Journal of Mathematics and Modeling in Finance (JMMF)
Vol. 2, No. 2, Summer & Autumn 2022
Research paper

Introduction a Method of Determining Returns to Scale in Network Data Envelopment Analysis

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Abstract:
In the process of evaluating the Decision Making Units, two factors of efficiency and production size can be used. When the production size of a unit is not optimal, its Returns To Scale (RTS) determines that changing the resources in another direction would enhance its productivity. In most previous research, RTS is considered to be increasing or decreasing, and frontier analysis is used to determine it. The concept of RTS in Network Data Envelopment Analysis (DEA) is so interesting. In this paper a method based on Most Productive Scale Size (MPSS) in several steps is developed, in addition to determining that RTS of units for each unit in directional manner, the shortest changes in resources for achieving the right size for network production is also obtained. In this approach, the computational complexity, and the ambiguity in units RTS is not present.

Keywords: Data Envelopment Analysis, Network Data Envelopment Analysis, Returns to Scale, Efficiency, Most Productive Scale Size.
MSC Classification: 90C08, 90C06, 62P05

1 Introduction

The concept of Returns to Scale (RTS) is an important topic in data envelopment analysis, since it identifies whether the expansion or contraction of the unit under assessment is beneficial. The quantity of the beneficial expansion or contraction is determined by introducing a MPSS. The estimation of RTS of DMUs using the data envelopment analysis method was investigated first by [3] and [4]. Banker introduced the definition of the RTS from classical economics into the framework of the DEA method and used the CCR-DEA model with radial measure to estimate the RTS of evaluated DMU. Two paths may be followed in treating returns to scale in DEA. The first path, developed by [7], determines RTS by a use of ratios of radial measures. These ratios are developed from model pairs which differ only in whether conditions of convexity and sub-convexity are satisfied. The second path stems from

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Received: 06/06/2022 Accepted: 09/09/2022
https://doi.org/10.22054/jmmf.2022.68634.1057

work by [3], [4] and [3]. This path is not restricted to radial measure models. [11] discussed three basic RTS methods and their modifications. They showed that the equivalence is between these different RTS methods. [2] searched for RTS in different DEA models. [10] researched MPSS patterns. Using this pattern, a unit would be able to reach its optimal size more easily and by small changes in its inputs and outputs. [14] proposed a definition of directional RTS in the DEA framework and estimated the directional RTS of research institutions using DEA models. In DEA, RTS of an inefficient unit is determined at its projected point on the frontier. If we have multi projection for one inefficient unit, our evaluation is not precise and may lead to erroneous RTS possibilities of DMUs. [10] solved this problem by defining the RTS of an inefficient DMU at its projected point that lies interior of the minimum face and which is based on this definition, it proposed an algorithm by extending the latest developed method of measuring RTS by the CCR model.

On the other hand, there are a lot of production technologies via multi-stage in nature whereby characterization of RTS in these technologies is important for firm managers. [12] proposed a slacks-based network DEA model, called Network SBM that can formally deal with intermediate products. Using this model, they could evaluate divisional efficiency along with the overall efficiency of decision-making units. In traditional DEA models there are two approaches for evaluating the efficiency of multi-division: Aggregation (Black box) and Separation. We now want to use this model to evaluate RTS in Network data envelopment analysis.

The efficiency of an organization is affected by two factors:

-The internal factors, which are the abilities of the organization to achieve maximum productivity, and they can be examined via efficiency evaluation methods. By having the production function in a specific production area, one can determine whether the examined organization is working under optimal conditions or some of its factors shows any weakness.

The production function is unknown in most cases and different science research from a variety of research fields study is required to estimate it. Data Envelopment Analysis (DEA) is a non-parametric technique that unlike the parametric approaches, estimates the constraints of the function, calculates the set of feasible production actions and uses its maximal and dominant boundary as a suitable estimation for the production function.

- The external factors are the external environmental conditions forced upon the organization and which affect the efficiency of the organization. One of these conditions could be production resources that lay in the disposal of the organization and determines the size of the organization. Human resources, budget, and equipment are among the resources which determine the size of production. Organizations with different sizes present different results.

Having a bigger production size does not always yield better results, but its often the case that small production size would prevent the better results.

Note that different known and unknowns environmental conditions could affect the productivity of an organization which are not necessarily dependent on each other.

If the basis for performance evaluations is set to be the homogeneous observed productive activities within a specific time period, the examination would be relative and the evaluation would be the frontier production function which could be estimated from various parametric approaches, or a Production Possibility Set (PPS) which is based on a set of accepted facts, which is determined in the data envelopment analysis.

2 Literature review

In this section, we give a brief summary of previous studies in the returns to scale and network in Data Envelopment Analysis (DEA).

First time Banker (1984) estimated the most productive scale size using DEA in 1984. He was able to show that for productive inefficiencies at the actual scale size the CCR efficiency measure also reflects any inefficiencies due to divergence from the most productive scale size. According to this idea in 1992, [4] proposed a new approach that was a partition of the optimal frontier into three parts corresponding, respectively to increasing, constant and decreasing returns to scale. Following this research in 2004 [13] studied congestion and returns to scale. In a 2011 study, [3] delimited approaches to be examined. In 2017 [14] researched estimating directional returns to scale in DEA.

[5] presented severe flaws of returns to scale in DEA and expressed that under certain circumstances, the classical RTS concept is the wrong indicator for scaling activities and even hides respective productivity improvement potentials. [8] researched the efficiency decomposition of the network DEA in variable returns to scale. He studied it in a two-stage network. Network DEA proposed by [6] in DEA at first, a system is like a black box and intermediate data was ignored so Fare presented network DEA to solve this problem. Scientists like [11], [9], [12],. Search about it.

We deal with n DMUs (j=1,, n) consisting of K divisions (k=1,,K).Let m_k, r_k be the numbers of inputs and outputs to division k , respectively. We denote the link leading from division k to division h by (k, h) and the set of links by L. The observed data are $\left\{x_j^k \in R^{m_k}\right\}(j = 1, ..., n, k = 1, ..., K)$ (Input resources to at division k), $\left\{y_j^k \in R^{r_k}\right\}(j = 1, ..., n, k = 1, ..., K)$ (output products from DMU_j at division k) and $\left\{z_j^{(k,h)} \in R^{t_{(k,h)}}\right\}(j = 1, ..., n, (k, h) \in l)$ (linking intermediate products from division k to division h) where $t_{(k,h)}$ is the number of items in link (k, h). In related works, the vectors X_j, Y_j are called the input and output vectors of the DMU_j and it's assumed that their values are additive. (This assumption is valid since it could easily be achieved via normalization).

