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Research paper

Design of a Pure Endowment Life Insurance Contract Based on Optimal Stochastic Control

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Abstract:
In this paper, we design a pure-endowment insurance contract and obtain the optimal strategy and consumption for a policyholder with CRRA utility function. In this contract, premiums are received from the policyholder at certain times. The insurer undertakes to pay the premiums by a certain guarantee rate, in addition, by investing in a portfolio of risky and risk free assets share invest profits. We used Variance Gamma process as a representative of infinite activity jump models and sensitivity of jump parameters in an uncertainty financial market has been studied. Also we compared results using by two forces of mortality.

Keywords: Optimal Strategy; Force of Mortality; Pure-Endowment; Infinite Activity Lévy Model.
MSC Classification: 62P05, 91B30, 93E20.

1 Introduction

There are a variety of products in the life insurance literature. These products differ in how the benefits are paid and maturity time. These products include term insurance, pure endowment and endowment. In this paper, we design a pure endowment contract by stochastic approach. A pure endowment is a type of life insurance policy in which an insurance company agrees to pay the insured a certain amount of money if the insured is still alive at the end of specified time period. In our designed contract, premiums are received from the policyholder at certain times. The insurer undertakes to pay the premiums by a certain guarantee rate, in addition, by investing in a portfolio of risky and risk free assets, share invest profits.

In this paper, we focus on the investment (in a finite timetable) for a person who buys this policy. In general, we consider a portfolio of risky and risk free asset and find an optimal strategy and consumption for this portfolio. Optimal allocation of capital among a set of financial assets under conditions of uncertainty

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and risk is a well-established research field in modern finance theory. In this aspect and before Merton [28] contribution to the field, most portfolio selection models have only considered one-period, static models based on Markowitz's mean variance [27] modern portfolio theory. Merton in [28] and [29], studied "the combined problem of optimal portfolio selection and consumption rules for an individual in a continuous-time model". As a particular case, he examined in detail the two-asset model (a risk free asset and a risky one) with constant relative risk-aversion or iso-elastic marginal utility. Yaari [38], consider a life insurance problem with uncertain lifetime. In this respect, Richard [32] was combined two above approaches that used sophisticated methods at an early date for the analysis of a life-cycle life insurance and consumption-investment problem in a continuous time model. In general, our contract is similar to an equity-linked life insurance. Typically, the policyholder pays either a single premium or a stream of periodic premiums during an accumulation phase. In return, the insurer guarantees a stream of periodic payments starting either immediately or at a future date. Barigou and Delong [3] defined equity-linked life insurance contracts and priced with multiple risk factors by neural networks. Ceci et al. [8] considered hedging problem of a unit-linked life insurance. They used an endowment insurance contract whose final value depends on the trend of a stock market where the premia the policyholder pays are invested. Kirkby and Nguyen [18] desined an equity-linked Guaranteed Minimum Death Benefit (GMDB) whose payoff depends on a dollar cost averaging (DCA) style periodic investment in the risky index, with rider premiums paid at regular intervals and derived closed-form valuation formulas under the fairly broad class of exponential Lévy models for the risky index, which includes Black-Scholes as a special case. In this paper, we used the jump-diffusion models for simulating stock price. Wang et al. [34] Priced an equity-linked death benefits contract by complex Fourier series method under regime-switching jump-diffusion models. Bosserhoff and Stadje [5] defined an optimal strategy problem with unit-link life insurance contract under mean-variance portfolio selection of an insurer and allowed for the incorporation of basis (mortality) risk. Wang et al. [35] studied the hedging problem of unit-linked life insurance contracts in an incomplete market presence of self-exciting effect, which is described by a Hawkes process. They demonstrated that jump clustering has a significant impact on the optimal hedging strategies. Huertas [16] using by fractional models, derived mathematical reserves of unit-linked insurance policies. Mathematical reserves are specified amounts of capital that an insurance company is legally obligated to allot to cover its expected claims in a given period.

