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Stochastic optimal control with Contingent Convertible Bond in banking industry

Asma Khadimallah[1], **Fathi Abid**[2]

[1] Faculty of Economic and Management Sciences, Probability and Statistics Laboratory, University of Sfax, Tunisia

khadimallahasma@gmail.com

[2] Faculty of Economic and Management Sciences, Probability and Statistics Laboratory, University of Sfax, Tunisia

fathi.abid@fsegs.usf.tn

Abstract:
This paper has potential implications for the management of the bank. We examine a bank capital structure with contingent convertible debt to improve financial stability. This type of debt converts to equity when the bank is facing financial difficulties and a conversion trigger occurs. We use a leverage ratio, which is introduced in Basel III to trigger conversion instead of traditional capital ratios. We formulate an optimization problem for a bank to choose an asset allocation strategy to maximize the expected utility of the bank's asset value. Our study presents an application of stochastic optimal control theory to a banking portfolio choice problem. By applying a dynamic programming principle to derive the HJB equation, we define and solve the optimization problem in the power utility case. The numerical results show that the evolution of the optimal asset allocation strategy is really affected by the realization of the stochastic variables characterizing the economy. We carried out a sensitivity analysis of risk aversion, time and volatility. We also reveal that the optimal asset allocation strategy is relatively sensitive to risk aversion as well as that the allocation in CoCo and equity decreases as the investment horizon increases. Finally, sensitivity analysis highlights the importance of dynamic considerations in optimal asset allocation based on the stochastic characteristics of investment opportunities.

Keywords: Contingent convertible bond, stochastic optimal control, asset allocation strategy, bank capital structure, optimization problem, power utility.
JEL Classifications: G21, C13, C15, C61.

1 Introduction

Several proposals aimed at enhancing the stability of the financial system include requiring banks to hold some form of contingent capital, i.e. capital that becomes available to a bank in the event of a crisis or financial difficulties. Variations of this idea differ in the choice of trigger for the activation of contingent capital and in the manner in which capital is held prior to a triggering event. The Dodd-Frank Financial Reform Bill of 2010 calls on regulators to study the potential effectiveness

[1] Corresponding author
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of contingent capital, and specific definitions of trigger events are offered in a recent advisory paper released by the Basel Committee on Banking Supervision.
[7] proposed reverse convertible debentures, a form of debt that converts into equity if a bank's capital ratio falls below a threshold. His proposal uses an equity ratio based on the market value of the bank's equity and the book value of its debt. [8] updated the proposal and renamed contingent capital certificates to securities. [13] and [21] propose contingent capital with a trigger that depends on both the health of an individual bank and the banking system as a whole. Convertible securities designed by the US Treasury for its capital assistance program can be considered a type of contingent capital in which banks hold the option to convert preferred stock into common stock and find it beneficial to do so if the market price of their action falls low enough; this contract is studied in [9]. Alternative proposals for the design of contingent capital have led to work on valuation. [13] assesses contingent capital with a dual trigger through a joint simulation of a bank's stock price and a market index. [18] compares several cases by simulation in a leaping diffusion model of a bank's assets. [1] obtain closed-form pricing expressions assuming that all debt has an infinite maturity and the conversion trigger is defined by a threshold level of assets. [19] also uses an asset-level trigger and obtains closed-form expressions with finite-maturity debt. [22] show that setting the conversion trigger at a stock price level can lead to multiple solutions or no solution for the market price of stock and convertible debt, raising questions about the viability of contracts designed with actions based on a market trigger. We develop a model to study contingent capital in the form of debt that converts to equity based on a capital ratio trigger. The bank is required to hold a minimum ratio of capital to total assets.
The optimization problem in the power utility case has attracted increasing attention from many investment institutions, including insurance companies, pension management institutions, and commercial banks. We undertake our valuation in a structural model, starting from the company's assets. The capital structure of the company is made up of deposits, contingent capital and equity.
Portfolio selection is one of the most difficult decision problems faced by banking institutions. The objective of bank managers is to choose an optimal structure of net worth by allocating assets and liabilities according to the proportions of income and costs. The first approach to solving a portfolio choice problem is the mean variance approach developed by Markowitz (1952) in a one-period decision model. It still has great importance in real applications and is widely applied in risk management departments of banks. The main reasons for this are the simplicity with which the algorithm can be implemented and the fact that it does not require any special knowledge of probability. Indeed, risk is only measured by variance, returns are normally distributed and bank managers use risk-averse utility functions. A criticism of the mean variance criterion is the assumption of the static nature of the financial market or myopic optimization character. This is an extreme simplification of reality that completely ignores the highly volatile behavior and dynamic

nature of prices. However, two main approaches dealing with the dynamic portfolio choice problem use continuous-time models.

The stochastic control theory developed by [14] and [15] was based on the solution of the HJB equation resulting from dynamic programming under the real-world probability measure. Several studies related to the dynamic portfolio choice problem in banking have recently surfaced in the literature (see, for example, [20], [10], [16], [17], [3]). In particular, [16] suggested an optimal portfolio choice and bank capital inflow rate that keeps the level of lending as close as possible to an actuarially determined benchmark process. In this paper, a general case of maximization problem with constant relative risk aversion utility function is discussed in order to determine an analytical solution for the associated HJB equation in the case with power utility function. [14] and [15] studied a problem of consumption and portfolio choice in continuous time and obtained optimal investment strategies under power utility and logarithmic utility by using the principle of dynamic programming. [23] studied an investment consumption problem with borrowing constraints. On this basis, [24] studied the problem of consumer investment with risky housing. [6] studied the consumption investment problem with HARA utility in incomplete markets. [4] studied the problem of consumption investment with transaction costs in a finite time horizon. Through deliberation and complicated calculations, [25] obtained the optimal consumption investment policies with non-exponential discounting and logarithmic utility. [11] focused on an inconsistent consumption investment problem and obtained instructive results. [11] studied the portfolio choice problem with stochastic interest rate by applying a stochastic control approach. [5] searched for the optimal investment problem with minimum guarantee. In this context, we study an optimal portfolio choice problem for a bank under a stochastic interest rate. We choose the power utility function because it is very tractable and the optimal asset allocation strategy is independent of the level of wealth.

