|^2 \leqslant \alpha + \beta\|x\|^2 + \gamma(t)e^{-\xi t}, \qquad (7)$$

for all $x \in \mathcal{R}^n$.

Also, for arbitrary $\delta > 0$, $\gamma(t)$ satisfies $\gamma(t) = o(e^{\delta t})$ [12].
Under Assumption 2, there exists a unique solution to (1) for any given initial value $x(0) = x_0 \in R^n$ [8]. The reason for presenting and proving the following theorem is that it reveals the upper bound for the probability of the process $x(t)$ remaining on a compact domain for a finite time $T > 0$. The bound will be used to derive the main convergence theorem of this paper.

Theorem 2.1. *There exists a unique, global solution $x(t)_{t \geq 0}$ to equation (1) for any given initial value $x(0) = x_0 \in R^n$, when Assumption 2 holds. Moreover, the solution has the properties that for any $T > 0$,*

$$E\|x(T)\|^2 \leq (\|x_0\|^2 + 2\alpha T + \gamma e^{-\xi T})exp(2\beta T), \qquad (8)$$

and

$$P(\tau_m < T) \leq (\frac{\|x_0\|^2 + 2\alpha T + \gamma e^{-\xi T})exp(2\beta T)}{m^2}. \tag{9}$$

For any positive integer m, where

$$\tau_m = inf\{t \geq 0, \|x(t)\| > m\}, \tag{10}$$

and

$$\gamma = sup\{\gamma(t)\ ,\ 0 < t < T\}. \tag{11}$$

Proof : Under Assumption 2., suppose there exists a unique solution $x(t)$ to equation (1) for any given initial value $x_0 \in \mathcal{R}^n$. Applying the Ito's formula to the function $V(x,t) = \|x\|^2$, the diffusion operator has been computed

$$LV(x,t) = 2 < x, f_\lambda(x) > + \|g(x)\|^2, \tag{12}$$

so by Assumption 2 we have

$$LV(x,t) \leq \alpha + \beta\|x\|^2 - \|h(x)\|^2 + \gamma(t)e^{-\xi t}, \tag{13}$$

or

$$LV(x,t) \leq \alpha + \beta\|x\|^2 + \gamma e^{-\xi t}. \tag{14}$$

Therefore

$$E\|x(t \wedge \tau_m)\|^2 \leq \|x_0\|^2 + 2\alpha T + \gamma e^{-\xi T} + \int_0^T 2\beta E\|x(s \wedge \tau_m)\|^2 ds, \tag{15}$$

and by using the Gronwall inequality we have

$$E\|x(T \wedge \tau_m)\|^2 \leq (\|x_0\|^2 + 2\alpha T + \gamma e^{-\xi T})exp(2\beta T). \tag{16}$$

Hence,

$$P(\tau_m > T) \leq \frac{E\|x(T \wedge \tau_m)\|^2}{m^2}. \tag{17}$$

By letting $m \longrightarrow \infty$ in (17) and applying Fatous lemma

$$E\|x(T)\|^2 \leq (\|x_0\|^2 + 2\alpha T + \gamma e^{-\xi T})exp(2\beta T). \tag{18}$$

Which gives the other assertion (8) and completes the proof.

3 The Compensated Split Step θ method

The equation (1) can be rewritten in the new form of SDEwJs

$$dx(t) = f_\lambda(x(t^-))dt + g(x(t^-))dw(t) + h(x(t^-))d\widehat{N}(t), \tag{19}$$

where $\widehat{N}(t)$ is a compensated Poisson process

$$\widehat{N}(t) = N(t) - \lambda t, \tag{20}$$

which $\widehat{N}(t)$ is a martingale and

$$f_\lambda(x) = f(x) + \lambda h(x). \tag{21}$$

It should be pointed out that f_λ also satisfies a one sided Lipschitz condition.
The CSSθ method for (19) by $Y_0 = x(0^-)$ can be defined with the constant step size $h = \Delta t$, such that $P_{\Delta t} = \{t_k = k\Delta t \ , k = 0, 1, 2, ...\}$ on $[0, \infty]$ and jump time, $J_t = \{t_i : i = 1, 2, 3, ..., l\} \subseteq [0, T]$, then for $P = P_{\Delta t} \cup J_t$ we have

$$\begin{aligned} y_n^* &= y_n + [(1-\theta)f_\lambda(y_n) + \theta f_\lambda(y^*)]\Delta t, \\ y_{n+1} &= y_n^* + g(y_n^*)\Delta w_n + h(y_n^*)\Delta \widehat{N}_n, \end{aligned} \tag{22}$$

where $\widehat{N}_n = \widehat{N}(t_{n+1}) - \widehat{N}(t_n)$.
Provided that where $\theta = 1$, the CSSθ method becomes the Compensated Split-Step Backward Euler (CSSBE) method in [2]. We will give the following lemma to answer the question of the existence of the numerical solution.
According to first equation in (22), the following can be deduced

$$y_n^* - \theta f_\lambda(y^*)\Delta t = y_n + (1-\theta)f_\lambda(y_n)\Delta t,$$

based on [8]. we can define $F : \mathcal{R}^n \to \mathcal{R}^n$ such that $F(x) = x - \theta f_\lambda(x)\Delta t$.
Clearly, X_{t_k} is F_{t_k}-measurable. Since the inverse function cannot be found explicitly, we can find the inverse function F^{-1} using root-finding algorithms, such as Newton's method or any function that satisfies the fixed point theorem.

Proposition 3.1. *Assume that $f : \mathcal{R} \longrightarrow \mathcal{R}$ satisfies Assumption 2 and let $\theta \in [0, 1]$, $0 < \Delta t < \frac{1}{\sqrt{k_\lambda \theta}}$, then in the first equation (22), we can uniquely solve the equation for y_n^*, with probability 1.*

proof: see [7]
We can achieve

$$y_n^* = F^{-1}(y_n + (1-\theta)f_\lambda(y_n)\Delta t), \tag{23}$$

and substitute it in the second equation, in order to solve the numerical method.
The additional parameter $\theta \in [0, 1]$ allows the implicitness of the numerical scheme to be controlled, which may lead to various asymptotic behaviours of the equation (3.4). Due to the presence of an implicit scheme, the availability of a unique solution y_{n+1}, where given $y_n$has to be proven for equation (22). To prove this, in addition to Assumption 2, we ask that function f_λ satisfies the one-sided Lipschitz condition.
There exists a constant $(\mu + \lambda L_h)$ such that

$$\| < x - y, f_\lambda(x) - f_\lambda(y) > \| \leqslant (\mu + \lambda L_h)\|x - y\|^2, \quad \forall x, y \in \mathcal{R}^n. \tag{24}$$

It follows the fixed point theorem that a unique solution y_{n+1} to equation (22) exists given y_n , provided that $\Delta t < \frac{1}{\theta L}$,(see [7]). In the next lemma, the fact that the second moment of the CSSθ method in (22) is bounded (Theorem 3.4). The stopping time technique has been employed in a similar way as in the proof of Theorem 2.1 to achieve the bound. The following lemma shows that boundedness of y_n moments are guaranteed, provided that the moments of F_{t_n} are bound.