Also, it's assumed that their values are non-zero and non-negative. With assigning values to the input and output factors, a measurement for the return at division k (R^k) could be obtained via:

$$R^k = \frac{\sum_{r=1}^{r_k} u_r^k y_r^k + \sum_{(k,h)\in L} \sum_{d=1}^{t^{(k,h)}} w^{(k,h)} z_d^{(k,h)}}{\sum_{i=1}^{m_k} v_i^k x_i^k + \sum_{(k,h)\in L} \sum_{d=1}^{t^{(k,h)}} w^{(k,h)} z_d^{(k,h)}} \tag{1}$$

In which v_i^k is the quantitative value of input $x_i^k (i = 1, ..., m_k), w^{(k,h)}$, $w^{(k,h)}$ are the quantitative values of intermediate values $z_d^{(k,h)} d = 1, ..., t^{(k,h)}$, $z_d^{(k,h)} d = 1, ..., t^{(k,h)}$ respectively and u_r^k is the quantitative value of output $y_r^k (i = 1, ..., r_k)$. Hence the returns of input and output vectors
$\left(x^k, z^{(h,k)}, z^{(k,h)}, y^k\right)$ in the hyper plane satisfying the equation

$$\sum_{r=1}^{r_k} u_r^k y_{rj}^k + \sum_{(k,h)\in L} \sum_{d=1}^{t^{(k,h)}} w_d^{(k,h)} z_{dj}^{(k,h)} - R^k \sum_{i=1}^{m_k} v_i^k x_{ij}^k - R^k \sum_{(h,k)\in L} \sum_{d=1}^{t^{(h,k)}} w_d^{(h,k)} z_{dj}^{(h,k)} = 0 \tag{2}$$

Which crosses the point $(x_o^k, z_o^{(h,k)}, z_o^{(k,h)}, y_o^k)$ and the origin. Based on the relative comparison, the Most Productivity Scale Size (MPSS) set, include DMUs which have the highest return among other DMUs, in other words:

$$MPSS^k = \left\{DMU_p \mid R_p^k = \max\left\{R_j^k\right\}\right\} \tag{3}$$

In most cases, the coefficient vectors $V^k = (v_1^k, v_2^k, ..., v_m^k)^T$,$U^k = (u_1^k, u_2^k, ..., u_s^k)^T$ and $w^{(k,h)} = (w_1^{(k,h)}, ..., w_{t^{(k,h)}}^{(k,h)})(k, h) \in L,, w^{(h,k)} = (w_1^{(h,k)}, ..., w_{t^{(h,k)}}^{(h,k)})(h, k) \in L$, are unknown and in DEA their values are adjusted relative to the unit under evaluation DMU_p, in a way that the best return among other units is achieved. It This conception can be assumed for every network. In other words, we researched weights for inputs, outputs and intermediate products which can be used to create the most returns for an estimated unit.

$$(V^{k*}, w^{*(k,h)}, w^{*(h,k)}, U^{k*}) = \arg \max_{(V,w^{(k,h)},w^{(h,k)},U)\geq(0,0,0,0)} \left\{ \frac{R_p^k}{\max\limits_j R_j^j} \right\} \tag{4}$$

Thus, if a unit is not in the MPSS situation relative to its corresponding coefficient vector, it would not be in this situation with any other coefficient vectors.

Consider n DMUs with m inputs and s outputs. The input and output vectors of DMU_j are $(j = 1, ..., n) X_j = (x_{1j}, ..., x_{mj})^t, Y_j = (y_{1j}, ..., y_{sj})^t$ where $X_j > 0, Y_j > 0$ By using the constant returns to score, convexity, and possibility postulates, the non-empty production possibility set (PPS) is defined as follows:

$$T_c = \left\{ (X, Y) : X \geq \sum_{j=1}^{n} \lambda_j X_j, Y \leq \sum_{j=1}^{n} \lambda_j Y_j, \lambda_j \geq 0, j = 1, ..., n \right\}$$

Using the production possibility set and input mitigation strategies oriented based was offered by Charls, Cooper and Rhodes for evaluating DMU_0

$$\begin{aligned}
&\text{Min } \theta \\
&st \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{i0} \quad i = 1, ..., m \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{r0} \quad r = 1, ..., s \\
&\lambda_j \geq 0, \quad j = 1, ..., n
\end{aligned}$$

The weights obtained from the model (3) are equivalent to the weights achieved via the CCR model at division k, and thus we have:

$$\begin{aligned}
&Ec_p^k = \text{Max} \sum_{r=1}^{s} u_r^k y_{rp}^k + \sum_{(k,h)\in L} \sum_{d=1}^{t^{(k,h)}} w_d^{(k,h)} z_{dp}^{(k,h)} \\
&-\sum_{i=1}^{m} v_i^k x_{ij}^k + \sum_{r=1}^{s} u_r^k y_{rj}^k - \sum_{(h,k)\in L} \sum_{d=1}^{t^{(h,k)}} w_d^{(h,k)} z_{dj}^{(h,k)} + \sum_{(k,h)\in L} \sum_{d=1}^{t^{(k,h)}} w_d^{(k,h)} z_{dj}^{(k,h)} \leq 0 \\
&\sum_{i=1}^{m} v_i^k x_{ip}^k + \sum_{(h,k)\in L} \sum_{d=1}^{t^{(h,k)}} w_d^{(h,k)} z_{dp}^{(h,k)} 1, \quad j = 1, ..., n, k = 1, ..., K, \\
&v_i^k \geq 0, u_r^k \geq 0, w_d^{(k,h)}, w_d^{(h,k)} \geq 0, k = 1, ..., K. \qquad (5)
\end{aligned}$$

In the model (4), Ec_p^k is the normalized return of the unit under evaluation DMU_p at division k which in the DEA literature is usually referred to as the efficiency score of CCR at division k. it's obvious that if $Ec_p^k = 1$, the $DMU_p$with its coefficient vector in equation (4) is located in MPSS. Thus, in situations which the coefficient vector is unknown, the $MPSS^k$ set is assumed to have the form:

$$MPSS^k = \left\{ DMU_p \left| Ec_p^k = 1 \right. \right\}$$

Regardless of the observed points, here we present the MPSS situation:

Definition 1: Relative to the set of units under evaluation at division k, the set of input vectors which are included in the convex combination of the input vectors of $MPSS^k$ points, is called an $MPSS^k$ region. Assuming that $J_{MPSS}^k$is the set including the indices of the MPSS units at division k, we define:

$$X_{MPSS}^k = \underset{j\in J_{MPSS}^k}{convex} \left\{ X_j^k \right\}$$

it's obvious that for any solution $(V^{k*}, w^{*(k,h)}, w^{*(h,k)}, U^{k*})$ achieved in model (4), the hyper plane

$$-\sum_{i=1}^{m_k} v_i^k x_{ij}^k + \sum_{r=1}^{r_k} u_r^k y_{rj}^k - \sum_{(h,k)} \sum_{d=1}^{t^{(h,k)}} w_d{}^{(h,k)} z_{dj}^{(h,k)} + \sum_{(k,h)\in L} \sum_{d=1}^{t^{(k,h)}} w_d{}^{(k,h)} z_{dj}^{(k,h)} = 0 \quad j = 1, ..., n \tag{6}$$

Includes all points (X, Z, Y) which represent the highest return if$X \in X_{MPSS}^k$. Every hyper plane in (5) is called and $MPSS^k$ hyper plane. The convex combination of every two MPSS points, which are located in the MPSS hyper plane, if observed, is an MPSS point, but if these points are located in different hyper planes this statement is not necessarily true. But this statement is different for MPSS points, meaning that the convex combination of every two MPSS points are in MPSS region.

3 Technical Scale efficiency

3.1 Technical efficiency

If a unit is not an MPSS, it does not necessarily mean that it does not possess the maximum productivity or that it is inefficient. Unsuitable production situation regarding the usage of resources and or the production size, prevent the inclusion of DMU in the MPSS set, but it does not prevent maximum production and efficiency. Efficiency is the optimal usage of resources in order to reach the maximum productivity and is completely dependent on the technology in that production area.

In 1984, [3] developed the basic DEA model for evaluating the efficiency named BCC which is based on this axiom that the production function is a concave, continuous and envelopment function. Instead of estimating the production function, they estimated the Production Possibility Set (PPS), with Variable Returns to Scale (VRS) based on including observations, free disposal in the input and output and convexity. The envelopment form of the BCC model at division k for calculating the efficiency score of the unit unit under evaluation DMU_p is presented in the following way:

$$\begin{aligned} &Ev_p^k = Min \quad \theta \\ &s.t \ \sum_{j=1}^{n} \lambda_j^k x_{ij}^k + s_i^{k-} = \theta x_{ip}^k \quad i = 1, ..., mk \end{aligned}$$

$$\sum_{j=1}^{n} \lambda_j^k y_{rj}^k - s_r^{k+} = y_{rp}^k \quad r = 1, ..., rk$$
$$\sum_{j=1}^{n} \lambda_j^k z_{dj}^{(k,h)} = z_{dp}^{(k,h)} \quad d = 1, ..., t^{(k,h)},$$

$$\sum_{j=1}^{n} \lambda_j^k z_{dj}^{(h,k)} = \theta z_{dp}^{(h,k)} \quad d = 1, ..., t^{(k,h)},$$
$$\sum_{j=1}^{n} \lambda_j^k = 1$$
$$\lambda_j^k \geq 0, s_i^{k-} \geq 0, s_r^{k+} \geq \quad j = 1, ..., n, i = 1, ..., mk, r = 1, ..., rk. \tag{7}$$

Ev_p^k is the efficiency score of the unit under evaluation. This unit is technically efficient if $Ev_p^k = 1$. Of-course, if for all optimal solutions of model we have $s_i^{k-} = s_r^{k+} = 0 (i = 1, ..., m_k, r = 1, ..., s_k)$

Then it is efficient in the pareto Koopmans context. The efficiency score achieved from (6) is at most $1Ev_p^k \leq 1$. Model (6) determines if unit DMU_p at division k, with using all the resources in its disposal have the Ev_p^k percentage of production power relative to the similar technologies and hence have the technical inefficiency of $(1-Ev_p^k)$at division k which in term is called inefficient. Although after appropriate evaluation of the outputs, the values s_i^{k-}, s_r^{k+} provides the opportunity of using less resources respectively and hence producing more results separately which is a complete evaluation criterion. The dual of (6) represent the support hyper plane on PPS in the point DMU_p which is:

$$\begin{aligned}
e_p^{BBC} &= \text{Max} \sum_{r=1}^{r_k} u_r^k y_{rp}^k + \sum_{d=1}^{t^{(k,h)}} w_d^{(k,h)} z_{dp}^{(k,h)} + u_0^k \\
s.t \quad & - \sum_{i=1}^{m_k} v_i^k x_{ij}^k + \sum_{r=1}^{r_k} u_r^k y_{rj}^k + \sum_{d=1}^{t^{(k,h)}} w_d^{(k,h)} z_{dj}^{(k,h)} - \sum_{d=1}^{t^{(h,k)}} w_d^{(h,k)} z_{dj}^{(h,k)} + u_0^k \leq 0 \\
& - \sum_{i=1}^{m_k} v_i^k x_{ip}^k - \sum_{d=1}^{t^{(k,h)}} w_d^{(k,h)} z_{dp}^{(k,h)} = 1 \\
& v_i^k \geq 0, u_r^k \geq 0, w_d^{(k,h)}, w_d^{(h,k)} \ : free \ \ u_o^k : free
\end{aligned} \tag{8}$$

for each solution like $(V^k, W^{(h,k)}, W^{(k,h)}, U^k)$ for model (7), the hyper plane

$$- \sum_{i=1}^{m_k} v_i^k x_{ij}^k + \sum_{r=1}^{r_k} u_r^k y_{rj}^k + \sum_{d=1}^{t^{(k,h)}} w_d^{(k,h)} z_{dj}^{(k,h)} - \sum_{d=1}^{t^{(h,k)}} w_d^{(h,k)} z_{dj}^{(h,k)} + u_0^k = 0 \tag{9}$$