Our goal is to present the numerical aspects of solving the Hamilton-Jacobi-Bellman (HJB) equation and focus on the results of the portfolio choice model from a practical point of view. This is driven by the need for banks to invest in assets with an acceptable level of risk and high returns. For example, if the yields on a specific loan turn out to be very high at the end of a loan contract period, the bank might regret not having allocated a large enough share of its capital to this type of loan. A dynamic portfolio position is particularly important in bank risk management, as most banks select an initial loan portfolio at the start of a loan period but often do not actively manage their portfolio afterward, unless a possibility of a fault does not arise. Another motivation for discussing the bank's optimal portfolio is the failure of spark risk management strategies and regulatory prescriptions to mitigate this risk.

One such requirement is the Basel Accord on Capital Adequacy Requirements, which stipulates that all major international banks hold capital in proportion to their perceived risks. A dynamic programming principle is used to derive the HJB

equation. We use the principle of dynamic programming and the method of separation of variables to obtain explicit solutions to optimal investment strategies. Finally, we give a numerical example to illustrate our results.
This article is organized as follows. Section 2 introduces the bank's capital structure and shows how to manage the main variables of a CoCo bond issue (including how to choose the trigger indicator and how to set the trigger point and conversion rate). In section 3, we model the dynamics of banking assets. Next, we define and solve the optimization problem in the case of a power utility. Indeed, we use the principle of dynamic programming to derive the HJB equation. In section 4, we illustrate our results numerically and in the last section, we draw the conclusion.

2 Capital structure and design of CoCo bonds

2.1 Bank capital structure

As in the model of [2] and [14], the bank is assumed to be funded by equity S, deposits P and a single zero-coupon bond with C as face value and T as due date. We also assume that the bank makes no payments to its shareholders and does not raise new shares before the bond matures. Unlike the Black-Scholes-Merton model, the bond is not entirely traditional. It is divided into two parts: the first is a debt convertible into equity (contingent convertible bond, CoCo) at zero coupon with αC_t^c as nominal value, while the second is a traditional zero-coupon bond with $(1-\alpha)C_t$, as a nominal value, where α is a coefficient that defines the triggering of the conversion if is equal to 1 or 0 otherwise.
As the bank's assets and liabilities must be equal at all times t, we have:

$$S_t + \alpha C_t^c + (1-\alpha)C_t + P_t = V_t \tag{1}$$

Where S_t represents the equity value, P_t are the deposits, C_t^c and C_t represent the values of the CoCo bond and the traditional bond respectively, with $C_t^c = C_t = C$, and V_t represents the value of bank assets. When $\alpha = 0$, the model is the same as the Black-Scholes and Merton model without any CoCo bond. When $\alpha = 1$, all of the debt is a CoCo bond.
The mechanism of a CoCo bond works as follows: if the trigger is not activated for the entire period $[0, T]$, the CoCo bond will behave like a traditional zero-coupon bond, otherwise, if the trigger is activated at time t^d with $t^d \in [0, T]$, the CoCo bond will be converted into equity with a predetermined conversion ratio β , which means that the CoCo bond with face value αC_t^c will be transformed into $\alpha\beta$ equity units.

2.2 Trigger level

How does this work in practice? So far, no conversion of CoCo bonds is based on leverage. To get an idea of the positioning of triggers in relation to regulatory ratios, one can look at CoCo bonds whose conversion is based on traditional capital ratios. Here, we assume that the trigger threshold x^d is set at 3%, the minimum level of Tier 1 leverage ratio required by Basel III. In our model, the Tier 1 leverage ratio is equal to S_t/V_t .
Let the fault time t^d be expressed by the first passage time, if the default occurs and the conversion will take place, which is defined as follows:

$$t^d = inf\left\{t \geq 0, \frac{S_t}{V_t} \leq x^d\right\} \tag{2}$$

If we represent the value of the CoCo bond at time t by αC_t^c and the value of the traditional bond by $(1-\alpha)C_t$, the trigger event can be defined as follows:

$$\begin{aligned}
&\tfrac{S_t}{V_t} \leq x^d \Leftrightarrow S_t \leq x^d V_t \\
&\Leftrightarrow V_t - \alpha C_t^c - (1-\alpha)C_t - P_t \leq x^d V_t \\
&\Leftrightarrow V_t \leq (\alpha C_t^c + (1-\alpha)C_t + P_t)/(1-x^d) = X^d
\end{aligned}$$

This means that the trigger event based on the leverage ratio (at the x^d level) is now transformed to a trigger event based on the value of the bank's assets (at the X^d level). The fact that X^d evolves with time t is inconvenient for us to continue our analysis. Consequently, we need to choose a trigger level, X, which does not depend on time t. Indeed, this constant trigger X must be greater than X^d, so that the conversion into shares can take place before the regulatory minimum is reached. Unlike risk-free bonds, CoCo bonds and traditional bonds are exposed to credit risk. So, other things being the same otherwise, they deserve higher yields to maturity than risk-free bonds, or they are worth less. Similarly, because CoCo bonds are exposed to higher risks than traditional bonds, they deserve the highest yields to maturity, or they are worth the least of the three types of bonds.
Let r be the risk-free interest rate, r_C and r_{C^c} the respective yields to maturity of the traditional bond and the CoCo bond, we have: $r_{C^c} \geq r_C \geq r$, and $C_0^c = Ce^{-r_{C^c}T} \leq C_0 = Ce^{-r_C T} \leq Ce^{-rT}$, where Ce^{-rT} is the current value of the risk-free bond. As a result, we have:

$$X^d = \frac{(\alpha C_t^c + (1-\alpha)C_t + P_t)}{(1-x^d)} \leq \frac{Ce^{-rT} + P}{1-x^d} = X \tag{3}$$