Lemma 3.2. *Under the Assumption 2 for $F(x) = x - \theta f_\lambda(x)\Delta t$ we have*

$$\|x\|^2 \leq (1 - 2\beta\theta\Delta t)^{-1}[\|F(x)\|^2 + (2\alpha + \gamma(t)e^{-\xi t})\theta\Delta t], \qquad \forall x \in \mathcal{R}^n. \tag{25}$$

Proof. Let $\|F(x)\|^2 =< F(x), F(x) >$ and using Assumption 2

$$\begin{aligned}\|F(x)\|^2 &=< F(x), F(x) >,\\ &=< x - \theta f_\lambda(x)\Delta t, x - \theta f_\lambda(x)\Delta t >,\\ &= \|x\|^2 - 2\theta\Delta t < x, f_\lambda(x) > +\theta^2\Delta^2\|f_\lambda(x)\|^2,\\ &\geq \|x\|^2 - 2\theta\Delta t(\alpha + \beta\|x\|^2 - \frac{1}{2}\|g(x)\|^2 - \frac{1}{2}\|h(x)\|^2 + \gamma(t)e^{-\xi t}),\\ &= \|x\|^2(1 - 2\theta\Delta t\beta) - 2\alpha\theta\Delta t + \|g(x)\|^2\theta\Delta t + \|h(x)\|^2\theta\Delta t,\\ &- \gamma(t)e^{-\xi t}\theta\Delta t \geq,\\ &= \|x\|^2(1 - 2\theta\Delta t\beta) - (2\alpha + \gamma(t)e^{-\xi t})\theta\Delta t,\end{aligned} \tag{26}$$

complete the proof.
The stopping time has been defined λ_m as follows

$$\lambda_m = \inf\{n : \|y_n\| > m\}, \tag{27}$$

the following lemma is not trivial when $\|y_n\| > m$.
For $n \in [0, \lambda_m]$, we have the following lemma.

Lemma 3.3. *Under Assumptions 2 for $p \geq 2$ and a sufficiently large integer m, there exists a constant $C(p, m)$, such that*

$$E[\|y_{t_k}\|^p 1_{[0,\lambda_m]}(n)] < C(p, m), \qquad for \quad n \geq 0. \tag{28}$$

Proof: From the equation (22)

$$\begin{aligned}y_n^* &= y_n + [(1 - \theta)f_\lambda(y_n) + \theta f_\lambda(y^*)]\Delta t,\\ y_{n+1} &= y_n^* + g(y_n^*)\Delta w_n + h(y_n^*)\Delta\widehat{N}_n.\end{aligned} \tag{29}$$

With Assumption 2 and $F(x) = x - \theta f_\lambda(x)\Delta t$,

$$\begin{aligned}y_{t_k}^* &= F(y_{t_k}) + (f_\lambda(y_{t_k}) + \theta f_\lambda(y_{t_k}^*))\Delta t,\\ F(y_{t_{k+1}}) &= y_{t_k}^* + g(y_{t_k}^*)\Delta w_{t_k} + h(y_{t_k}^*)\Delta\widehat{N}_{t_k}.\end{aligned} \tag{30}$$

It can be written as

$$\begin{aligned}\|F(y_{t_{k+1}})\|^2 &=< F(y_{t_{k+1}}), F(y_{t_{k+1}}) > \\ &=< F(y_{t_k}) + (f_\lambda(y_{t_k}) + \theta f_\lambda(y^*_{t_k}))\Delta t + g(y^*_{t_k})\Delta w_{t_k} + h(y^*_{t_k})\Delta \widehat{N}_{t_k} \\ &F(y_{t_k}) + (f_\lambda(y_{t_k}) + \theta f_\lambda(y^*_{t_k}))\Delta t + g(y^*_{t_k})\Delta w_{t_k} + h(y^*_{t_k})\Delta \widehat{N}_{t_k} > \\ &= \|F(y_{t_k})\|^2 + \|_\lambda(y_{t_k})\|^2\Delta t^2 \\ &+ \theta^2\|f_\lambda(y^*_{t_k})]\|^2\Delta t^2 + (\|g(y_{t_k})\|^2 + \theta\|g(y_{t_k})\|^2)\Delta t \\ &+ 2 < F(y_{t_k}), f_\lambda(y_{t_k})\Delta t > + 2 < F(y_{t_k}), \theta f_\lambda(y^*_{t_k})\Delta t > + \Delta M_{t_k},\end{aligned} \tag{31}$$

by setting $f_\lambda(y^*_{t_k}) = f_\lambda(y_{t_k}) + \epsilon$,

$$\begin{aligned}\|F(y_{t_{k+1}})\|^2 \le& \|F(y_{t_k})\|^2 + \|f_\lambda(y_{t_k})\|^2\Delta t^2 \\ &+ \theta^2\|f_\lambda(y_{t_k})]\|^2\Delta t^2 + \|\theta\epsilon\Delta t\|^2 \\ &+ 2 < y_{t_k}, f_\lambda(y_{t_k}) > \Delta t - 2\theta\|f_\lambda(y_{t_k})]\|^2\Delta t^2 \\ &+ 2 < y_{t_k}, \theta f_\lambda(y_{t_k})\Delta t > -2\theta^2\|f_\lambda(y_{t_k})]\|^2\Delta t^2 + 2 < F(y_{t_k}), \theta\epsilon\Delta t > \\ &+ (\|g(y_{t_k})\|^2 + \theta\|g(y_{t_k})\|^2)\Delta t + \Delta M_{t_k},\end{aligned} \tag{32}$$

hence, based on Assumption 2

$$\begin{aligned}\|F(y_{t_{k+1}})\|^2 \le& \|F(y_{t_k})\|^2 + (1 - 2\theta - \theta^2)\|f_\lambda(y_{t_k})\|^2\Delta t^2 + \|\theta\epsilon\Delta t\|^2 \\ &+ 2 < y_{t_k}, f_\lambda(y_{t_k}) > \Delta t + 2 < y_{t_k}, \theta f_\lambda(y_{t_k})\Delta t > \\ &+ 2 < y_{t_k}, \theta\epsilon\Delta t > (\|g(y_{t_k})\|^2 + \theta\|g(y_{t_k})\|^2)\Delta t + \Delta M_{t_k},\end{aligned} \tag{33}$$

consequently we have the new condition for θ such that

$$(1 - 2\theta - \theta^2) \le 0, \tag{34}$$

resulting in

$$\begin{aligned}\|F(y_{t_{k+1}})\|^2 \le& \|F(y_{t_k})\|^2 + \|\theta\epsilon\Delta t\|^2 \\ &+ (\alpha + \beta\|y_{t_k}\|^2 - \|h(x)\|^2 + \gamma(t)e^{-\xi t})(1 + \theta) \\ &+ 2 < y_{t_k}, \theta\epsilon\Delta t > + \Delta M_{t_k}.\end{aligned} \tag{35}$$