is a supporting hyperplane in the coordinates of DMU_pon the PPS. The hyper plane (8) has some important tips in the evaluating the efficient unitDMU_p:
_ if $DMU_p \notin MPSS^k$, then in each optimal solution of (7) we have $u_0'^k \neq 0$. Although the vice versa does not always hold.
_ Since the set of solutions of a linear programming is convex, if there exist two different solutions of (7) with positive and negative $u_0'^k \neq 0$, then in the optimal solution it's zero and hence$DMU_p \in MPSS^k$.
_if $u_0'^k > 0$then $-\sum_{i=1}^{m_k} v_i^k x_{ij}^k + \sum_{r=1}^{r_k} u_r^k y_{rj}^k + \sum_{d=1}^{t^{(k,h)}} w_d^{(k,h)} z_{dj}^{(k,h)} - \sum_{d=1}^{t^{(h,k)}} w_d^{(h,k)} z_{dj}^{(h,k)} < 0$ meaning that if the hyper plane (8) is transformed parallel so that it crosses the origin, then the coordinates of DMU_pwould be above it, that is the production average (current efficiency) of the unit is more than the production rate.
_if $u_0'^k < 0$ then $-\sum_{i=1}^{m_k} v_i^k x_{ij}^k + \sum_{r=1}^{r_k} u_r^k y_{rj}^k + \sum_{d=1}^{t^{(k,h)}} w_d^{(k,h)} z_{dj}^{(k,h)} - \sum_{d=1}^{t^{(h,k)}} w_d^{(h,k)} z_{dj}^{(h,k)} > 0$ meaning that if the hyper plane (8) is transformed parallel so that it crosses the origin, then the coordinates of DMU_pwould be below it, that is the production average (current efficiency) of the unit is less than the production rate.
These tips will be used in section 5.

3.2 Scale efficiency

It's obvious that $Ec_p^k \leq Ev_p^k$ and hence if $Ec_p^k = 1$ then $DMU_p$is technically efficient. Having this in mind, the scale efficiency score could be achieved via

$$e_p^{SE} = \frac{Ec_p^k}{Ev_p^k} \tag{10}$$

This score combines both the performance evaluation and the optimal activity size. Having $e_p^{SE} = 1$ means that the unit under evaluation have the most productivity scale size, and it's called scale efficient. Otherwise in case scale$e_p^{Scale} < 1$, the unit is scale inefficient and is not in the optimal production situation. If the unit is not in the optimal production situation, the production size relative to the optimal production situation cannot be determined from the scale efficiency score. In other words, is the production resources higher or lower than the optimal value?

4 Returns to Scale

The location of a unit located on the frontiers of production, relative to the optimal production region in MPSS, is called Returns to Scale. If a unit reaches the MPSS region with increasing its resources, it's said that the unit shows an increasing returns to scale. In other words, increase in the resources lead to the increase in return. If decreasing the resources pushes the unit to MPSS, it's said that the unit shows decreasing returns to scale and decreasing the resources would lead to higher return. In other cases, the returns to scale is considered to be constant which may ignore the changes that does not affect the input and output simultaneously.

One way to determine the RTS of an efficient technical point located on the frontier of productivity, is to compare the marginal product and average product which is referred to as the Scale Elasticity. $e_p^{Scalelasticity}$which in the case of single input-output can be calculated via,

$$e_p^{Scalelasticity} = \frac{{dy}/{dx}}{{y}/{x}} \,|(x,y) = (x_p, y_p) \tag{11}$$

If the production rate is higher (lower) than the average production rate, increasing (decreasing) the resources would cause in higher return. If they are equivalent, changes in the resources would not affect the return. Hence:
_ If $e_p^{Scalelasticity} > 1$ increasing returns to scale for the $DMU_p$is detected.
_ If $e_p^{Scalelasticity} < 1$ decreasing returns to scale for the $DMU_p$is detected.
_ If $e_p^{Scalelasticity} = 1$ returns to scale for the DMU_premains constant.
This factor cannot be calculated in the case of multiple input-output unless the appropriate changes relative to the input-output is used in order to transform it to a single input-output form. Suppose that:

$$\beta(\alpha) = \max\{\beta \mid ((\alpha+1)X_p, (\beta+1)Y_p \in PPS\} \tag{12}$$

Such that the multiple input-output vector (X,Y) is turned into the single input-output vector$(\alpha, (\beta+1))$. Note that the corresponding vector to the unit under evaluation DMU_p is$(\alpha, (\beta+1)) = (0,0)$. In this case the scale elasticity could be calculated via:

$$e_p^{Scalelasticity} = \frac{d\beta(\alpha)}{d\alpha} \mid \alpha = 0 \tag{13}$$

Among all possible changes, equation (12) only considers the increase or decrease relative to the input and output and ignores other changes in the resources, while it's possible that changes in other directions may increase the return.
Using support hyperplanes on PPS, could be a solution to the mentioned problem. For every optimal solution, the hyperplanes (8) are estimations of production rate relative to the small changes in the resources. As we seen in the section 3, the sign of u_0' is the relation between production rate and production average, but we should have in mind that:
_ If for some optimal solutions of (7) we have $u_0'^k > 0$ it means that DMU_pshows decreasing RTS.
_ If for all optimal solutions of (7) we have $u_0'^k < 0$it means that DMU_pshows increasing RTS.
_ If for all optimal solutions of (7) we have $u_0'^k = 0$ it means that the unit under evaluation $DMU_p$is in MPSS and its RTS and changes in resources is not necessary.
Using the scale efficiency, one could determine if the activity size of a unit is optimal or not. And if it's not optimal, using the supporting hyper planes in (7) could

determine whether increasing or decreasing the production size would lead into increasing the return. The question that we are after in this research is that:
Is it possible that changes in resources in non- radial and inappropriate directions would lead into the increasing efficiency? If so, what directions?
In other words, in this paper we are looking for a change in the resources which is not necessarily appropriate, and may lead into increasing, decreasing and or leaving some resources unchanged but in that direction the return and productivity could be increased.