Where P is the face value of the deposits. Then, the constant trigger X can be expressed by:

$$X = \frac{Ce^{-rT} + P}{1-x^d} \tag{4}$$

2.3 Conversion rate

The conversion ratio β indicates the number of shares into which each CoCo bond will be converted. It is the ratio of the conversion amount to the conversion price. Specifically, the present value of the CoCo bond αC will be transformed into $\alpha\beta$ equity units whose value at t^d is $\alpha\beta S_{t^d}$, which is equivalent to saying $\alpha C^c_{t^d}$, where t^d is the trigger time. Then, the conversion ratio is given by:

$$\beta = \frac{C^c_{t^d}}{S_{t^d}} \tag{5}$$

If the conversion of the CoCo bond is triggered at time 0, the wealth of the CoCo bond holders will be the same before and after the conversion. This means that at time $t^d = 0$, we have:

$$\beta = \frac{C^c_0}{S_0} = \frac{C^c_0}{V_0 - \alpha C^c_0 - (1-\alpha)C_0 - P_0} \tag{6}$$

3 Stochastic optimal control

We consider a continuous-time dynamic model in which the bank holds assets (uses of funds) and has liabilities (sources of funds) that behave stochastically.

$$V_t = S_t + C^c_t + P_t \tag{7}$$

We assume that S_t is the equity value that follows a geometric Brownian motion such that

$$dS_t = \mu_s S_t dt + \sigma_s S_t dW_{s,t} \tag{8}$$

Where μ_s denotes the drift of the process under the neutral risk measure, σ_s is the cash flow volatility, $W_{s,t}$ is the standard Brownian motion.
In the Vasicek model, the instantaneous dynamics of the interest rate r_t is modeled by an Ornstein-Uhlenbeck process

$$dr_t = \kappa(\theta - r)dt + \sigma_r dW_{r,t} \tag{9}$$

With κ the degree of mean reversion, θ is the long term mean and σ_r is the volatility of the interest rate. Note that $(\kappa, \theta, \sigma_r) \in R^+R^+R^+$ and constants.
Under the neutral risk probability Q, this process induces a price, at time t, of zero-coupon bonds with maturity T, which is calculated according to the following formula:

$$\begin{aligned} P(t,T) \quad &= exp\{-R_\infty(T-t) + (R_\infty - r)\tfrac{1-e^{-\kappa(T-t)}}{\kappa} \\ &-\tfrac{\sigma_r^2}{4\kappa^3}(1-e^{-\kappa(T-t)})^2\} \end{aligned} \tag{10}$$

$$R_\infty = \theta + \frac{\sigma_r \lambda_r}{\kappa} - \frac{\sigma_r^2}{2\kappa^2}$$

With R_∞ and λ_r represents the yield to maturity of a zero-coupon bond and the constant interest rate risk premium respectively.
The bank deposit is assumed to be in this study as a risk-free asset satisfying the following equation:

$$\frac{dP_t}{P_t} = rdt \tag{11}$$

The CoCo bond is essentially a debt contingent on the interest rate, we choose to represent it by means of a geometric Brownian motion which makes the problem more analytically manageable. So, the price dynamics of the CoCo bond, C_t^c, is assumed to be:

$$dC_t^c = \mu_c C_t^c dt + \sigma_c C_t^c dW_{c,t} \tag{12}$$

With μ_c, σ_c and $W_{c,t}$ are the drift, the volatility and a standard Brownian motion of the process C_t^c.
Suppose the bank's shareholders can control the volatility of asset values through an expensive stochastic risk control technology. In fact, bank management must strategically allocate equity to maximize shareholders' ultimate wealth before conversion. Indeed, the change in the value of bank assets is reflected in the change in equity and CoCos bonds, which encourages the bank to maximize the return on the portfolio of assets in relation to the risk.
The shareholders are assumed to have a constant relative risk aversion (CRRA) utility function. Indeed, we formulate the optimization problem and derive the proxy from its solution. We wish to choose an asset allocation strategy in order to maximize the expected utility of the value of the bank's assets at a future date $T > 0$.
Let $\omega_s(t)$ and $\omega_c(t)$ be the proportions invested in equity and the CoCo bond, respectively. Moreover, $\omega_p(t)$ is the proportion invested in the risk-free asset, so $\omega_p(t) = 1 - \omega_s(t) - \omega_c(t)$.
We consider that V_t is the value of bank assets at time t. Due to the independence of Brownian motions and self-financing assumptions, the value of assets can be expressed as the following stochastic process:

$$\begin{aligned} \frac{dV_t}{V_t} &= \omega_p(t)\frac{dP_t}{P_t} + \omega_s(t)\frac{dS_t}{S_t} + \omega_c(t)\frac{dC_t^c}{C_t^c} \\ &= (1 - \omega_s(t) - \omega_c(t))rdt + \omega_s(t)\mu_s dt + \omega_s(t)\sigma_s dW_{s,t} + \omega_c(t)\mu_c dt \\ &\quad + \omega_c(t)\sigma_c dW_{c,t} \\ &= (r_t + (\mu_s - r)\omega_s(t) + (\mu_c - r)\omega_c(t))dt + \omega_s(t)\sigma_s dW_{(s,t)} \\ &\quad + \omega_c(t)\sigma_c dW_{c,t} \end{aligned} \tag{13}$$

This equation admits a unique strong solution satisfying.
Then, for a well-posed optimization problem, we assumed an additional assumption

on the admissible controls defined below. The set of admissible controls is given by:

$$\mathcal{A} = \{\omega(.) = \omega(t)_{t\in[0,T]}, \quad \int_0^T (\omega_s(t)\sigma_s)^2 + (\omega_c(t)\sigma_c)^2 dt < +\infty, \quad \mathbb{P} - a.s.\}$$

Mathematically, the stochastic optimal control problem can be stated as follows: $\underbrace{max}_{\omega(.)\in\mathcal{A}} \mathbb{E}(u(V(T)))$, where $u(.)$ is a utility function

$$\begin{cases} V_t = \omega_s(t)S_t + \omega_c(t)C_t^c + (1-\omega_s(t)-\omega_c(t))P_t \\ V_t < \frac{S_t}{x^d} \\ \omega_i(t) \geq 0 \end{cases} \tag{14}$$