The main results of this paper is about investing of the received premiums in financial markets. Following Kung and Yang [21], we consider an optimal investment strategy and consumption for a policyholder who has purchased a pure endowment contract. In this literature, Li et al. [23] calculated the optimal insurance and reinsurance problems for an insurer with risk process under the heston model. Simular to them, we used jump-diffusion processes for modeling risky assets. Liang and

Lu [24] used short-noice process to model stock price in the equity-linked life insurance. Another applications of optimal control (strategy and consumption) is in defining to pension fund whether Defined Contribution (DC) or Defined Benefit (DB). This topic still receive a considerable attention from authors. For instance, Xu and Gao [36] provided a closed-form solution for the optimal portfolio control problem of a DC pension. Dong and Zheng [14] applied the concavification and dual control method to solve an optimal investment problem for a DC pension fund. Yao et al. [37] considered the stochastic inflation rate which described by a discrete-time of the Ornstein-Uhlenbeck process to derive an analytical expression for the efficient investment strategy for a DC pension fund. Dong and Zheng [13] for a DC pension fund whose its manager is a loss averse person derived an optimal investment strategy in terms of the dual controlled process and the dual value function.

In recent work of pure endowment contract can be found in ceci et al. [7] when the insurance company has a limited information on the mortality intensity of the policyholder. They priced the pure endowment contract via BSDEs under partial information. And in last paper, Baños et al. [4] considered an unit-linked insurance policy and evaluated variance and interest rate risk in the insurance markets.

The rest of this article is organized as follows. Section 2 collects some elements that play vital roles in the rest of this article. Using the stochastic optimal control method, Section 3 calculates the optimal investment strategy for an asset which its stock dynamic has a jump process. In Section 4, the numerical implementation of the results have been given.

2 Preliminaries

In the paper we design a pure endowment contract by stochastic approach. A pure endowment is a type of life insurance policy in which an insurance company agrees to pay the insured a certain amount of money if the insured is still alive at the end of a specific time period. In our designed contract, the premiums $P(t)$ are received from the policyholder at certain times. The insurer undertakes to pay the premiums by guarantee rate g, in addition, by investing in a portfolio of risky and risk free assets, share invest profits with rate τ. We assume that in maturity time T, the policyholder is alive. It should be noted that the two guarantee and participation rates are updated at the beginning of each year. In this contract, we create an account for the policyholder in which we put the return of investment. We name this account C_t^+ and defined:

$$C_t^+ = \begin{cases} P & t=0 \\ (1+g)C_{t-1} + \tau\Big[W_t^{\pi^*,c^*} - (1+g)C_{t-1}\Big]^+ & t=1,...,T \end{cases}$$

Where $[A]^+ = \max[A, 0]$ and $W_t^{\pi^*,c^*}$ is the wealth account under optimal strategy and consumption (π^*,c^* respectively) that we will introduce in the following. Another account designed for the policyholder is the reserve account, which we indicate with R_t^+, and is a cover for the policyholder's account when the market is in a bad situation. In other words, we can write

$$R_t^+ = \begin{cases} 0 & t = 0 \\ W_t^{\pi_t^*} - (1+g)C_{t-1} - \tau\Big[W_t^{\pi^*,c^*} - (1+g)C_{t-1}\Big]^+ & t = 1, ..., T \end{cases}$$

We remind that in pure endowment contracts, only in the insured account at the end of the contract is considered. Hence we write the policyholder's account at the maturity:

$$C_T = (1+g)^T C_0 + \tau \sum_{t=1}^{T} \Big[W_t^{\pi^*,c^*} - (1+g)C_{t-1}\Big]^+ (1+g)^{T-t}. \tag{1}$$

Next, with the premiums received from the policyholder, we create an investment portfolio of risky and risk free assets. We start the model's description by assuming that an expected utility maximizing, risk-averse economic agent makes investment decisions in a continuous-time setting in a finite time horizon $[0, T]$ in a market modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$. All the processes in the paper are adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ which describes the flow of information in the given time period. Let us assume that the market composes of two underling securities: (**1**) A safe and risk free asset (e.g. a bond or a bank account) described by

$$\frac{dB(t)}{B(t)} = r(t)dt, \quad B(0) = 1, \tag{2}$$

for a locally deterministic interest rate process $r(t)$ and (**2**) A risky asset (e.g. a stock) specified by the stochastic dynamics for its return as

$$\frac{dS(t)}{S(t)} = \Big(\mu(t) + \frac{1}{2}\sigma(t)^2\Big)dt + \sigma(t)dZ(t) + \int_{-\infty}^{\infty} (e^x - 1)N(dt, dx), \quad S(0) = S_0, \tag{3}$$

where $\mu(t)$ and $\sigma(t)$ are two adapted processes, respectively, representing the drift and diffusion parts of the rate of return and $Z(t)$ is a standard Brownian motion and $N(\cdot, \cdot)$ is a Poisson random counting measure with the compensator $\Pi(\cdot, \cdot)$ both defined on $\mathbb{R}^+ \times \mathbb{R}$. Moreover, we assume that the compensator $\Pi(\cdot, \cdot)$ for any measurable random function $\Phi(t, x) := \Phi(\omega, t, x)$ satisfies

$$\mathbb{E}\left(\int_{\mathbb{R}} \Phi(t, x)N(dt, dx)\right) = \int_{\mathbb{R}} \Phi(t, x)\Pi(t, dx)dt. \tag{4}$$