By considering [8], we have

$$(\sum_{i=1}^{n} a_i)^{p/2} \le 4^{p/2-1}(\sum_{i=1}^{n} a_i^{p/2}), \quad for \quad a_i \ge 0. \tag{36}$$

Using the above inequality, the following can be obtained

$$\begin{aligned}\|F(y_{t_{k+1}})\|^p \le& 4^{p/2-1}(\|F(y_{t_k})\|^p + \|\theta\epsilon\Delta t\|^p \\ &+ ((\alpha + \beta\|y_{t_k}\|^2 + \gamma(t)e^{-\xi t})(1 + \theta)^{p/2} \\ &+ (2\|y_{t_k}\|\|\theta\epsilon\Delta t\|)^{p/2} + \|\Delta M_{t_{k+1}}\|^{p/2}.\end{aligned} \tag{37}$$

As a consequence

$$\begin{aligned} E[\|F(y_{t_{k+1}})\|^p 1_{[0,\lambda_m]}(n)] \leq 4^{p/2-1}(& E[\|F(y_{t_k})\|^p 1_{[0,\lambda_m]}(n)] \\ &+ \|\theta\epsilon\Delta t\|^p + (\alpha + \beta m^2 + \gamma e^{-\xi T})(1+\theta)^{p/2} \\ &+ (2m\|\theta\epsilon\Delta t\|)^{p/2} + E[\|\Delta M_{t_{k+1}}\|^{p/2} 1_{[0,\lambda_m]}(n)]). \end{aligned} \tag{38}$$

Now we can write ΔM_{t_k} as follows

$$\begin{aligned} \Delta M_{t_{k+1}} &= \|g(y^*_{t_k})\Delta w_{t_k}\|^2 + \|h(y^*_{t_k})\Delta \widehat{N}_{t_k}\|^2 \\ &- (\|g(y_{t_k})\|^2 + \theta\|g(y_{t_k})\|^2)\Delta t + 2 < F(y_{t_k}), g(y^*_{t_k})\Delta w_{t_k} > \\ &+ 2 < F(y_{t_k}), h(y^*_{t_k})\Delta \widehat{N}_n > + 2 < f_\lambda(y_{t_k})\Delta t, \theta f_\lambda(y^*_{t_k})\Delta t > \\ &+ 2 < f_\lambda(y_{t_k})\Delta t, g(y^*_{t_k})\Delta w_{t_k} > + 2 < f_\lambda(y_{t_k})\Delta t, h(y^*_{t_k})\Delta \widehat{N}_{t_k} > \\ &+ 2 < \theta f_\lambda(y_{t_k})\Delta t, g(y^*_{t_k})\Delta w_{t_k} > + 2 < \theta f_\lambda(y^*_{t_k})\Delta t, h(y^*_{t_k})\Delta \widehat{N}_{t_k} > \\ &+ 2 < g(y^*_{t_k})\Delta w_{t_k}, h(y^*_{t_k})\Delta \widehat{N}_{t_k} > . \end{aligned} \tag{39}$$

From equation (36), (39) and Cauchy inequality

$$\begin{aligned} E[\|\Delta M_{t_{k+1}}\|^{p/2}]1_{[0,\lambda_m]}(n) \leq 4^{p/2-1}E(&\|g(y^*_{t_k})\Delta w_{t_k}\|^p + \|h(y^*_{t_k})\Delta \widehat{N}_{t_k}\|^p \\ &- (\|g(y_{t_k})\|^p + \theta^{p/2}\|g(y_{t_k})\|^p)(\Delta t)^{p/2} \\ &+ 2^{p/2}\|F(y_{t_k})\|^{p/2}\|g(y^*_{t_k})\Delta w_{t_k}\|^{p/2} \\ &+ 2^{p/2}\|F(y_{t_k})\|^{p/2}\|h(y^*_{t_k})\Delta \widehat{N}_n\|^{p/2} \\ &+ 2^{p/2}\|f_\lambda(y_{t_k})\Delta t\|^{p/2}\|\theta f_\lambda(y^*_{t_k})\Delta t\|^{p/2} \\ &+ 2^{p/2}\|f_\lambda(y_{t_k})\Delta t\|^{p/2}\|g(y^*_{t_k})\Delta w_{t_k}\|^{p/2} \\ &+ 2^{p/2}\|f_\lambda(y_{t_k})\Delta t\|^{p/2}\|h(y^*_{t_k})\Delta \widehat{N}_{t_k}\|^{p/2} \\ &+ 2^{p/2}\|\theta f_\lambda(y_{t_k})\Delta t\|^{p/2}\|g(y^*_{t_k})\Delta w_{t_k}\|^{p/2} \\ &+ 2^{p/2}\|\theta f_\lambda(y^*_{t_k})\Delta t\|^{p/2}\|h(y^*_{t_k})\Delta \widehat{N}_{t_k}\|^{p/2} \\ &+ 2^{p/2}\|g(y^*_{t_k})\Delta w_{t_k}\|^{p/2}\|h(y^*_{t_k})\Delta \widehat{N}_{t_k}\|^{p/2})1_{[0,\lambda_m]}(n). \end{aligned} \tag{40}$$

Due to Assumption 2, $\|F(x)\|$ and $\|g(x)\|$ has been bounded for $\|y\| < m$. Where, there exists a constant $C(m,p)$, such that

$$\begin{aligned} E[\|\Delta M_{t_{k+1}}\|^{p/2}]1_{[0,\lambda_m]}(n) \leq& \\ & E[m_1 + m_2\|g(y^*_{t_k})\Delta w_{t_k}\|^{p/2} \\ & + m_3\|h(y^*_{t_k})\Delta \widehat{N}_{t_k}\|^{p/2}]1_{[0,\lambda_m]}(n), \end{aligned} \tag{41}$$

where m_1,m_2 and m_3 are constant. Now by holding this holder inequality

$$\begin{aligned} E[\|\Delta M_{t_{k+1}}\|^{p/2}]1_{[0,\lambda_m]}(n) \leq C(m,p)[&1 + (E\|g(y^*_{t_k})\|^p 1_{[0,\lambda_m]}(n))^{1/2}(E\|\Delta w_{t_k}\|^p)^{1/2} \\ &+ (E\|h(y^*_{t_k})\|^p 1_{[0,\lambda_m]}(n))^{1/2}(E\|\Delta \widehat{N}_{t_k}\|^p)^{1/2}]. \end{aligned} \tag{42}$$