We define the following value function:

$$H(t,r,V) = \underbrace{max}_{\omega(.)\in\mathcal{A}} \mathbb{E}(u(V(T)))$$

with the following boundary condition $H(T,r,V) = u(V)$
Stochastic control methods are obvious candidates for solving continuous-time portfolio problems. This is moreover the approach of [13]. The Hamilton-Jacobi-Bellman equation associated with the optimization problem can be written directly as follows:

$$\begin{aligned} &H_t + H_r(\kappa(\theta - r)) + \tfrac{1}{2}H_{rr}\sigma_r^2 + \underbrace{max}_{\omega(.)\in\mathcal{A}}[VH_v(r + \big(\mu_s - r)\omega_s(t) + \big(\mu_c - r)\omega_c(t)) \\ &+\tfrac{1}{2}H_{VV}V^2\big((\omega_s(t)\sigma_s)^2 + (\omega_c(t)\sigma_c)^2\big) + VH_{Vr}(\omega_s(t)\sigma_s) + \omega_c(t)\sigma_c\big)\sigma_r)] = 0 \end{aligned} \tag{15}$$

Where H_t, H_r, H_V, H_{rr},H_{VV} and H_{Vr} are the first-order, second-order, and mixed partial derivatives of the value function with respect to the variables t, r and V. Using first-order maximization conditions for optimal investment strategies (see appendix A), we get

$$\begin{cases} \omega_s^*(t) = -\frac{(\mu_s - r)H_v}{\sigma_s{}^2VH_{VV}} - \frac{\sigma_r H_{Vr}}{\sigma_s VH_{VV}} \\ \omega_c^*(t) = -\frac{(\mu_c - r)H_V}{\sigma_c{}^2VH_{VV}} - \frac{\sigma_r H_{Vr}}{\sigma_c VH_{VV}} \\ \omega_p^*(t) = 1 - \omega_s^*(t) - \omega_c^*(t) \end{cases} \tag{16}$$

By putting (16) in (15), we obtain the equation of HJB as follows:

$$\begin{aligned} &H_t + H_r(\kappa(\theta - r)) + \tfrac{1}{2}H_{rr}\sigma_r^2 \\ &+VH_v(r + \big(\mu_s - r)(-\tfrac{(\mu_s - r)H_v}{\sigma_s{}^2VH_{VV}} - \tfrac{\sigma_r H_{Vr}}{\sigma_s VH_{VV}}) + (\mu_c - r)(-\tfrac{(\mu_c - r)H_V}{\sigma_c{}^2VH_{VV}} - \tfrac{\sigma_r H_{Vr}}{\sigma_c VH_{VV}})\big) \\ &+\tfrac{1}{2}H_{VV}V^2\big(((-\tfrac{(\mu_s - r)H_V}{\sigma_s{}^2VH_{VV}} - \tfrac{\sigma_r H_{Vr}}{\sigma_s VH_{VV}})\sigma_s)^2 + ((-\tfrac{(\mu_c - r)H_V}{\sigma_c{}^2VH_{VV}} - \tfrac{\sigma_r H_{Vr}}{\sigma_c VH_{VV}})\sigma_c)^2\big) \\ &+VH_{Vr}\big((-\tfrac{(\mu_s - r)H_V}{\sigma_s{}^2VH_{VV}} - \tfrac{\sigma_r H_{Vr}}{\sigma_s VH_{VV}})\sigma_s + \big(-\tfrac{(\mu_c - r)H_V}{\sigma_c{}^2VH_{VV}} - \tfrac{\sigma_r H_{Vr}}{\sigma_c VH_{VV}})\sigma_c\big)\sigma_r = 0 \end{aligned} \tag{17}$$

In this article, we assume that the degree of investor risk aversion can be described by the power utility. We use a change of variable technique to study optimal investment strategies.
In order to solve this type of Partial Differential Equations, we try to use the separability condition. In [15], the separability condition represents a common assumption in the attempt to solve explicitly optimal portfolio problems.
Indeed, we choose a power utility function to obtain a smooth analytical solution to the maximization problem.
The power utility is given by: $u(V) = V^\gamma/\gamma$, $\gamma < 1$ and $\gamma \neq 0$, where γ is the risk aversion factor.
The value function H can be rewritten as follows:

$$H(t,r,V) = (V^\gamma/\gamma)f(t,r), \qquad f(T,r) = 1$$

The partial derivatives of the previous equation are as follows:

$$\begin{aligned} H_t &= (V^\gamma/\gamma)f_t \\ H_V &= V^{\gamma-1}f \\ H_{VV} &= (\gamma-1)V^{\gamma-2}f \\ H_r &= (V^\gamma/\gamma)f_r \\ H_{rr} &= (V^\gamma/\gamma)f_{rr} \\ H_{Vr} &= V^{\gamma-1}f_r \end{aligned}$$

Substituting partial derivatives of the value function H into HJB's equation, leads to a second order Partial Differential Equations for f of the following form:

$$\begin{aligned} &(V^\gamma/\gamma)f_t + (V^\gamma/\gamma)f_r(\kappa(\theta-r)) + \tfrac{1}{2}(V^\gamma/\gamma)f_{rr}\sigma_r^2 \\ &+V^\gamma f(r + \big(\mu_s - r)(-\tfrac{(\mu_s-r)V^{\gamma-1}f}{\sigma_s{}^2(\gamma-1)V^{\gamma-1}f} - \tfrac{\sigma_r V^{\gamma-1}f_r}{\sigma_s(\gamma-1)V^{\gamma-1}f}) \\ &+(\mu_c - r)(-\tfrac{(\mu_c-r)V^{\gamma-1}f}{\sigma_c{}^2(\gamma-1)V^{\gamma-1}f} - \tfrac{\sigma_r V^{\gamma-1}f_r}{\sigma_c(\gamma-1)V^{\gamma-1}f})\big) \\ &+\tfrac{1}{2}(\gamma-1)V^\gamma f\big(((-\tfrac{(\mu_s-r)V^{\gamma-1}f}{\sigma_s{}^2(\gamma-1)V^{\gamma-1}f} - \tfrac{\sigma_r V^{\gamma-1}f_r}{\sigma_s(\gamma-1)V^{\gamma-1}f})\sigma_s)^2 \\ &+((-\tfrac{(\mu_c-r)V^{\gamma-1}f}{\sigma_c{}^2(\gamma-1)V^{\gamma-1}f} - \tfrac{\sigma_r V^{\gamma-1}f_r}{\sigma_c(\gamma-1)V^{\gamma-1}f})\sigma_c)^2\big) \\ &+V^\gamma f_r\big((-\tfrac{(\mu_s-r)V^{\gamma-1}f}{\sigma_s{}^2(\gamma-1)V^{\gamma-1}f} - \tfrac{\sigma_r V^{\gamma-1}f_r}{\sigma_s(\gamma-1)V^{\gamma-1}f})\sigma_s \\ &+\big(-\tfrac{(\mu_c-r)V^{\gamma-1}f}{\sigma_c{}^2(\gamma-1)V^{\gamma-1}f} - \tfrac{\sigma_r V^{\gamma-1}f_r}{\sigma_c(\gamma-1)V^{\gamma-1}f})\sigma_c\big)\sigma_r = 0 \end{aligned} \tag{18}$$

After simplification and by eliminating the dependence of V^γ/γ, the equation becomes:

$$\begin{aligned} &f_t + f_r(\kappa(\theta-r)) + \tfrac{1}{2}f_{rr}\sigma_r^2 \\ &+\gamma f(r + \big(\mu_s - r)(-\tfrac{(\mu_s-r)}{\sigma_s{}^2(\gamma-1)} - \tfrac{\sigma_r f_r}{\sigma_s(\gamma-1)f}) + (\mu_c - r)(-\tfrac{(\mu_c-r)}{\sigma_c{}^2(\gamma-1)} - \tfrac{\sigma_r f_r}{\sigma_c(\gamma-1)f})\big) \\ &+\tfrac{1}{2}\gamma(\gamma-1)f\big(((-\tfrac{(\mu_s-r)}{\sigma_s{}^2(\gamma-1)} - \tfrac{\sigma_r f_r}{\sigma_s(\gamma-1)f})\sigma_s)^2 + ((-\tfrac{(\mu_c-r)}{\sigma_c{}^2(\gamma-1)} - \tfrac{\sigma_r f_r}{\sigma_c(\gamma-1)f})\sigma_c)^2\big) \\ &+\gamma f_r\big((-\tfrac{(\mu_s-r)}{\sigma_s{}^2(\gamma-1)} - \tfrac{\sigma_r f_r}{\sigma_s(\gamma-1)f})\sigma_s + \big(-\tfrac{(\mu_c-r)}{\sigma_c{}^2(\gamma-1)} - \tfrac{\sigma_r f_r}{\sigma_c(\gamma-1)f})\sigma_c\big)\sigma_r = 0 \end{aligned} \tag{19}$$

By hypothesis, the solution of PDE, H, therefore has the following form: $f(t,r) = g_t exp(A_t r)$ with g and A are regular functions, $g(T) = 1$ and $A(T) = 0$.
The partial derivatives of the above function are:

$$f_t = A'_t r g_t exp(A_t r) + g' exp(A_t r)$$

$$f_r = g_t A_t exp(A_t r)$$

$$f_{rr} = g_t A_t^2 exp(A_t r)$$

$$\begin{aligned}
&A'_t r g_t exp(A_t r) + g'_t exp(A_t r) + g_t A_t exp(A_t r)(\kappa(\theta - r)) + \tfrac{1}{2} g_t A_t^2 exp(A_t r){\sigma_r}^2 \\
&+\gamma g_t exp(A_t r)(r + \big((\mu_s - r)(-\tfrac{(\mu_s - r)}{{\sigma_s}^2(\gamma-1)} - \tfrac{\sigma_r A_t}{\sigma_s(\gamma-1)}) + (\mu_c - r)(-\tfrac{(\mu_c - r)}{{\sigma_c}^2(\gamma-1)} - \tfrac{\sigma_r A_t}{\sigma_c(\gamma-1)}\big) \\
&+\tfrac{1}{2}\gamma(\gamma-1) g_t exp(A_t r)\big(((-\tfrac{(\mu_s - r)}{{\sigma_s}^2(\gamma-1)} - \tfrac{\sigma_r A_t}{\sigma_s(\gamma-1)})\sigma_s)^2 + ((-\tfrac{(\mu_c - r)}{{\sigma_c}^2(\gamma-1)} - \tfrac{\sigma_r A_t}{\sigma_c(\gamma-1)})\sigma_c)^2\big) \\
&+\gamma g_t A_t exp(A_t r)\big((-\tfrac{(\mu_s - r)}{{\sigma_s}^2(\gamma-1)} - \tfrac{\sigma_r A_t}{\sigma_s(\gamma-1)})\sigma_s + \big(-\tfrac{(\mu_c - r)}{{\sigma_c}^2(\gamma-1)} - \tfrac{\sigma_r A_t}{\sigma_c(\gamma-1)})\sigma_c\big)\sigma_r = 0
\end{aligned} \tag{20}$$