Hereafter now, we assume that the jump part and the Brownian part in the stock dynamic are independent. Note that the measure $\Pi(t, dx)$ specifies the intensity of the aggregate jump arrival rate and for practical purposes, we could assume it to depend on a deterministic/stochastic state variable ν_t via $\Pi(t, dx) = \Pi(\nu_t, dx)$.

We may think of this state variable as representing the current level of the market or business activity (e.g. trading volume or liquidity) or some other related micro-level indicator and assume that it follows a stochastic differential equation of the form

$$d\nu_t = m(\nu_t, t)dt + \sigma(\nu_t, t)dZ_t^\nu.$$

Where $m(\nu_t, t)$, $\sigma(\nu_t, t)$ are the drift and diffusion parts of the state variable ν_t and Z_t^ν is a standard Brownian motion. In such a situation, we will assume that the drift μ, the diffusion σ and the interest rate r are all deterministic functions of time and the state variable. We should note that: in the special case that the jump arrival rate is proportional to the state according to

$$\Pi(\nu_t, dx) = \nu_t\Pi(dx). \tag{5}$$

Financial models with jumps can be decomposed as the jump-diffusion models and models with infinite number of jumps in every interval, say infinite activity models. The regular price for the jump-diffusion models can be obtained by a diffusion process, which its jumps punctuated at random intervals. Such the jumps represent rare events-crashes and large drawdown. See Merton [29] and the Kou [19] for some examples on such approach. For the infinite activity models, since dynamics of jumps is already rich enough to generate nontrivial small-time behavior, one does not need to introduce a Brownian component. Moreover, Madan [26] among other authors, has been argued that such infinite activity models give a more realistic description of the price process at various time scales. It's worthwhile mentioning that, many models from this class can be constructed via a Brownian subordination, which gives them additional analytical tractability compared to jump-diffusion models. In this article, we consider a class of the infinite activity models.

(Variance-Gamma Model: Infinite Activity Case) The Variance-Gamma model could be considered as an extension of the Brownian motion process with drift which is obtained by a random time change specified by a gamma process as:

$$X_t = \theta\tau_t + \rho Z(\tau_t),$$

where θ and ρ are some given constants and for fixed $l > 0$, and $\nu > 0$, the gamma process $\tau_t = \gamma_t(l, \nu)$ has mean rate $l\nu$ and variance rate $l^2\nu$. We also note in passing that the last (integral) term in (3) indicates the presence of jumps in stock price dynamics, first considered by Merton [30], where he assumed that the stock follows a jump-diffusion process with a Poisson (slow) arrival rate (see also Liu et al. [23] for a recent study about implications of jumps in pricing and volatility on investment strategies). However, we consider here the more realistic choice of Lévy processes with extremely fast (potentially infinite) jump rates. The specific example in our mind is the Variance-Gamma (VG) process which is a pure jump Lévy process with an infinite arrival rate of small jumps, first introduced to the literature by Madan and Seneta [25].

In the VG model, the Lévy compensator measure could be represented as

$$\Pi(dx) = \frac{1}{x} e^{-\frac{x}{\lambda_u}} I_{\{x>0\}} - \frac{1}{x} e^{\frac{x}{\lambda_d}} I_{\{x<0\}},$$

where λ_u and λ_d are the posetive solutions from $\lambda_u - \lambda_u = \theta l$ and $\lambda_u \lambda_u = \frac{1}{2}\rho^2 l$. That is $\lambda_u = \frac{1}{2}\left(\sqrt{\theta^2 l^2 + 2\rho^2 l} + \theta l\right)$ and $\lambda_d = \frac{1}{2}\left(\sqrt{\theta^2 l^2 + 2\rho^2 l} - \theta l\right)$.

We know that the VG process is a pure jump Lévy process. In other words the volatility in the stock dynamic is zero ($\sigma = 0$). For some (deterministic) measure $\Pi(\cdot)$, we will be able to simplify many of the presented results in the paper.