Since there exists a positive constant $C(p)$, such that $E\|\Delta w_{t_k}\|^p \bigwedge E\|\widehat{N}_{t_k}\|^p \leq C(p)$, the following can be deduced

$$E[\|F(y_{t_k})\|^p]1_{[0,\lambda_m]}(n) \leq C(p,m).$$

The coefficients of the equation (19) satisfy the polynomial growth condition. In this regard, there exists a pair of constants $k \geq 1$ and $C(k) > 0$ such that

$$\|f(x)\| \vee \|g(x)\| \vee \|h(x)\| \leq C(k)(1+\|x\|^k), \qquad \forall x \in \mathcal{R}^n. \tag{43}$$

It is time to establish the fundamental results of this paper that reveals the boundedness of the equation (19), under Assumptions 2 and 3.

Theorem 3.4. *Let Assumptions 2, 3, 3 hold, and* $\sqrt{2}-1 \leq \theta \leq 1$. *Then, for any* $T > 0$, *there exists a constant* $C(T) > 0$ *such that the CSSθ scheme has the following property*

$$\sup_{\Delta t \leq \Delta t^*} \Big(\sup_{0 \leq t_k \leq T} E\|y_{t_k}\|^2\Big) < C(T).$$

Proof. By (31), we can represent the CSS θ scheme (22) as

$$F(y_{t_{k+1}}) = F(y_{t_k}) + (f_\lambda(y_{t_k}) + \theta f_\lambda(y^*_{t_k}))\Delta t + g(y^*_{t_k})\Delta w_{t_k} + h(y^*_{t_k})\Delta \widehat{N}_{t_k}.$$

Consequently, writing $\|F(y_{n+1})\|^2 =< F(y_{n+1}), F(y_{n+1}) >$ and utilizing Assumption 2,

$$\begin{aligned}\|F(y_{t_{k+1}})\|^2 \leq{}& \|F(y_{t_k})\|^2 + \|\theta\epsilon\Delta t\|^2 \\ &+ (\alpha + \beta\|y_{t_k}\|^2 - \|h(x)\|^2 + \gamma(t)e^{-\xi t})(1+\theta)\Delta t \\ &+ 2 < y_{t_k}, \theta\epsilon\Delta t > + \Delta M_{t_{k+1}}.\end{aligned} \tag{44}$$

By the cauchy inequality and $\|h(x)\|^2(1+\theta)\Delta t \geq \|\theta\epsilon\Delta t\|^2$,

$$\begin{aligned}\|F(y_{t_{k+1}})\|^2 \leq{}& \|F(y_{t_k})\|^2 + (\alpha + \gamma(t)e^{-\xi t})(1+\theta)\Delta t \\ &+ \beta(1+\theta)\|y_{t_k}\|^2\Delta t + 2\alpha_1\|y_{t_k}\|\Delta t + \Delta M_{t_{k+1}},\end{aligned} \tag{45}$$

is a local martingale.

$$\begin{aligned}\|F(y_{t_{k+1}})\|^2 \leq{}& \|F(y_{t_k})\|^2 + (\alpha + \gamma(t)e^{-\xi t})(1+\theta)\Delta t, \\ &+ \beta(1+\theta)(\|y_{t_k}\|^2 + 2\frac{\alpha_1}{\beta(1+\theta)}\|y_{t_k}\|)\Delta t + \Delta M_{t_{k+1}},\end{aligned} \tag{46}$$

or

$$\begin{aligned}\|F(y_{t_{k+1}})\|^2 \leq{}& \|F(y_{t_k})\|^2 + (\alpha + \gamma(t)e^{-\xi t})(1+\theta)\Delta t \\ &- \frac{\alpha_1^2}{\beta(1+\theta)}\Delta t + \Delta M_{t_{k+1}}.\end{aligned} \tag{47}$$

Let N be any non-negative integer such that $N\Delta t \leq T$. By summing up both sides of inequality (47) from $k=0$ to$N\wedge\lambda_m$, the expression below can be deduced

$$\begin{aligned}\|F(y_{t_{N\wedge\lambda_m+1}})\|^2 &\leq \|F(y_{t_0})\|^2+(\alpha+\gamma)(1+\theta)T\\ &-\frac{\alpha_1^2}{\beta(1+\theta)}T+\sum_{J=0}^{l}\sum_{k_J=0}^{(N\wedge\lambda_m)_J}\Delta M_{t_{k+1}},\end{aligned} \tag{48}$$

where $J=\{t_i : i=1,2,3,...,l\}$ is jump time, then

$$\begin{aligned}\|F(y_{t_{N\wedge\lambda_m+1}})\|^2 &\leq \|F(y_{t_0})\|^2+(\alpha+\gamma)(1+\theta)T\\ &-\frac{\alpha_1^2}{\beta(1+\theta)}T+\sum_{J=0}^{l}\sum_{k_J=0}^{N_J}\Delta M_{t_{k+1}}1_{[0,k_m]}(n).\end{aligned} \tag{49}$$

Applying Lemma 3.2, Assumption 3 and noting that y_{t_k} and $1_{[0,k_m]}(n)$ are F_{t_k}-measurable while Δt_k is independent of F_{t_k}. Finally, the equation presented below can be derived

$$\|F(y_{t_{N\wedge\lambda_m+1}})\|^2 \leq \|F(y_{t_0})\|^2+(\alpha+\gamma)(1+\theta)T-\frac{\alpha_1^2}{\beta(1+\theta)}T, \tag{50}$$

in equation (50) where $N\Delta t\leq T$, $m\to\infty$ and applying Fatous lemma, we get

$$\|F(y_{t_{N+1}})\|^2 \leq \|F(y_{t_0})\|^2+(\alpha+\gamma)(1+\theta)T-\frac{\alpha_1^2}{\beta(1+\theta)}T, \tag{51}$$

by Lemma 3.2, the assertion follows.