5 Proposed method

As it was discussed before, in the production possibility set with variable returns to scale, the maximum efficiency is achieved inX_{MPSS}. So, based on definition 1

$$X^k{}_{MPSS} = \left\{ X \left| X = \sum_{j \in J_{MPSS}{}^k} \mu_j^k X_j^k, \sum_{j \in J_{MPSS}{}^k} \mu_j^k = 1, \mu_j^k \geq 0 (j \in J_{MPSS}{}^k) \right. \right\} \tag{14}$$

By refer to CCR model, we know that exist at least one efficient DMU with constant link and constant return to scale in every division. The resources in PPS which have shorter distance to the $X^k{}_{MPSS}$ set, if efficient operation, have higher return. Hence for determining the RTS of a unit under evaluation which is outside the MPSS position, it's enough to determine the shortest production position distance to the $X^k{}_{MPSS}$ using:

$$d_{x^k} = (d_1, ..., d_m)^T = \frac{1}{\sum_{i=1}^{m_k} |\alpha_i^*|} (\alpha_1^*, ..., \alpha_{m_k}^*)^T \tag{15}$$

In which $\alpha^* = (\alpha_1^*, ..., \alpha_{m_k}^*)^T$ is one of the optimal solutions for the problem:

$$\begin{aligned}
\text{Min} \quad & \sum_{i=1}^{m_k} |\alpha_i| \\
s.t \quad & \sum_{j \in J^k{}_{MPSS}} \lambda_j^k x_{ij}^k = x_{ip}^k + \alpha_i \qquad i = 1, ..., m_k, k = 1, ..., K, \\
& \sum_{j \in J^k{}_{MPSS}} \lambda_j^k y_{rj}^k = y_{rp}^k + \beta_r \qquad r = 1, ..., r_k, k = 1, ..., K, \\
& \sum_{j \in J^k{}_{MPSS}} \lambda_j^k z_{dj}^{(k,h)} = z_{dp}^{(k,h)} + \gamma_d \qquad d = 1, ..., l, k = 1, ..., K, \\
& \sum_{j \in J^k{}_{MPSS}} \lambda_j^k z_{dj}^{(k,h)} = z_{dp}^{(k,h)} + \gamma_d \qquad d = 1, ..., l, k = 1, ..., K, \\
& \sum_{j \in J^k{}_{MPSS}} \lambda_j^k = 1 \qquad k = 1, ..., K, \\
& \lambda_j^k \geq 0, \quad j \in J^k, i = 1, ..., m_k, r = 1, ..., r_k, k = 1, ..., K.
\end{aligned} \tag{16}$$

The point $X^k{}_p - \alpha^*$ is in MPSS position and has the shortest distance to the $X^k{}_p$position. The production technology from point $X^k{}_p$and in the direction of d_{x^k}shows upgrade the return. Model (15) could easily become a linear model:

$$
\begin{aligned}
\text{Min} \ & \sum_{i=1}^{m_k} \alpha_i^{'} + \alpha_i^{''} \\
s.t \ & \sum_{j \in J^k{}_{MPSS}} \lambda_j^k x_{ij}^k = x_{ip}^k + \alpha_i^{'} - \alpha_i^{''} \qquad i = 1, ..., m_k \\
& \sum_{j \in J^k{}_{MPSS}} \lambda_j^k y_{rj}^k = y_{rp}^k + \beta_r^{'} - \beta_r^{''} \qquad r = 1, ..., r_k, \\
& \sum_{j \in J^k{}_{MPSS}} \lambda_j^k z_{dj}^{(k,h)} = z_{dp}^{(k,h)} + {\gamma_d}^{'(k,h)} - {\gamma_d}^{''(k,h)} \qquad d = 1, ..., l, (k,h) \in L \\
& \sum_{j \in J^k{}_{MPSS}} \lambda_j^k z_{dj}^{(h,k)} = z_{dp}^{(h,k)} + {\gamma_d}^{'(h,k)} - {\gamma_d}^{''(h,k)} \qquad d = 1, ..., l, (h,k) \in L \\
& \sum_{j \in J^k{}_{MPSS}} \lambda_j^k = 1 \\
& \lambda_j^k \geq 0, \alpha_i^{'}, \alpha_i^{''} \geq 0, \beta_r^{'}, \beta_r^{''} \geq 0, \gamma_d^{'}, \gamma_d^{''} \geq 0, j \in J^k{}_{MPSS}, i = 1, ..., m_k, r = 1, ..., r_k.
\end{aligned}
$$

In which

$$
\begin{cases}
\alpha_i = \alpha_i^{'} - \alpha_i^{''}, & \alpha_i^{'}\alpha_i^{''} = 0, \quad i = 1, ..., m_k \\
\beta_r = \beta_r^{'} - \beta_r^{''}, & \beta_r^{'}\beta_r^{''} = 0, \quad r = 1, ..., r_k \\
\gamma_d = \gamma_d^{'} - \gamma_d^{''}, & \gamma_d^{'}\gamma_d^{''} = 0, \quad d = 1, ..., l
\end{cases}
$$

$\alpha^* = (\alpha_1^*, ..., \alpha_{m_k}^*), \beta^* = (\beta_1^*, ..., \beta_{r_k}^*)^T, \gamma^* = (\gamma_1^*, ..., \gamma_l^*)$is an optimal solution for (16), it's obvious that the
projected point

$$(X_p^{k'}, Z_p^{k'}, Y_p^{k'}) = (X_p^k + \alpha^*, Z_p^{k'} + \gamma^*, Y_p^k + \beta^*)$$

is in the MPSS position and is Pareto-Koopmans efficient. The unit under evaluation could reach the MPSS of the projection point $(X_p^{k'}, Z_p^{k'}, Y_p^{k'})$in (17) with changing its input resources from $X^k{}_p$in the direction $d_{x^k}$with magnitude$\sigma = \sum_{i=1}^{m_k} |\alpha_i^*|$. Since $DMU_p$is out of the MPSS position, vector $\alpha^* \neq 0$, but it's not necessarily non-positive or non-negative, the same also holds for the changing output results vector obtained from this transformation, β^*. Hence the concept of increasing and or decreasing returns to scale is irrelevant here and only the changing and or constant returns to scale could be relevant. All units which their resource situation is in$X^k{}_{MPSS}$, have a constant return to scale, otherwise they have a variable return to scale. Anyway, in this transformation the return would increase regarding to (1). Although it's possible that$\beta^* = 0$. The transformation direction would be:

$$d_{X^k} = \frac{1}{\sum_{i=1}^{m_k} |\alpha_i^*| + \sum_{r=1}^{r_k} |\beta_r^*| + \sum_{d=1}^{l} |\gamma_d^*|} ((\alpha_1^*, ..., \alpha_{m_k}^*)^T, (\beta_1^*, ..., \beta_{r_k}^*)^T, (\gamma_1^*, ..., \gamma_d^*)^T)$$