Suppose $\{W(t)\}_{t\in[0,T]}$ denotes the wealth process of the investor representing the total accumulated wealth at time t. We need the following definitions before any further progress.

Definition 2.1. (i) A **portfolio process** (or portfolio strategy) is a real-valued progressively measurable process $\{\pi(t)\}_{t\in[0,T]}$ with

$$\int_0^T |\pi(t)W(t)|^2 dt < \infty, \quad \mathbb{P} - \text{a.s.}$$

(ii) A **consumption process** is a non-negative real-valued progressively measurable process $\{c(t)\}_{t\in[0,T]}$ with

$$\int_0^T c(t)dt < \infty, \quad \mathbb{P} - \text{a.s.}$$

The **cumulative consumption** up to time t will be denoted by $C(t)$ and is defined by

$$C(t) = \int_0^t c(u)du.$$

In the sequel, we assume that the investor maintains a **self-financing portfolio** by allocating his/her wealth among the two underlying assets in such a way that any wealth change is only due to consumption or gains/losses from investment in the bond and the stock. In this respect, We are following Iscanoglu-Cekic [17] to define the wealth process $W^{\pi,c} := \{W^{\pi,c}(t)\}_{t\in[0,T]}$ corresponding to a self-financing portfolio/consumption strategy (π, c) will be the (unique) solution of the stochastic differential equation

$$dW^{\pi,c}(t) = ((r(t) + \pi(t)(\mu(t) + \frac{1}{2}\sigma(t)^2 - r(t)))W^{\pi,c}(t) - c(t))dt \qquad (6)$$

$$+P(t)dt + \pi(t)W^{\pi,c}(t)\sigma(t)dZ(t) + \pi(t-)W^{\pi,c}(t-)\int_{-\infty}^{\infty}(e^x - 1)N(dt, dx),$$

Where $W^{\pi,c}(0) = P(0)$.
A self-financing strategy (π, c) is said to be **admissible** if $W^{\pi,c}(t) \geq 0,\ \mathbb{P} - \text{a.s.}$, for all $t \geq 0$. The set of all admissible strategies will be denoted by $\mathcal{A}$.

Definition 2.2. CRRA utility function In this definition, we introduce two popular utility functions and calculate optimal policy for their. One of the most important utility functions is CRRA, so that, these utility functions have two utility function in their self and the following is displayed:

$$U(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma} & \gamma \neq 1, \gamma > 0, \\ \\ \log(x) & \gamma = 1, \end{cases} \tag{7}$$

in which $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ and $U(x) = \log(x)$ are power and logarithm utility, respectively.

In the following, we will design a pure endowment insurance product that is connected to investing in stock markets. As mentioned earlier, in pure endowment product, the condition of the policyholder surviving at the end of the contract is one of the main conditions for the execution of the contract. Hence, we consider the probability distribution for the survival of the policyholder. The function $\bar{F}(t)$, which is called the survivor function, is defined to be the probability that the lifetime is greater than or equal to t,

$$\bar{F}(t) = P\left(\tau \geq t\right) = 1 - F(t)$$

The hazard function represents the instantaneous death rate for the policyholder who has survived to time t, and it is defined by:

$$\lambda(t) = \lim_{\delta t \to 0} \frac{P\left(t \leq \tau < t + \delta t \,|\tau \geq t\right)}{\delta t} = \frac{f(t)}{\bar{F}(t)},$$

From this it follows that

$$\lambda(t) = -\frac{d}{dt}\ln(\bar{F}(t)),$$

in which case the survivor function is given by:

$$\bar{F}(t) = \exp\left\{-\int_0^t \lambda(u)du\right\},$$

and the probability density function is related to the hazard rate by:

$$f(t) = \lambda(t)\exp\left\{-\int_0^t \lambda(u)du\right\}.$$

We now introduce some additional notation associated with the random variable τ. Denote by $f(s,t)$ the conditional probability density for death at time s conditional upon the policyholder being alive at time $t \leq s$, so that

$$f(s,t) = \frac{f(s)}{\bar{F}(t)} = \lambda(s)\exp\left\{-\int_t^s \lambda(u)du\right\},$$

And denote by $F(s,t)$ the conditional probability for the policuholder being alive at time s conditional upon being alive at time $t \leq s$, so that

$$F(s,t) = \frac{\bar{F}(s)}{\bar{F}(t)} = \exp\left\{-\int_t^s \lambda(u)du\right\}.$$