4 Forward-Backward Euler-Maruyama (FBEM) S-cheme

In this section, a continuous extension of a numerical method has been introduced which enables us to use the powerful continuous-time stochastic analysis in order to formulate theorems on numerical approximations and useful in the proof of Forthcoming Theorem 4.2. Now with the extended formula from [8], the following can be defined

$$\begin{aligned}\eta(t) &:= t_k \qquad t\in[t_k,t_{k+1}),\ k\geq 0,\\ \eta_+(t) &:= t_{k+1} \quad t\in[t_k,t_{k+1}),\ k\geq 0.\end{aligned} \tag{52}$$

Suppose continuous version of the θ-EM for jump equation is given by

$$\begin{aligned}y(t) &= y_{t_0}+\theta\int_0^t f(y_{\mu_+(s)})ds+(1-\theta)\int_0^t f(y_{\mu(s)})ds\\ &+\int_0^t g(y_{\mu(s)})dw(s)+\int_0^t h(y_{\mu(s)})d\widehat{N}(s), \qquad t\geq 0.\end{aligned} \tag{53}$$

According to findings presented by [8], $y(t)$ is not F_t-adapted since it does not meet the fundamental requirements of the classical stochastic analysis. For more detail, a new numerical method has been introduced and labelled as the Forward-Backward Euler-Maruyama (FBEM) method.
We define the discrete FBEM by

$$\begin{aligned} y^*_{t_k} &= \hat{y}_{t_k} + f(y_{t_k})\Delta t \\ \hat{y}_{t_{k+1}} &= y^*_{t_k} + g(y^*_{t_k})\Delta w_{t_k} + h(y^*_{t_k})\Delta \widehat{N}_{t_k}. \end{aligned} \tag{54}$$

And the continuous FBEM by

$$\hat{y}_{t_{k+1}} = \hat{y}_{t_0} + \int_0^t f(y_{\mu(s)})ds + \int_0^t g(y_{\mu(s)})dw(s) + \int_0^t h(y_{\mu(s)})d\widehat{N}(s), \qquad t \geq 0. \tag{55}$$

Note that the continuous and discrete FBEM schemes coincide at the grid points, that is, $\hat{y}(t_k) = \hat{y}_{t_k}$, for $k \geq 0$.

4.1 Strong Convergence on the Compact Domain

It this section, the strong convergence theorem has been proven. This has been carried out by showing that both schemes of the FBEM (54) and the CSSθ (22) stay close to each other on a compact domain. Then, an estimation has been made of the probability that both continuous FBEM (54) and CSSθ (22) will not explode on a finite time interval.

Lemma 4.1. *Under Assumptions 2, 3, 3, $\sqrt{2}-1 < \theta \leq 1$, any integer $p \geq 2$ and $m \geq \|y_0\|$, there exists a constant $C(m,p)$ such that*

$$E[\|\widehat{y}_{t_k} - y_{t_k}\|^p 1_{[0,\lambda_m]}(n)] \leq C(p,m)^p, \qquad \forall k \in N. \tag{56}$$

Proof: From both methods, the FBEM (54) and the $CSS\theta$ (22)

$$\widehat{y}_{t_{k+1}} - y_{t_{k+1}} = \widehat{y}_{t_k} - y_{t_k} + \theta[f_\lambda(y_{t_k}) - f_\lambda(y^*_{t_k})]\Delta t, \tag{57}$$

with Minkowski inequality in norm p

$$\|\widehat{y}_{t_{k+1}} - y_{t_{k+1}}\|_p \leq \|\widehat{y}_{t_k} - y_{t_k}\|_p + \|\theta[f_\lambda(y_{t_k}) - f_\lambda(y^*_{t_k})]\Delta t\|_p, \tag{58}$$

by summing up both sides of the equation

$$\|\widehat{y}_{t_N} - y_{t_N}\|_p \leq \theta \sum_{k=0}^{N-1} \|f_\lambda(y_{t_k}) - f_\lambda(y^*_{t_k})\Delta t\|_p, \tag{59}$$

expectation of both sides

$$E[\|\widehat{y}_{t_N} - y_{t_N}\|^p_p 1_{[0,\lambda_m]}(n)] \leq 4^{p-1}\theta \sum_{k=0}^{N-1} E[\|f_\lambda(y_{t_k}) - f_\lambda(y^*_{t_k})\|\Delta t\|^p_p 1_{[0,\lambda_m]}(n)], \tag{60}$$

from $f_\lambda(y^*_{t_k}) = f_\lambda(y_{t_k}) + \epsilon$ we then see easily that there exists a constant $C(p,m) \geq 0$, such that

$$E[\|\widehat{y}_{t_N} - y_{t_N}\|_p^p 1_{[0,\lambda_m]}(n)] \leq C(m,p)\Delta t^p, \tag{61}$$

and (56) obtained.
The following Theorem provides us with a similar estimate for the distribution of the first passage time for the continuous FBEM (53) and CSSθ (22) methods that we have obtained for the SDEwJs (1) in Theorem 2.1.

Theorem 4.2. *Under the Assumptions 2, 3 , 3 and $\sqrt{2}-1 \leq \theta \leq 1$ for any given $\epsilon > 0$, there exists a positive integer N_0 such that for every $m \geq N_0$, we can find a positive number $\Delta t_0 = \Delta t_0(m)$ provided that $\Delta t \leq \Delta t_0$.*

$$P(\vartheta_m < T) \leq \epsilon, \qquad for \;\; T > 0.$$

Where $\vartheta_m = \inf\{t > 0 : \|\widehat{X}(t)\| \geq m \;\;,\;\; or \;\|X_{\eta(t)}\| > m\}$.

Proof: By the Ito lemma for SDEs with Poisson jump

$$\begin{aligned}
\|\widehat{X}(T\wedge\vartheta_m)\|^2 &= \|X_0\|^2 + \int_0^{T\wedge\vartheta_m} (2 < \widehat{X}(s), f(X_{\eta(s)}) > + \|g(X_{\eta(s)})^2)ds \\
&+ 2\int_0^{T\wedge\vartheta_m} < \widehat{X}(s), g(X_{\eta(s)}) > dw(s) + \int_0^{T\wedge\vartheta_m} (\widehat{X}(s) - \widehat{X}(s_-))dN(s) \\
&= \|X_0\|^2 + \int_0^{T\wedge\vartheta_m} (2 < \widehat{X}(s) - X_{\eta(s)} + X_{\eta(s)}, f(X_{\eta(s)}) > + \|g(X_{\eta(s)})\|^2)ds \\
&+ 2\int_0^{T\wedge\vartheta_m} < \widehat{X}(s), g(X_{\eta(s)}) > dw(s) + \int_0^{T\wedge\vartheta_m} (\widehat{X}(s) - \widehat{X}(s_-))dN(s) \\
&\leq \|X_0\|^2 + \int_0^{T\wedge\vartheta_m} (2 < X_{\eta(s)}, f(X_{\eta(s)}) > + \|g(X_{\eta(s)})\|^2)ds \\
&+ \int_0^{T\wedge\vartheta_m} \|\widehat{X}(s) - X_{\eta(s)}\|\|f(X_{\eta(s)})\|ds \\
&+ 2\int_0^{T\wedge\vartheta_m} < \widehat{X}(s), g(X_{\eta(s)}) > dw(s) + \\
&\int_0^{T\wedge\vartheta_m} \|\widehat{X}(s) - \widehat{X}(s_-)\|dN(s),
\end{aligned}$$