Using (14) we would have

$$d = \frac{1}{\sum_{i=1}^{m_k}|\alpha_i^*| + \sum_{r=1}^{r_k}|\beta_r^*| + \sum_{d=1}^{l}|\gamma_d^*|}(\sum_{i=1}^{m_k}|\alpha_i^*|\, d_{x^k}, (\beta_1^*, ..., \beta_{r_k}^*)^T, (\gamma_1^*, ..., \gamma_d^*)^T) =$$

$$\frac{\sum_{i=1}^{m_k}|\alpha_i^*|}{\sum_{i=1}^{m_k}|\alpha_i^*| + \sum_{r=1}^{r_k}|\beta_r^*| + \sum_{d=1}^{l}|\gamma_d^*|}(d_{x^k}, \frac{1}{\sum_{i=1}^{m_k}|\alpha_i^*|}(\beta_1^*, ..., \beta_{r_k}^*)^T, \frac{1}{\sum_{i=1}^{m_k}|\alpha_i^*|}(\gamma_1^*, ..., \gamma_d^*)^T)$$

So, the rate of change in the output corresponding to one unit change in the direction d_{x^k} is equivalent to $\mathrm{Rate}(d_{x^k}) = \frac{1}{\sum_{i=1}^{m_k}|\alpha_i^*| + \sum_{r=1}^{r_k}|\beta_r^*| + \sum_{d=1}^{l}|\gamma_d^*|}$ in the direction:

$$d_{y^k} = (d_{m_k+1}, ..., d_{m_k+r_k})^T = \frac{1}{\sum_{r=1}^{r_k}|\beta_r^*|}(\beta_1^*, ..., \beta_{r_k}^*)^T$$

$$d_{z^k} = (d_{r_k+1}, ..., d_{r_k+l})^T = \frac{1}{\sum_{d=1}^{l}|\gamma_d^*|}(\gamma_1^*, ..., \gamma_l^*)^T$$

Only of$\beta^* \neq 0, \gamma^* \neq 0$, otherwise$Rate(d_{x^k}) = 0$.

Note 1: in the above method, the shortest path to the MPSS set is used as a measurement to determine the RTS situation, while all directions that lead to the MPSS could be used too. Hence the RTS situation of a frontier point is also dependent on the chosen direction. As previously mentioned, in most methods available determining the appropriate RTS direction (equivalently here $\alpha^* > 0, \beta^* > 0, \gamma^* > 0$) is used, thus the comparison results of that methods with the proposed method could differ.

Note 2: The returns to scale concept is one of the features of the production function. The efficient points are used as estimated points of this function and is used in RTS studies. Although the projection of the inefficient points on the frontier of PPS:

$$(\overline{X_p^k}, \overline{Z_p^{(h,k)}}, \overline{Z_p^{(k,h)}}, \overline{Y_p^k}) = (\theta^* X_p^k, \theta^* Z_p^{(h,k)}, Z_p^{(k,h)}, Y_p^k) \tag{17}$$

which (θ^*) is an optimal solution for (6), could be used to determine the RTS of an estimated point from the production function with the same resources. In most studies, the result of RTS, (19) is assigned to the inefficient unit(X_p^k, Z_p^k, Y_p^k).
Here we are going to investigate this further using the example article [12]:

6 Example

Fig 1 exhibits typical integrated electric utility companies consisting of generation, transmission and distribution divisions. The generation division (division1) uses several inputs such as capital, labour and fuel (input1) and produces electric power. Then it becomes an intermediate input for the transmission division (link1-2). In the transmission division (division2) , companies utilize capital, labour and purchased power inputs(input2) as well as the intermediate inputs from generation division (link1-2) .Electricity through transmission lines is sent to distribution division as intermediate output(link2-3) or sales to large customers(output2) that do not utilize distribution line. The distribution division (division3) uses capital and labour inputs (input3) and the intermediate input from the transmission division (link2-3) and provides electricity to small customers (output3).

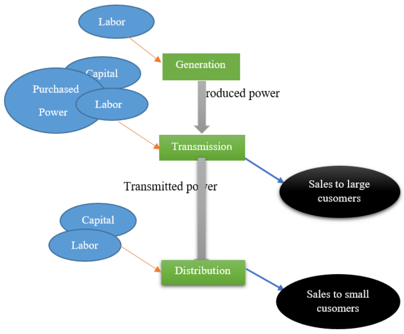

Figure 1: Vertically integrated electric power companies

In this example DMUs K, L, M and N have a same network structure. Input, output and links are as follow:

Div1 (generation):

Input1: labour input (number of employees)

Div2 (transmission):

Input2: labour input (number of employees)

Output2: electric power sold to large customers

Div3 (distribution):

Input3: (number of employees)

Output3: electric power sold to small customers

Link1-2: electric power generated (output from generation division and input to transmission division)

Link2-3: electric power sent (output from transmission division and input to distribution division)

Table 1 denotes this data as below:

Here is an example of a table.

First we are using model (4) for division 1.We know that dmu1 is CCR efficiency so it is in MPSS region. Units in MPSS region have fixed RTS status C and other units have variable RTS status V. After running model (16) on the projection of units on the PPS, we can find minimum distance to a set of ${X^k}_{MPSS}$ as ${\alpha_1}^* > 0$that we can see variable directiond_x, rate of change $\sigma$and output rate change in this direction $Rate(d_x)$ based on result of this model. As you can see variable direction isn't just in increase or decrease direction proportionate to inputs.

In division2, dmus2, 3 and 4 are in MPSS and ${X^k}_{MPSS}$in inputs region obtained

Table 1: Data for four DMUs

DMU	Div1	Div2		Div3		Link	
	Input1	Input2	Output2	Input3	Output3	Link1-2	Link2-3
K	3	10	2	5	2	8	2
L	14	1	1	5	5	9	5
M	16	2	2	11	4	7	4
N	19	0.5	2	7	4	11	4

from convex composition of inputs vectors of these three points. We run model (16) on the projection of DMU1 and obtained minimum distance to a set of $X^k{}_{MPSS}$ $\alpha_1{}^*$,d_x and $Rate(d_x)$ such as division1. Finally, we can conclude variable direction isn't just in increase or decrease direction proportionate to inputs.

In division3, DMUs 1, 2, 3 and 4 are in MPSS and $X^k{}_{MPSS}$in inputs region obtained from convex composition of inputs vectors of this points.