3 Creating an Investment Portfolio for the Policyholder

In this section, we are creating a portfolio of assets for a policyholder and finding an optimal strategy and consumption with two utility functions. Following Merton [28] and to formulate the problem of choosing optimal portfolio selection and consumption rules (π^* and c^*, respectively),

$$J(W,t) = \sup_{\{\pi(s),c(s)\}_{0\leq s\leq t}} \mathbb{E}_t\left(\bar{F}(T,t)\left(U(W^{\pi,c}(T)) + \int_t^T U(s,c(s))ds\right)\right), \quad (8)$$

in which $\mathbb{E}_t$ denotes the conditional expectation operator relative to σ-algebra $\mathcal{F}_t$ and $U(.)$ and $U(t,.)$ are some (generalized) utility functions and $\bar{F}(T,t)$ is survival distribution function.

In order to derive the optimality equations, we employ the dynamic programming principle and stochastic optimal control theory leading us to the following nonlinear Hamilton-Jacobi-Bellman **(HJB)** partial differential equation

$$\begin{aligned}\Omega(W,\pi,c,t) = J_t - \lambda(t)J + \sup_{\{\pi(t),c(t)\}} \{&(r + \pi(\mu + \frac{1}{2}\sigma^2 - r))W_t J_W \quad (9)\\ &-cJ_W + U(t,c) + \frac{1}{2}\pi^2\sigma^2 W_t^2 J_{WW}\\ &+\int_{-\infty}^{\infty}\Big[J\big(W(1+\pi(e^x-1)),t\big) - J(W,t)\Big]\Pi(\nu_t,dx)\Big\} = 0,\end{aligned}$$

where J_W and J_t denote the first partial derivatives of $J(W,t)$ w.r.t. W and t, and similarly for higher derivatives. A candidate for the optimal strategy and consumption (π^*, c^*) is obtained by taking partial derivatives of $\Omega(W,\pi,c,t)$ with respect to π_t and c_t, and equate to 0 (the first-order condition, say FOC, for Ω):

$$(\mu + \tfrac{1}{2}\sigma^2 - r)W_t J_W + \pi^*\sigma^2 W_t^2 J_{WW} + \int_{-\infty}^{\infty}\frac{\partial}{\partial\pi}J(W(1+\pi^*(e^x-1)),t)\Pi(\nu_t,dx) = 0. \quad (10)$$

and

$$\frac{\partial}{\partial c}U(t,c^*) - J_W = 0 \quad (11)$$

Thereafter, we can calculate the optimal strategy and cunsumption for power utility function cases. A general form of power utility function when the time is finite horizon, as derive by:

$$U(W_t) = \mathrm{e}^{-\alpha T}\frac{{W_t}^{1-\gamma}}{1-\gamma}, \quad U(t,c) = \mathrm{e}^{-\alpha t}\frac{c^{1-\gamma}}{1-\gamma}, \tag{12}$$

Theorem 3.1. *Consider the following assumptions: Assume that the coefficients μ, σ and r in (2) and (3) are driven by a constant state $\nu_t \equiv \nu$, i.e.*

- *Following by Aït-Sahalia et al. [1], we guess the value function in the following form:*
$$J(W_t,t) = U(W_t)f^{\gamma}(t), \tag{13}$$
in which $f(t)$ is the deterministic function:
$$f(t) = \left(\int_0^t e^{-\frac{(\alpha+\Psi(\pi^*))(T-s)}{\gamma}+\frac{1}{\gamma}\int_0^s \lambda(u)du} ds\right) e^{\frac{1}{\gamma}\left(\int_0^t \lambda(s)ds - \Psi(\pi^*)(T-t)\right)} \tag{14}$$
in which
$$\begin{cases} \Psi(\pi^*) = (1-\gamma)\left(\left(r+\pi^*\left(\mu+\frac{1}{2}\sigma^2-r\right)\right)-\frac{\gamma{\pi^*}^2\sigma^2}{2}\right)+\Phi(\pi^*) \\ \Phi(\pi^*) = \int_{-\infty}^{\infty}\left((1+\pi^*(e^x-1))^{1-\gamma}-1\right)\Pi(\nu_t,dx) \end{cases}$$
With boundary condition $f(T) = 1$.
- *Also assume that there is deterministic function π^* that solves the following equations:*
$$-\gamma\sigma^2\pi^* + (\mu+\frac{1}{2}\sigma^2-r)+\int_{-\infty}^{\infty}(1+\pi^*(\mathrm{e}^x-1))^{-\gamma}(\mathrm{e}^x-1)\Pi(\nu_t,dx) = 0. \tag{15}$$
- *and the optimal cunsumption solves the following equations:*
$$c^*(t) = \mathrm{e}^{\frac{\alpha(T-t)}{\gamma}}\frac{W_t}{f(t)}. \tag{16}$$