Eliminating the dependence of $exp(A_t r)$ and simplifying, we get

$$\begin{aligned}
&g'_t + \underbrace{[A'_t + \gamma - \kappa A_t]}_{\rho_t} r g_t + \big[\tfrac{1}{2} A_t^2 {\sigma_r}^2 + \theta\kappa A_t + \gamma\big((\mu_s - r)(-\tfrac{(\mu_s - r)}{{\sigma_s}^2(\gamma-1)} - \tfrac{\sigma_r A_t}{\sigma_s(\gamma-1)})\big) \\
&+(\mu_c - r)(-\tfrac{(\mu_c - r)}{{\sigma_c}^2(\gamma-1)} - \tfrac{\sigma_r A_t}{\sigma_c(\gamma-1)}\big) + \tfrac{1}{2}\gamma(\gamma-1)\big(((-\tfrac{(\mu_s - r)}{{\sigma_s}^2(\gamma-1)} - \tfrac{\sigma_r A_t}{\sigma_s(\gamma-1)})\sigma_s)^2 \\
&+\big((-\tfrac{(\mu_c - r)}{{\sigma_c}^2(\gamma-1)} - \tfrac{\sigma_r A_t}{\sigma_c(\gamma-1)})\sigma_c)^2\big) + \gamma A_t\big((-\tfrac{(\mu_s - r)}{{\sigma_s}^2(\gamma-1)} - \tfrac{\sigma_r A_t}{\sigma_s(\gamma-1)})\sigma_s \\
&+\big(-\tfrac{(\mu_c - r)}{{\sigma_c}^2(\gamma-1)} - \tfrac{\sigma_r A_t}{\sigma_c(\gamma-1)})\sigma_c\big)\sigma_r] g_t = 0
\end{aligned} \tag{21}$$

We can calculate the function A_t so that $\rho_t = 0$,

$$A'_t - \kappa A_t = -\gamma \tag{22}$$

We note that the resolution of this equation is determined by the solution of a linear differential equation of order one which has the following form:

$$A_t = \frac{\gamma}{\kappa}[1 - exp(\kappa(T - t))] \tag{23}$$

Replacing A_t in (21), we get:

$$g'_t + \epsilon_t g_t = 0 \tag{24}$$

With

$$\begin{aligned}
\varepsilon_t \quad &= \tfrac{1}{2} A_t^2 {\sigma_r}^2 + \theta\kappa A_t + \gamma\big((\mu_s - r)(-\tfrac{(\mu_s - r)}{{\sigma_s}^2(\gamma-1)} - \tfrac{\sigma_r A_t}{\sigma_s(\gamma-1)})\big) \\
&+(\mu_c - r)(-\tfrac{(\mu_c - r)}{{\sigma_c}^2(\gamma-1)} - \tfrac{\sigma_r A_t}{\sigma_c(\gamma-1)}\big) \\
&+\tfrac{1}{2}\gamma(\gamma-1)\big(((-\tfrac{(\mu_s - r)}{{\sigma_s}^2(\gamma-1)} - \tfrac{\sigma_r A_t}{\sigma_s(\gamma-1)})\sigma_s)^2 + \big((-\tfrac{(\mu_c - r)}{{\sigma_c}^2(\gamma-1)} - \tfrac{\sigma_r A_t}{\sigma_c(\gamma-1)})\sigma_c)^2\big) \\
&+\gamma A_t\big((-\tfrac{(\mu_s - r)}{{\sigma_s}^2(\gamma-1)} - \tfrac{\sigma_r A_t}{\sigma_s(\gamma-1)})\sigma_s + \big(-\tfrac{(\mu_c - r)}{{\sigma_c}^2(\gamma-1)} - \tfrac{\sigma_r A_t}{\sigma_c(\gamma-1)})\sigma_c\big)\sigma_r
\end{aligned}$$

Then the solution of a homogeneous linear differential equation is in the following form:

$$g_t = exp[\varepsilon(t) - \varepsilon(T)] \tag{25}$$

With $\varepsilon(t)$ is the primitive of ϵ_t (see appendix B). Hence the value function H is given by

$$H_t = (V^\gamma/\gamma)exp[(\varepsilon(t) - \varepsilon(T)) + (\frac{\gamma}{\kappa}(1 - exp(\kappa(T-t)))r] \tag{26}$$

with

$$H_v = V^{\gamma-1}exp[(\varepsilon(t) - \varepsilon(T)) + (\frac{\gamma}{\kappa}(1 - exp(\kappa(T-t)))r]$$

$$H_{VV} = (\gamma-1)V^{\gamma-2}exp[(\varepsilon(t) - \varepsilon(T)) + (\frac{\gamma}{\kappa}(1 - exp(\kappa(T-t)))r]$$

$$H_{Vr} = V^{\gamma-1}(\frac{\gamma}{\kappa}((1 - exp(\kappa(T-t)))exp[(\varepsilon(t) - \varepsilon(T)) + (\frac{\gamma}{\kappa}(1 - exp(\kappa(T-t)))r]$$

Then the optimal proportions can be rewritten in the following form

$$\begin{cases} \omega_s^*(t) = -\frac{(\mu_s - r_t)}{(\gamma-1){\sigma_s}^2} - \frac{\sigma_r}{(\gamma-1)\sigma_s}(\frac{\gamma}{\kappa}(1 - exp(\kappa(T-t))) \\ \omega_c^*(t) = -\frac{(\mu_c - r_t)}{(\gamma-1){\sigma_c}^2} - \frac{\sigma_r}{(\gamma-1)\sigma_c}(\frac{\gamma}{\kappa}(1 - exp(\kappa(T-t))) \\ \omega_p^*(t) = 1 - \omega_s^*(t) - \omega_c^*(t) \end{cases} \tag{27}$$

This equation represents the optimal proportions of the power utility function which are directly related to the rate of interest and continuous time.

4 Numerical example

In this section, we give a numerical example with a maturity $T = 10$ years and a risk-free interest rate $r = 0.05$ to illustrate the optimal investment strategy in the context of a power utility. The numerical example is based on MATLAB software. In order to analyze the impact of the parameters on the optimal strategies, we assume that the main parameters are given by $\mu_s = 0.06$; $\mu_C = 0.08$; $\sigma_s = 0.3$; $\sigma_C = 0.5$; $\sigma_r = 0.0003$; $\kappa = 0.00037$; $\theta = 0.044$ and $\gamma = 0.5$. From the strategy expressions above, we can see that the optimal investment strategy can be influenced by many market parameters, such as risk aversion factor, volatility, and time. In this section, we will analyze the sensitivity of optimal investment strategies to the parameters γ, σ_r and T, respectively.