7 Financial Example

In connection with the proposed model, in this section, we will review 20 stock companies in the years 1398, 1399:
The year 1398 is the first division and the year1399 is the second division of the example. The Inputs are the capital and NAV of the companies, the link in this example is the year-end profit of the companies and the output is the market value. Table 5,6 denotes this data as below:

Table 2: Results in first division

DMU	Ec_p^1	Ev_p^1	e_p^{SE}	X_p^1	**RTS**	$(\overline{X},\overline{Z}^{(h,k)},\overline{Z}^{(k,h)},\overline{Y})$	$(\alpha^*,\gamma_1^*,\gamma_2^*,\beta^*)$	d_x	σ_x	Rate(d_x)
dmu1	1	1	1	In	C	(3,8,2,0)	-	-	-	-
dmu2	0.24	0.6	0.4	out	V	(3.36,9,5,0)	(0.36,0,-1,0)	1	0.36	0
dmu3	0.16	1	0.16	out	V	(2.56,7,4,0)	(0.44,1,0,0)	1	0.44	0
dmu4	0.22	1	0.22	out	V	(4.18,11,4,0)	(1.18,0,-3,0)	1	1.18	0

Table 3: Results in second division

DMU	Ec_p^2	Ev_p^2	e_p^{SE}	X_p^2	**RTS**	$(\overline{X},\overline{Z}^{(h,k)},\overline{Z}^{(k,h)},\overline{Y})$	$(\alpha^*,\gamma_1^*,\gamma_2^*,\beta^*)$	d_x	σ_x	Rate(d_x)
dmu1	0.87	1	0.87	Out	V	(8.7,6.96,2,2)	(6.7,0.4,2,0)	-	6.7	0.11
dmu2	1	1	1	In	C	(1,9,5,1)	(0, 0.67,-1,1)	-	-	-
dmu3	1	1	1	In	C	(2,7,4,2)	-	-	-	-
dmu4	1	1	1	In	C	(0.5,11,4,2)	-	-	-	-

Table 4: Results in third division

DMU	Ec_p^3	Ev_p^3	e_p^{SE}	X_p^3	**RTS**	$(\overline{X},\overline{Z}^{(h,k)},\overline{Z}^{(k,h)},\overline{Y})$	$(\alpha^*,\gamma_1^*,\gamma_2^*,\beta^*)$	d_x	σ_x	Rate(d_x)
dmu1	1	1	1	In	C	(5,2,0,2)	-	-	-	-
dmu2	1	1	1	In	C	(5,5,0,5)	-	-	-	-
dmu3	1	1	1	In	C	(11,4,0,4)	-	-	-	-
dmu4	1	1	1	In	C	(7,4,0,4)	-	-	-	-

Table 5: Data for 20 DMUs in year1398

	i2	i1	link	o1
DMU	Capital (million Rials)	NAV (billion rials)	Net profit (loss)	Market value (billions Rials)
DMU1	16250000	213581	16297952	246885
DMU2	12500000	125310	5705679	105,125
DMU3	4500000	123845	8248906	53,910
DMU4	2400000	31765	2627044	17,856
DMU5	4000000	37050	2196857	47,960
DMU6	6000000	34634	12855563	40,440
DMU7	1300000	6200	2550754	12,678.9
DMU8	4500000	63045	13918606	43,650
DMU9	10000000	61942	5648116	319,200
DMU10	27000000	243970	27895155	411,750
DMU11	15000000	361614	67965678	372,690
DMU12	1500000	32967	1448585	33,075
DMU13	1600000	8201	1096422	18,270
DMU14	10675000	30973	1911212	64,156.75
DMU15	1110300	3479	1339663	56,848.47
DMU16	17500000	376700	21193436	195,650
DMU17	1150000	12205	1260933	9,701.4
DMU18	6000000	46387	13525134	33,540
DMU19	2750000	58401	5440362	43,862.5
DMU20	6691000	158335	10580807	86,760

Table 6: Data for 20 DMUs in year1399

	i2	i1	link	o1
DMU	Capital (million Rials)	NAV (billion rials)	Net profit (loss)	Market value (billions Rials)
DMU1	27250000	387506	47046834	222633
DMU2	18700000	122620	22048791	82,055.6
DMU3	4500000	81421	8603032	40,455
DMU4	3888000	30672	4355258	17,317.152
DMU5	4000000	28462	2913605	38,280
DMU6	6000000	55882	7164753	34,440
DMU7	3900000	10660	2697182	13,070
DMU8	4500000	49809	10218841	30,424
DMU9	30000000	131375	35158434	213,300
DMU10	27000000	638055	59826420	476,193.6
DMU11	45000000	409126	85702635	257,400
DMU12	4800000	78630	3054628	23,644.8
DMU13	3000000	20960	6250463	11,286
DMU14	10675000	82912	4398862	49,872
DMU15	42620178	86116	1510516	32,092.994
DMU16	17500000	320489	28006426	153,475
DMU17	1150000	10704	1795987	8,019.6
DMU18	12000000	49987	13281123	27,936
DMU19	2750000	50821	6746742	48,427.5
DMU20	12000000	189043	15681110	85,560