Proof. We use FOC condition from HJB equation to compute strategy policy (π^* and c^*). In this case utility function introduce in (12) and J as following by:

$$J(W_t,t) = e^{-\alpha T}\frac{{W_t}^{1-\gamma}}{1-\gamma}f^{\gamma}(t),$$

and

$$\begin{cases} J_W = e^{-\alpha T}f^{\gamma}(t){W_t}^{-\gamma} \\ J_{WW} = -\gamma e^{-\alpha T}f^{\gamma}(t){W_t}^{-\gamma-1} \\ J_t = \gamma e^{-\alpha T}\frac{{W_t}^{1-\gamma}}{1-\gamma}f^{\gamma-1}(t)f'(t) \\ U(t,c_t^*) = \frac{e^{\alpha(T-t)}e^{\frac{\alpha(T-t)(1-\gamma)}{\gamma}}}{f(t)}.\frac{e^{-\alpha T}{W_t}^{1-\gamma}}{1-\gamma}f^{\gamma}(t) \end{cases} \tag{17}$$

by substituting (17) in HJB equation and factoring of $\frac{e^{-\alpha T} W_t{}^{1-\gamma}}{1-\gamma} f^{\gamma}(t)$, we have:

$$f'(t) - \frac{(\lambda(t) - \Psi(\pi^*))}{\gamma} f(t) + e^{\frac{\alpha(T-t)}{\gamma}} = 0$$

Finally, by solving this ODE and applying the boundary condition $f(T) = 1$, $f(t)$ will be achieved. □

Remark 3.2. Due to above theorem, the age of policyholder and the mortality rate are only effective in calculating the optimal consumption but not in the optimal strategy.

The following lemma provides the optimal strategy under Variance Gamma model.

Lemma 3.3. *Under the Variance Gamma model the optimal strategy* π^* *given by Theorem* (3.1) *is, respectively, solution of the following equations*

$$0 \quad = \quad [(\mu - r) + M(\pi^*)$$

where

$$M(\pi^*) = \int_0^1 \left[(1 - \pi^* t)^{-\gamma} \left(\frac{t(1-t)^{\frac{1}{\lambda_d}}}{\ln(1-t)} \right) - \left(\frac{1 - t(1-\pi^*)}{1-t} \right)^{-\gamma} \left(\frac{-t(1-t)^{\frac{1}{\lambda_u}}}{(1-t)\ln(1-t)} \right) \right] \frac{dt}{1-t}.$$

Proof. To compute the integral part of Equation (15), one may separate such integral into two parts and used the Variance Gamma measure (with $\sigma = 0$)

$$\begin{aligned} M(\pi^*) = &- \int_{-\infty}^{0} (1 + \pi^*(e^x - 1))^{-\gamma} (e^x - 1) \frac{1}{x} e^{\frac{x}{\lambda_d}} dx \\ &+ \int_0^{\infty} (1 + \pi^*(e^x - 1))^{-\gamma} (e^x - 1) \frac{1}{x} e^{-\frac{x}{\lambda_u}} dx. \end{aligned}$$

In the next step, we change variable for positive part of integral to $x = -\ln(1-t)$ and for negative part of integral to $x = \ln(1-t)$ and get the desired result. □

Following by the Gaussian integration method, we can numerically solve the optimal strategy which reported in the numerical section.

Example.1 In this example, we use De Moivre Force of mortality with maximum age ω for computing optimal consumption in power utility case

$$\lambda(t) = \frac{1}{\omega - x - t}, \quad 0 < t < \omega$$

and

$$c^*(t) = \mathrm{e}^{\frac{\alpha(T-t)}{\gamma}} \frac{W_t}{f(t)}.$$

where

$$f(t) = \xi^{-(\gamma+1)} \left(\frac{2(\omega-x)}{\omega-x-t}\right)^{\frac{1}{\gamma}} \exp\left(\xi(\omega-x-T) - \frac{\psi(\pi^*)}{\gamma}(T-t)\right)$$
$$\times \left(\Gamma\left(\gamma+1, \xi(\omega-x-t)\right) - \Gamma\left(\gamma+1, \xi(\omega-x)\right)\right) \tag{18}$$

where Γ is the lower incomplete gamma function.