Table 1: Optimal proportions invested in CoCo, equity and deposits

$\omega_s^*(t)$	0.0164	0.0174	0.0184	0.0194	0.0204	0.0214	0.0224	0.0234	0.0244	0.0254
$\omega_c^*(t)$	0.0635	0.0641	0.0647	0.0653	0.0659	0.0665	0.0671	0.0677	0.0683	0.0689
$\omega_p^*(t)$	0.9201	0.9185	0.9168	0.9152	0.9136	0.9120	0.9104	0.9088	0.9072	0.9056

Table 1 shows how the evolution of the optimal asset allocation strategy is actually affected by the realization of the stochastic variables characterizing the economy. The optimal asset allocation strategy shows that the optimal proportion invested in CoCo and equity increases over time. In particular, the proportion of the CoCo bond increases from an initial value of around 6.35% to just over 6.89%, while the proportion invested in equity increases from an initial value close to 1.64% to a proportion of approximately 2.54%. However, deposits play a residual role in the optimal composition of the bank. At the beginning of the investment period, the need for a conservative strategy to create a higher level of wealth and lower risk leads to a high proportion of deposits, while the investment in CoCo and equity is very weak.
To test the sensitivity of the optimal strategy to changes in the various underlying variables, we performed a sensitivity analysis of risk aversion, time and volatility.
Figure 1 shows that the proportions of CoCo and equity increase with γ, while the

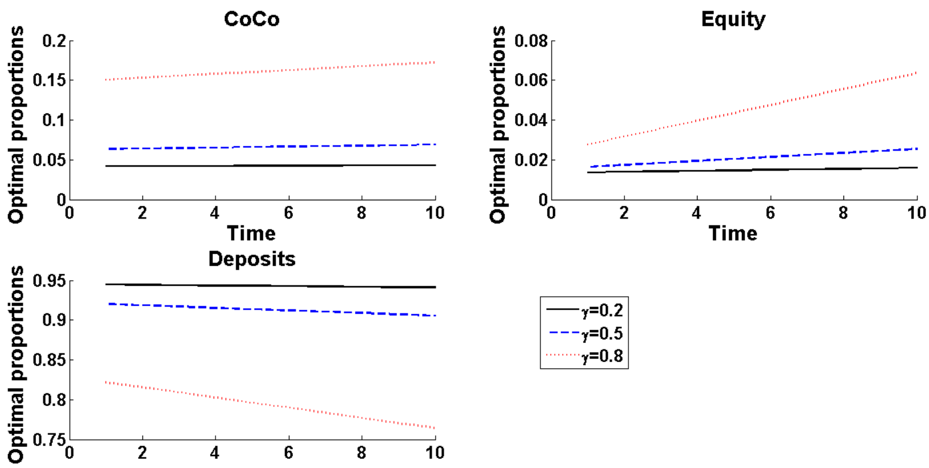

Figure 1: Effect of degree of risk aversion

proportion of deposits decreases with γ. From the point of view of utility theory, the risk aversion coefficient for the power utility is given by $1-\gamma$. This means that the degree of risk aversion of investors will decrease when the value of γ increases. For a predefined investment horizon and higher risk aversion, i.e. If $\gamma = 0.2$, the allocation is constant over time and the investment behavior seems to stabilize until maturity. In addition, the optimal proportions of CoCo and equity are higher when the γ value increases, which means a decrease in the degree of risk aversion. Therefore, the optimal asset allocation strategy is quite sensitive to risk aversion.

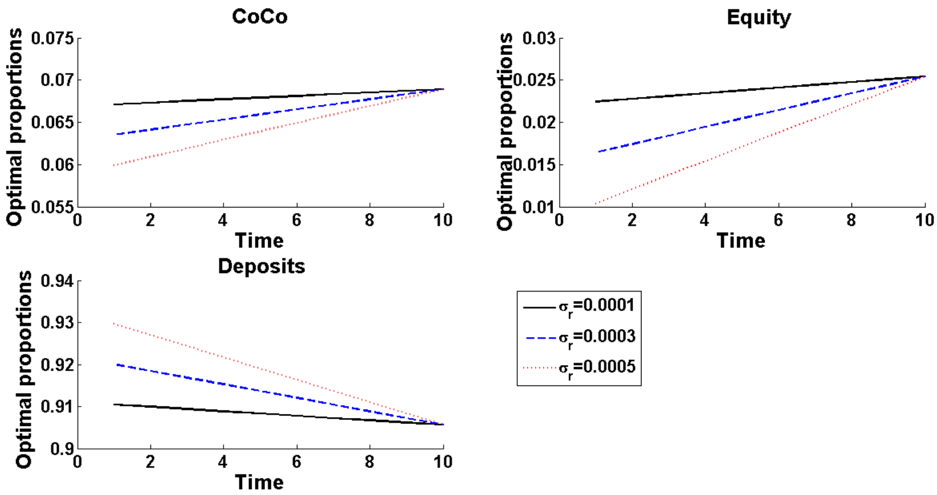

Figure 2: Effect of interest rate volatility

Figure 2 shows that the CoCos and equity optimal proportions decrease as the interest rate volatility increases. Therefore, CoCo bonds and equity can be used for the purpose of hedging interest rate uncertainty. The optimal weights evolution is an increasing function with the time horizon for different values of interest rate volatility and these weights are higher when the interest rate volatility is low.