Table 7: Results in first division

DMU	Ec_p^1	Ev_p^1	e_p^{SE}	X_p^1	**RTS**	$(\overline{X},\overline{Z}^{(h,k)},\overline{Z}^{(k,h)},\overline{Y})$	$(\alpha^*,\gamma_1^*,\gamma_2^*,\beta^*)$	d_x	σ_x
Dmu1	0.86	1	0.86	out	V	(333255.16,23435000,40460277.24,0, 222633)	(0,0,-413952,0)	-	0
Dmu2	0.72	0.48	1.5	out	V	(88286.4,13464000,15875129.52,0, 82,055.6)	(0,0,3522349.02,0)	-	0
Dmu3	0.79	2.46	0.32	out	V	(64322.59,3555000,6796395.28,0, 40,455)	(0,0,1255874.59,0)	-	0
Dmu4	0.62	4.47	0.13	out	V	(19016.64,2410560,3440653.82,0, 17,317.152)	(0,0,-1054398.09,0)	-	0
Dmu5	1	2.67	0.37	in	V	(28462,4000000,2913605,0, 38,280)	(0,0,0,0)	-	0
Dmu6	0.77	3.27	0.23	out	C	(43029.14,4620000,5516859.81,0, 34,440)	(0,0,2204396.9)	-	0
Dmu7	0.92	3.78	0.24	out	C	(98072,3588000,2481407.44,0, 13,070)	(1.32,0,4256347,0)	1.14	1.32
Dmu8	1	1.25	0.8	in	V	(49809,4500000,10218841,0, 30,424)	(0,0,-124700.4,0)	-	0
Dmu9	1	1	1	in	V	(131375,30000000,35158434,0, 213,300)	(0,0,0,0)	-	0
Dmu10	1	1	1	in	V	(638055,27000000,59826420,0, 476,193.6)	(0,0,0,0)	-	0
Dmu11	0.89	1	0.89	out	C	(364122.14,40050000,76275345.15,0, 257,400)	(1.52,277751,-50544201,15000000)	1.23	1.52
Dmu12	0.58	1	0.58	out	V	(45605.4,2784000,1771684.24,0, 23,644.8)	(0,0,7002107.1,0)	-	0
Dmu13	1	1	1	in	V	(20960,3000000,6250463,0, 11,286)	(0,0,0,0)	-	0
Dmu14	0.69	0.65	1.06	out	V	(57209.28,3035214.78,0, 49,872)	(0, 0,9947078.6,0)	-	0
Dmu15	0.63	1	0.63	out	C	(54253.08,26850712.14,951625.08,0, 32,092.994)	(1.26,0,33647918,12620178)	1.12	1.26
Dmu16	0.7	1	0.7	out	V	(224342.3,12250000,19604498.2,0, 153,475)	(0,0,5688281.5,0)	-	0
Dmu17	0.83	1	0.83	out	V	(8884.32,954500,1490669.21,0, 8,019.6)	(1.64,0.4950755,0)	1.27	1.64
Dmu18	1	5.1	0.19	in	V	(499987,12000000,13281123,0, 27,936)	(0,0,0,0)	-	0
Dmu19	1	4.32	0.23	in	V	(50821,2750000,6746742,0, 48,427.5)	(0,0,0,0)	-	0
Dmu20	0.6	2.41	0.25	out	V	(113425.8,7200000,9408666,0, 85,560)	(0,0,6081097.7,0)	-	0

Table 8: Results in second division

DMU	Ec_p^1	Ev_p^1	e_p^{SE}	X_p^1	**RTS**	$(\overline{X},\overline{Z}^{(h,k)},\overline{Z}^{(k,h)},\overline{Y})$	$(\alpha^*,\gamma_1^*,\gamma_2^*,\beta^*)$	d_x	σ_x
Dmu1	0.86	1	0.86	out	V	(333255.16,23435000,40460277.24,0, 222633)	(0,0,-413952,0)	-	0
Dmu2	0.72	0.48	1.5	out	V	(88286.4,13464000,15875129.52,0, 82,055.6)	(0,0,3522349.02,0)	-	0
Dmu3	0.79	2.46	0.32	out	V	(64322.59,3555000,6796395.28,0, 40,455)	(0,0,1255874.59,0)	-	0
Dmu4	0.62	4.47	0.13	out	V	(19016.64,2410560,3440653.82,0, 17,317.152)	(0,0,-1054398.09,0)	-	0
Dmu5	1	2.67	0.37	in	V	(28462,4000000,2913605,0, 38,280)	(0,0,0,0)	-	0
Dmu6	0.77	3.27	0.23	out	C	(43029.14,4620000,5516859.81,0, 34,440)	(0,0,2204396.9)	-	0
Dmu7	0.92	3.78	0.24	out	C	(98072,3588000,2481407.44,0, 13,070)	(1.32,0,4256347,0)	1.14	1.32
Dmu8	1	1.25	0.8	in	V	(49809,4500000,10218841,0, 30,424)	(0,0,-124700.4,0)	-	0
Dmu9	1	1	1	in	V	(131375,30000000,35158434,0, 213,300)	(0,0,0,0)	-	0
Dmu10	1	1	1	in	V	(638055,27000000,59826420,0, 476,193.6)	(0,0,0,0)	-	0
Dmu11	0.89	1	0.89	out	C	(364122.14,40050000,76275345.15,0, 257,400)	(1.52,277751,-50544201,15000000)	1.23	1.52
Dmu12	0.58	1	0.58	out	V	(45605.4,2784000,1771684.24,0, 23,644.8)	(0,0,7002107.1,0)	-	0
Dmu13	1	1	1	in	V	(20960,3000000,6250463,0, 11,286)	(0,0,0,0)	-	0
Dmu14	0.69	0.65	1.06	out	V	(57209.28,3035214.78,0, 49,872)	(0, 0,9947078.6,0)	-	0
Dmu15	0.63	1	0.63	out	C	(54253.08,26850712.14,951625.08,0, 32,092.994)	(1.26,0,33647918,12620178)	1.12	1.26
Dmu16	0.7	1	0.7	out	V	(224342.3,12250000,19604498.2,0, 153,475)	(0,0,5688281.5,0)	-	0
Dmu17	0.83	1	0.83	out	V	(8884.32,954500,1490669.21,0, 8,019.6)	(1.64,0.4950755,0)	1.27	1.64
Dmu18	1	5.1	0.19	in	V	(499987,12000000,13281123,0, 27,936)	(0,0,0,0)	-	0
Dmu19	1	4.32	0.23	in	V	(50821,2750000,6746742,0, 48,427.5)	(0,0,0,0)	-	0
Dmu20	0.6	2.41	0.25	out	V	(113425.8,7200000,9408666,0, 85,560)	(0,0,6081097.7,0)	-	0

8 Conclusion

Regard to presented models for determine the optimize productive size, we got the result that RTS of a DMU can't be proportional and it may determine in the different directions.

In all of the improvement directions of productive size, conversions for set of X^k_{MPSS} in shortest direction is a clear criterion for determine situation of RTS at point.

Here we examine Network RTS in a separate state. However, the detection of RTS subunits in the network cannot be independent of each other and increasing, decreasing or constantan the RTS of one subunit is conditional on the status of other relevant subunits of RTS. That is, if the RTS is an incremental subunit, the RTS subunits associated with it must be allowed to be increased. In other word, increasing or decreasing inputs of a subunit will have the effect of changing the inputs and outputs of the subunits associated with this subunit.

Eliminating this drawback in detecting RTS subunits of a network can be considered a topic for future research.

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How to Cite: Zeinab Sinaeinasab[1], Hadi Bagherzadeh Valami[2], *Introduction a Method of Determining Returns to Scale in Network Data Envelopment Analysis*, Journal of Mathematics and Modeling in Finance (JMMF), Vol. 2, No. 2, Pages:15–34, (2022).