by Assumption 2,

$$\begin{aligned}
\|f(x)\|^2 &\leq 2(C(m)\|x\|^2 + \|f(0)\|^2, \\
\|g(x)\|^2 &\leq 2(C(m)\|x\|^2 + \|g(0)\|^2, \\
\|h(x)\|^2 &\leq 2(C(m)\|x\|^2 + \|h(0)\|^2,
\end{aligned}$$

for $\|X\| \leq m$.

It can be expressed as

$$E\|\widehat{X}(T\wedge\vartheta_m)\|^2 \leq \|X_0\|^2 + 2\alpha_1 T + 2\beta E\int_0^{T\wedge\vartheta_m} \|X_{\eta(s)} - \widehat{X}(s) + \widehat{X}(s)\|^2 ds,$$
$$+ C(m)E\int_0^{T\wedge\vartheta_m} \|X_{\eta(s)} - \widehat{X}(s))\| ds.$$

For the rest of the proof see Theorem 4.2 in [8].

5 Linear mean-square stability

This section focuses on linear mean-square stability. Here, we are concerned with the regime where $t \to \infty$ with Δt fixed. Following the approach used in the deterministic case, we examine the behaviour of the method on a linear test equation. We consider the case where f, g and h in the equation (1) are scalar and multiplicatively linear, that is

$$dS(t) = aS(t^-)dt + bS(t^-)dW(t) + cS(t^-)dN(t), \tag{62}$$

where a, b and c are real constants, we assume $S(0) \neq 0$ with probability one. Note that equation (62) is a natural generalization of both the classical linear equation used to study the stability of methods for deterministic ODEs. We also remark that equation (62) has been proposed as a model in mathematical finance [10].

Theorem 5.1. *The equation (62) is mean-square stability if and only if*

$$a < \frac{\lambda}{2}, \quad \textit{and} \quad -1-\sqrt{1-\frac{2a}{\lambda}} < c < -1+\sqrt{1-\frac{2a}{\lambda}}. \tag{63}$$

Proof: The equation (5.1) has solution

$$S(t) = S(0)exp\{a-\frac{b^2}{2})t + bW(t)\}(1+c)^{N(t)}, \tag{64}$$

for $c \neq -1$, [11, 13].
Using $E((1+c)^{2N(t)}) = exp(\lambda c(2+c)t)$, we have

$$\begin{aligned} E[S(t)^2] &= E[S(0)^2 exp(2(a-\frac{b^2}{2})t]E[exp(2bW(t)]E[(1+c)^{2N(t)}] \\ &= E(S(0)^2)exp((2a+b^2+\lambda c(2+c))t). \end{aligned} \tag{65}$$

Hence, mean-square stability (of the zero solution) for $c \neq -1$ in (62) may be characterized by

$$Lim_{t\longrightarrow\infty} E|S(t)|^2 = 0 \quad \Leftrightarrow \quad 2a+b^2+\lambda c(2+c) < 0. \tag{66}$$

And it is straightforward to check that (66) remains true when $c = -1$.
We can rewrite (66) as follow

$$b^2 < -(2a + \lambda c(2 + c)),$$

where $b^2 > 0$, so

$$\begin{aligned} 2a + \lambda c(2 + c) < 0, \\ c^2 + 2c + \frac{2a}{\lambda} < 0, \\ (c + 1)^2 < 1 - \frac{2a}{\lambda}, \end{aligned} \tag{67}$$

then

$$a < \frac{\lambda}{2}, \quad \textit{and} \quad -1 - \sqrt{1 - \frac{2a}{\lambda}} < c < -1 + \sqrt{1 - \frac{2a}{\lambda}}. \tag{68}$$

From Theorem 5.1, few comments are in order regarding the parameters in (66).

- (i) The scalar Poisson intensity, $\lambda > 0$,
- (ii) For any $b \in R^n$,
$$2a + \lambda c(2 + c) < 0,$$
- (iii) The diffusion parameter b does not matter, but for the drift parameter $a < \frac{\lambda}{2}$ and the jump parameter c
$$-1 - \sqrt{1 - \frac{2a}{\lambda}} < c < -1 + \sqrt{1 - \frac{2a}{\lambda}},$$
- (iv) It is interesting to note that if $a \simeq \frac{\lambda}{2}$ or $(a \to \frac{\lambda}{2})$ we must have $c \simeq -1$, then
$$2a + b^2 + \lambda c(2 + c) \simeq \lambda + b^2 - \lambda,$$
and the drift parameter must be zero $(b \simeq 0)$.
- (v) If the jump parameter $c = 0$ or $c = -2$ then the drift parameter $a < 0$.
- (vi) If $a \simeq -\frac{b^2}{2}(a \to -\frac{b^2}{2})$ then the jump parameter $-2 < c < 0$.
- (vii) If $b = c = \lambda$ then $c^3 + 3c + 2a = 0$ and with $c = c - 1$ we can rewrite $c^3 - 3c + 2a + 2 < 0$, so with $\Delta = a(a + 2)$ we have three situation, such that
  - (a) If $a < -2$ or $a > 0$ then $\Delta > 0$, and we have only one real jump parameter c.
  - (b) If $a = -2$ or $a = 0$ then $\Delta = 0$, and we have two real jump parameter c.
  - (c) If $-2 < a < 0$ then $\Delta < 0$, and we have three real jump parameter c.

(viii) If $a = b = \lambda$, then $a^2 + a((c+1)^2 + 1) < 0$.

(a) If $a > 0$ then $a + ((c+1)^2 + 1) < 0$ and It is contradiction.

(b) If $a < 0$ then $a + ((c+1)^2 + 1) > 0$ and we choose the jump parameter c such that $(c+1)^2 < -(1+a)$.

(ix) If $a = b$ then

$$2a + a^2 + \lambda c(c+2) < 0,$$

$$(1+a)^2 < 1 - \lambda c(c+2),$$

we know $(1+a)^2 > 0$, then

$$1 - \lambda c(c+2) > 0,$$

$$(c+1)^2 < 1 - \frac{1}{\lambda},$$

so $\lambda > -1$.