Example.2 In this example, we use Gompertz Force of mortality for computing optimal consumption in power utility case

$$\lambda(t) = BC^{x+t}, \quad t > 0$$

and

$$c^*(t) = \mathrm{e}^{\frac{\alpha(T-t)}{\gamma}} \frac{W_t}{f(t)}.$$

where

$$f(t) = \left(\int_0^t e^{-\frac{(\alpha+\Psi(\pi^*))(T-s)}{\gamma} + \frac{B(C^{x+s}-C^x)}{\gamma \ln(C)}} ds\right) e^{\frac{1}{\gamma}\left(\frac{B(C^{x+s}-C^x)}{\ln(C)} - \Psi(\pi^*)(T-t)\right)}$$

4 Numerical Results

Now for simulating W_t, we follow the theorem thad introduced by Applebaum [2] which said that the integral of $\int_{-\infty}^{\infty} y(x)N(t,dx)$ in wealth equation has compound poisson distribution. In generally, every compound poisson distribution has a intensity λ and jump size distribution $F(x)$. For example in Jump-Diffusion of Merton, the jump size distribution is normal or in Kou model is double exponential. In this paper, we use infinite activity model for stock price behavior, such as Variance-Gamma (VG) Lévy process. Following by Cont and Tankov [10], we can approximate the infinite activity variance gamma process by compound poisson process with intensity $U(\varepsilon) = \int_{\varepsilon}^{\infty} \Pi(dx)$ and jump size distribution $P^{\varepsilon}(x) = \frac{\Pi(dx)}{U(\varepsilon)} 1_{x \geq \varepsilon}$, in which $\Pi(dx)$ is VG Lévy measure which introduce in section 2. Recall that, the Lévy measure for VG process is:

$$\Pi(dx) = \frac{1}{x} e^{-\frac{x}{\lambda_u}} 1_{\{x>0\}} - \frac{1}{x} e^{\frac{x}{\lambda_d}} 1_{\{x<0\}}$$

The intensity and jump size distribution in this model have two part of positive and negative jump, such that:

$$\begin{cases} U_1(\varepsilon) = \int_{\varepsilon}^{\infty} \frac{e^{-x/\lambda_u}}{x} dx = -Ei(-\frac{\varepsilon}{\lambda_u}) \\ \\ U_2(\varepsilon) = -\int_{-\infty}^{-\varepsilon} \frac{e^{x/\lambda_d}}{x} dx = -Ei(-\frac{\varepsilon}{\lambda_d}) \end{cases}$$

in which, Ei is representation of exponential integral which introduced by Gradshteyn and Ryzhik [15] and the jump size distribution is:

$$\begin{cases} P_1^\varepsilon(x) = \frac{e^{-x/\lambda_u}}{xU_1(\varepsilon)} 1_{x\geq\varepsilon} \\ \\ P_2^\varepsilon(x) = -\frac{e^{x/\lambda_d}}{xU_2(\varepsilon)} 1_{x\leq-\varepsilon} \end{cases}$$

Finaly, for all process, we have:

$$P^\varepsilon(x) = P_1^\varepsilon(x) + P_2^\varepsilon(x) \qquad U(\varepsilon) = U_1(\varepsilon) + U_2(\varepsilon)$$

In this paper, to generate jump size of this distribution, we use the rejection method that introduced by Cont and Tankov [10]. For this purpose, for positive jump distribution and for all $x \geq \varepsilon$, it is obvious that:

$$P_1^\varepsilon(x) \leq f_1^\varepsilon(x) \frac{\lambda_u e^{-\varepsilon/\lambda_u}}{\varepsilon U_1(\varepsilon)},$$

where $f_1^\varepsilon(x) = \frac{e^{-\frac{(x-\varepsilon)}{\lambda_u}}}{\lambda_u} 1_{x\geq\varepsilon}$ is a probability density function. Following by Cont and Tankov(2004), $f_1^\varepsilon(x)$ has a survival function $F_1^\varepsilon(x) = e^{-\frac{(x-\varepsilon)}{\lambda_u}} 1_{x\geq\varepsilon}$ and inverse survival function $F_{1\varepsilon}^{-1}(x) = \varepsilon - \lambda_u \ln(x)$. Random variables with distribution $P_1^\varepsilon(x)$ may be simulated using the rejection method as follows by Devroye [12]. Simular posetive jump distribution, we use this method for negative jump distribution. Observe that for all $x \leq -\varepsilon$ we have:

$$\begin{cases} P_2^\varepsilon(x) \leq f_2^\varepsilon(x) \frac{\lambda_d e^{-\varepsilon/\lambda_d}}{\varepsilon U_2(\varepsilon)} \\ \\ F_{2\varepsilon}^{-1}(x) = \lambda_d \ln(x) - \varepsilon \end{cases}$$