Theorem 5.2. *The CSSθ applied to (62) is Mean-square stable if and only if*

$$Lim_{t\longrightarrow\infty} E|S(t)|^2 = 0 \quad \Leftrightarrow \quad \Delta t < \frac{1}{a + \lambda c}. \tag{69}$$

Proof. From (22) we have

$$Y_{n+1} = (\frac{1 + (1-\theta)(a+\lambda c)\Delta t}{1 - \theta(a+\lambda c)\Delta t})(1 + b\Delta W_n + c\widehat{\Delta N})Y_n,$$

nothing that $E(\Delta W) = 0$, $E(\Delta W)^2 = \Delta t$, $E(\widehat{\Delta N}) = 0$ and $E(\widehat{\Delta N})^2 = \lambda\Delta t$, so we can write

$$E|Y_{n+1}|^2 = (\frac{1 + (1-\theta)(a+\lambda c)\Delta t}{1 - \theta(a+\lambda c)\Delta t})^2(1 + b^2\Delta t + \lambda c^2\Delta t)E|Y_n|^2. \tag{70}$$

We conclude that $Lim_{n\longrightarrow\infty} E|Y_n|^2 = 0$, if

$$(\frac{1 + (1-\theta)(a+\lambda c)\Delta t}{1 - \theta(a+\lambda c)\Delta t})^2(1 + b^2\Delta t + \lambda c^2\Delta t) < 1,$$

but $(1 + b^2\Delta t + \lambda c^2\Delta t) > 1$, then

$$\frac{1 + (1-\theta)(a+\lambda c)\Delta t}{1 - \theta(a+\lambda c)\Delta t} < 1, \tag{71}$$

$$1 + \frac{(a+\lambda c)\Delta t}{1 - \theta(a+\lambda c)\Delta t} < 1,$$

$$\frac{(a+\lambda c)\Delta t}{1 - \theta(a+\lambda c)\Delta t} < 0.$$

With elimination $(a+\lambda c)\Delta t$ and substitute $\nabla = \frac{1}{(a+\lambda c)\Delta t}$, we have

$$\frac{1}{\nabla - \theta} < 0, \tag{72}$$

we have $(\nabla - \theta) < 0$, then

$$\Delta_1 t < \frac{1}{a+\lambda c}, \tag{73}$$

and prove the theorem.

Corollary 5.3. *If we use the split step θ method (non compensated), the equation (73) can be replace by*

$$\Delta_2 t < \frac{1}{a}.$$

5.1 Numerical experiments

In this section, we present several numerical experiments that corroborate the strong convergence. Two examples have been presented in order to test the designed program in MATLAB simulating environment. Here, the random numbers generated by using the command $randn(N1)$. The command $randn(N1)$ creates an $N1$ step matrix of independent $N(0;1)$ samples. In order to make a repeatable simulation, Matlab allows generating the same random numbers with similar initial states. Here, the initial value with $rng('default')$ and after that $randn(N1)$ has been set, Hence, different simulations can be performed by resetting the initial value.

Consider a simple case of equation (1), where $f(x(t)) = ax(t)$, $g(x(t)) = bx(t)$ and $h(x(t)) = cx(t)$, (see [11, 13])

$$dx(t) = ax(t^-)dt + bx(t^-)dw(t) + cx(t^-)dN(t), \qquad t \geq 0, \tag{74}$$

with $x(0) = 1$, and exact solution,

$$x(t) = x(0)exp((a - \frac{b}{2})t + bw(t))(1+c)^{N(t)}.$$

Clearly, the operators f,g and h satisfy the Assumption 2 for $(\alpha, \beta, \gamma) = (1, 6, 1)$.

Example 5.4. $a = 1$, $b = 1$, $c = 0.5$,$\lambda = 1$.

Remark 5.5. From theorem 5.1 part 8, Example 5.4 is convergent, but isn't mean square stable, whilst $\Delta t < \frac{1}{1.5} = \frac{2}{3} = 0.67$

Example 5.6. $a = 2$, $b = 2$, $c = -0.9$,$\lambda = 9$.

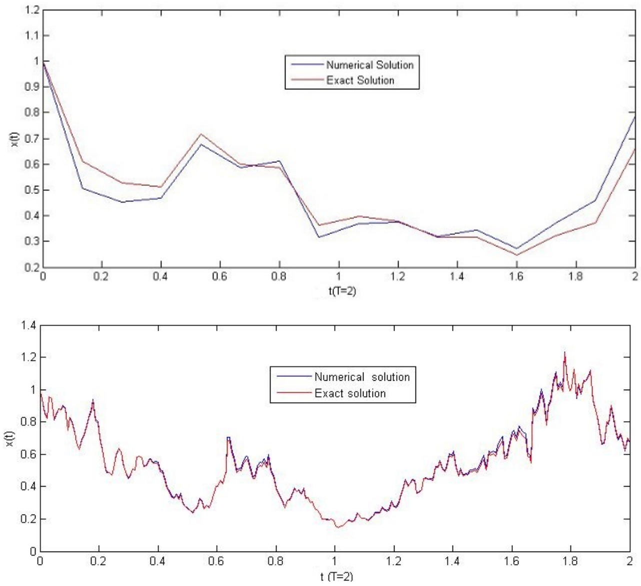

Figure 1: $CSS\theta$ method of Example 5.4

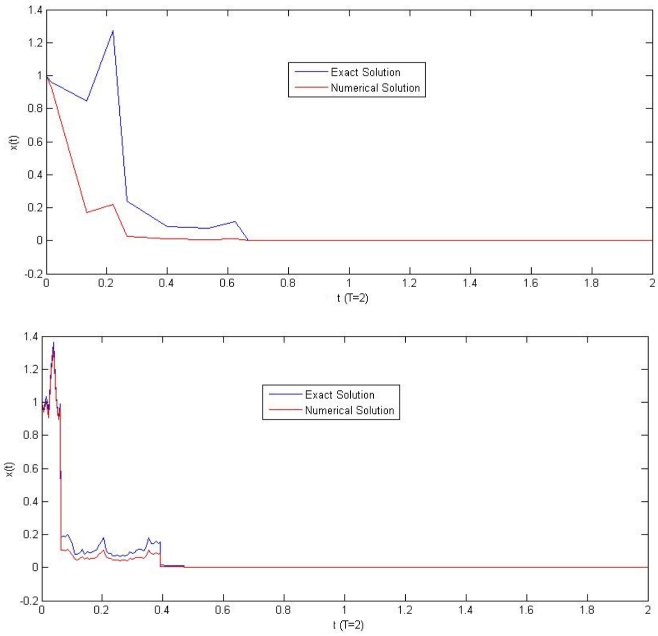

Figure 2: $CSS\theta$ method of Example 5.6

Therefore, it proves that assumptions 2, 3 and 4 have been satisfied. Consequently, the approximate solution will converge to the true solution for any $t \geq 0$, in the sense of Theorem 3.4, provided that Δt is sufficiently small. To confirm our convergence order, we apply the CSSθ methods to Examples 5.4 and 5.6 and plot them in Figures 1 and 2. Using two different step sizes: $\Delta t = 2^{-q}$, with $q = 4, 8$ and $\theta = 0.5$.