Also, Vahabi and Payandeh Najafabadi [33] developed the following two algorithm to generate random variables from $P_1^\varepsilon(x)$ and $P_2^\varepsilon(x)$, respectively. In this work, we design a pure-endowment contract in such a way that the benefits of this contract depend on two markets, one is risk free and the other is risky. In this regard, we calculated the optimal strategy and optimal consumption rate for this policyholder over a 20-year time horizon. The simulation results show that the optimal consumption rate behavior is the same as the Merton's papers [29] and [28], but with the difference that the effect of jumps in stock asset on the consumption rate is quite clear. In Figures 1-2, we represent the optimal consumption rate behavior for both 30 and 60 years of age.

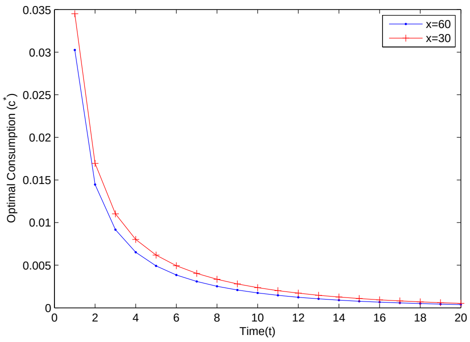

Figure 1: The behavior of optimal consumption with De Moivre Force of mortality for T=20, $\omega = 100$, $\gamma = 2.2$, $\mu = 0.28$, $\theta = -0.00799$, $\rho = 0.864$, $l = 0.6$, r=0.04 and α=0.05 in power utility function with Variance Gamma jump model for two personal ages.

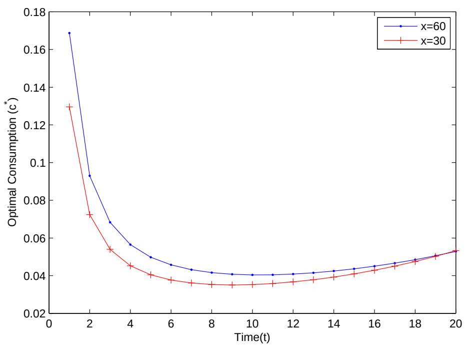

Figure 2: Figure4. The behavior of optimal consumption with Gompertz force of mortality for T=20, C=1.01, B=0.01 , $\gamma = 2.2$, $\mu = 0.28$, $\theta = -0.00799$, $\rho = 0.864$, $l = 0.6$, r=0.04 and α=0.05 in power utility function with Variance Gamma jump model for two personal ages

Table 1, summarize the optimal strategy simulations for the power utility function and different values of Variance Gamma jump parameters. All comparisons is based on the results of Monte Carlo simulations with 200 time steps in each year.

Table 1: Optimal allocation under the power utility function in a risky stock based on the moment data reported in Campbell (1997) with $T = 20,\ r = 0.04,\ P = 0.7,$ $\mu = 0.28,\ \tau = 0.12,\ g = 0.02$ and $\gamma = 3$ under Variance-Gamma Model . In the simulation, there are 200 steps in each year.

θ	l	ρ	π^*
-0.00799	0.6	0.864	0.3901
-0.0007	0.5	0.872	0.3947
-0.004282	0.5	0.794	0.3943
0.00245	0.4	0.671	0.4019
0.001514	0.4	0.632	0.4021

5 Conclusion and suggestion

Specifically, we examine the impact of one special Lévy process categories on the optimal investment strategy and consumption of a risk-averse investor over finite horizons. Similar to Cont and Tankov [10], we employed jump parameters in our model to take into account both uncertainty and the risk of falling stock prices. We considered two popular forces of mortality in our models. The application of our findings in pure endowment contracts has been studied. For future studies, we suggest a family of the stochastic differential equation for modeling the force of mortality.

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