Remark 5.7. From Theorem 5.1 part 9 the example II convergence, mean square stable and $\Delta t < \dfrac{1}{2 - 0.81} = 0.840$.

Example 5.8. This example has been constructed to demonstrate the effectiveness and experiment of theoretical results of our theory.

$$dx(t) = (-2x - \frac{5}{2}x^3(t))dt + x(t)^2 dw(t) + (1 + x^2(t))dN(t), \tag{75}$$

where$f(x) = -2x - \frac{5}{2}x^3$, $g(x) = x^2$, $h(x) = 1 + x^2$ with initial value $x_0 = 1$. Setting $\lambda = 2$ for the Poisson process intensity and solve this example over $T = 2$. Also the operators f,g and h satisfy the Assumption 2 for $(\alpha, \beta, \gamma) = (1, 1, 1)$. Note that the coefficients in example satisfy assumption 2, 3, 3 and consequently, the approximate solution will converge to the true solution for any $t \geq 0$. The exact

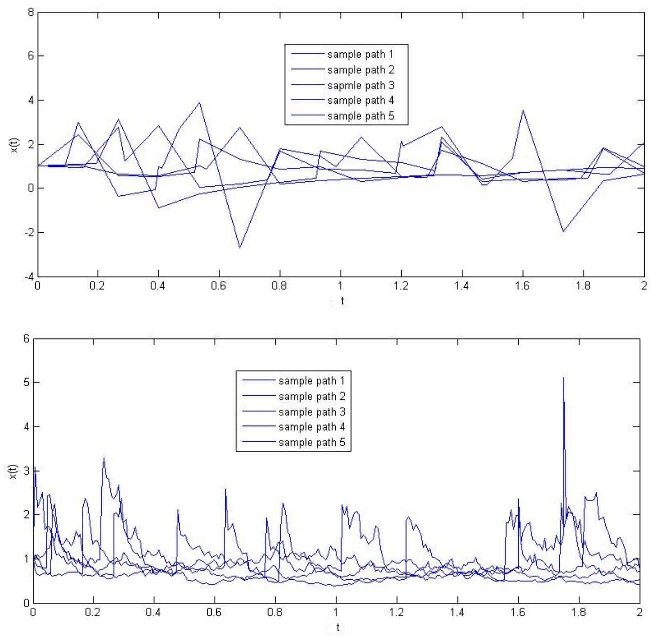

Figure 3: $CSS\theta$ method of Example 5.8

solution of Example 5.8 for some sample path and the CSSθ approximations has been illustrated in Figure 3. Then, the numerical solutions with two different step sizes: $\Delta t = 2^{-q}$, with $q = 4, 8$ and $\lambda = 2, \theta = 0.5$ has been simulated.

6 Conclusion

In this paper, the CSSθ and FBEM methods for Numerical solutions of SDEs with Poisson jump has been presented and analyzed. We showed the strong convergence of CSS θ method under coercivity condition, where $\sqrt{2}-1 < \theta \leq 1$, the drift term f has a one-sided Lipschitz condition , the diffusion term g and jump term h satisfy global Lipschitz condition. Also we proved the p-th moments of the numerical solution are bounded for $p > 2$ on compact domain. Furthermore, we discussed about stability of SDEwJs with constant coefficients and presented new useful relations between their coefficients.

Bibliography

[1] D. Bates, *Jumps and stochastic volatility: exchange rate processes implicit in deutsche mark options*, Rev. Financ. Studies, 9 (1996), pp. 96-107.

[2] D. J. Higham and P. E. Kloeden, *Numerical methods for nonlinear stochastic differential equations with jumps*, Numer. Math. 101(2005), pp.101-119.

[3] D. J. Higham and P. E. Kloeden, *Convergence and stability of implicit methods for jump diffusion system*, Int. J. Numer. Anal. Model, 3(2006), pp. 125-140.

[4] D. J. Higham and P. E. Kloeden, *Strong convergence rates for backward Euler on a class of nonlinear jump-diffusion problems*, J. Comput. and Appl. Math., 205(2007), pp. 949-956.

[5] M. Hutzenthaler, A. Jentzen and P.E. Kloeden. *Strong convergence of an explicit numerical method for SDEs with non-globally lipschitz continuous coefficients.* to appear in The Annals of Applied Probability, 22(2012), pp. 1611-1641.

[6] M. Hutzenthaler, A. Jentzen, and P.E. Kloeden. *Strong and weak divergence in finite time of Eulers method for stochastic differential equations with non-globally Lipschitz continuous coefficients.* Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science, 467(2013), pp. 1563-2011.

[7] X. Mao, *Stochastic Differential Equations and Applications*, Horwood Pub Ltd, 2007.

[8] X. Mao and L. Szpruch, *Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally Lipschitz continuous coefficients*, J. Comput. and Appl. Math., 238 (2012), pp. 14-28.

[9] W. Mao, S. You and X. Mao, *On the asymptotic stability and numerical analysis of solutions to nonlinear stochastic differential equations with jumps*, J. Comput. and Appl. Math., 301 (2016), pp. 1-15.

[10] E. Platen and N. Bruti-Liberati, *Numerical Solution of Stochastic Differential Equations with Jumps in Finance*, Springer-Verlag, Berlin, 2010.

[11] J. Tan, Z. Mu and Y. Guo, *Convergence and stability of the compensated split-step θ method for stochastic differential equations with jumps*, Advances in Difference Equations, 2014 (2014), pp. 1-19.

[12] L. Ronghua, W. K. Pangb and W. Qinghe,Numerical analysis for stochastic age-dependent population equations with Poisson jumps,J. Math. Anal. Appl. 327, 12141224(2007).

[13] X. Wang and S. Gan, *Compensated stochastic theta methods for stochastic differential equations with jumps*, Appl. Numer. Math., 60 (2010), pp. 877-887.

How to Cite: Yasser Taherinasab[1], Ali Reza Soheili[2], Mohammad Amini[3], *Mean-square stability and convergence of compensated split-step θ-method for nonlinear jump diffusion systems*, Journal of Mathematics and Modeling in Finance (JMMF), Vol. 1, No. 1, Pages:83–101, (2021).