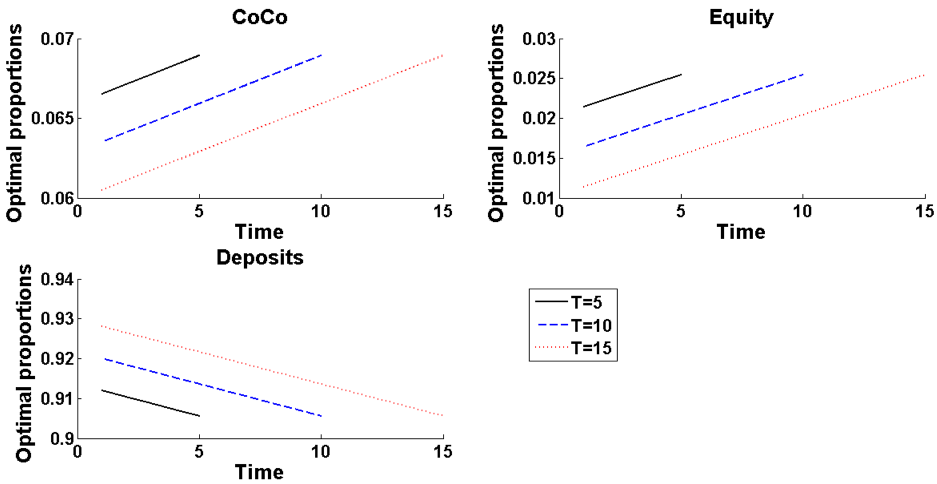

Figure 3: Effect of time

Figure 3 shows how the proportions change over time for a given value of gamma ($\gamma = 0.5$). With horizons ranging from 5 years to 15 years the proportion in bank account increases and remains positive. However, the CoCo and equity allocation decreases as the investment horizon increases.

5 Conclusion

This paper has great potential implications for the management of the bank. Contingent capital in the form of debt that converts to equity when a bank faces financial difficulties has been proposed as a mechanism to improve financial stability and avoid costly government bailouts. The specific proposals vary in the choice of conversion trigger and conversion mechanism.
We analyze the case of contingent capital with a leverage ratio trigger. Indeed, leverage ratios are used for the first time to trigger equity conversion instead of traditional capital ratios. Based on non-risk-weighted assets, leverage ratios are less biased than traditional capital ratios which are based on risk-weighted assets. They are therefore more effective in warning of an early risk. In this sense, CoCo bonds with leverage ratios can better strengthen the solvency of banks and thus better improve financial stability.
Our work presents an application of stochastic control theory to a banking portfolio choice problem. Applying a dynamic programming principle, we find a closed-form solution for the power utility function. A case study is given to illustrate our results and analyze the effect of the parameters on the optimal asset allocation strategy. Sensitivity analysis highlights the importance of dynamic considerations in optimal asset allocation based on the stochastic characteristics of the investment opportunity set.

Appendices

Appendix A: Derivation of (16)
By applying the first order conditions, the necessary condition for a relative extremum (maximum or minimum) is that the first-order derivative be zero, that is, the derivative of (15) be zero. For the proportions invested in equity and the CoCo bond, we derive (15) with respect to $\omega_s(t)$ and $\omega_c(t)$ respectively, and we obtain the following system:

$$\begin{cases} VH_V(\mu_s - r) + H_{VV}V^2\omega_s^*(t)\sigma_s^2 + VH_{Vr}\sigma_s\sigma_r = 0 \\ VH_V(\mu_c - r) + H_{VV}V^2\omega_c^*(t)\sigma_c^2 + VH_{Vr}\sigma_c\sigma_r = 0 \\ \omega_p^*(t) = 1 - \omega_s^*(t) - \omega_c^*(t) \end{cases}$$

Then, we get

$$\begin{cases} \omega_s^*(t) = -\frac{(\mu_s - r)H_v}{{\sigma_s}^2 VH_{VV}} - \frac{\sigma_r H_{Vr}}{\sigma_s VH_{VV}} \\ \omega_c^*(t) = -\frac{(\mu_c - r)H_V}{{\sigma_c}^2 VH_{VV}} - \frac{\sigma_r H_{Vr}}{\sigma_c VH_{VV}} \\ \omega_p^*(t) = 1 - \omega_s^*(t) - \omega_c^*(t) \end{cases}$$

Appendix B: Derivation of the primitive of ϵ_t

$$\varepsilon(t) = \frac{1}{2}(\frac{\gamma}{\kappa})^2[t - 2\frac{exp(\kappa(T-t))}{\kappa} + \frac{exp(2\kappa(T-t))}{2\kappa}]\sigma_r^2 + \theta\gamma[t - \frac{exp(\kappa(T-t))}{\kappa}]$$
$$-\frac{\gamma}{\gamma-1}[-\frac{1}{2}((((\frac{(\mu_s-r)}{\sigma_s^2})^2 t + 2\frac{(\mu_s-r)}{\sigma_s^2}\frac{\sigma_r}{\sigma_s}\frac{\gamma}{\kappa}[t - \frac{exp(\kappa(T-t))}{\kappa}]$$
$$+(\frac{\sigma_r}{\sigma_s})^2(\frac{\gamma}{\kappa})^2[t - 2\frac{exp(\kappa(T-t))}{\kappa} + \frac{exp(2\kappa(T-t))}{2\kappa}])\sigma_s^2$$
$$+((\frac{(\mu_c-r)}{\sigma_c^2})^2 t + 2\frac{(\mu_c-r)}{\sigma_c^2}\frac{\sigma_r}{\sigma_c}\frac{\gamma}{\kappa}[t - \frac{exp(\kappa(T-t))}{\kappa}]$$
$$+(\frac{\sigma_r}{\sigma_c})^2(\frac{\gamma}{\kappa})^2[t - 2\frac{exp(\kappa(T-t))}{\kappa} + \frac{exp(2\kappa(T-t))}{2\kappa}])\sigma_c^2)$$
$$+((\frac{(\mu_s-r)}{\sigma_s^2}\frac{\gamma}{\kappa}[t - \frac{exp(\kappa(T-t))}{\kappa}] + \frac{\sigma_r}{\sigma_s}(\frac{\gamma}{\kappa})^2[t - 2\frac{exp(\kappa(T-t))}{\kappa} + \frac{exp(2\kappa(T-t))}{2\kappa}])\sigma_s$$
$$+(\frac{(\mu_c-r)}{\sigma_c^2}\frac{\gamma}{\kappa}[t - \frac{exp(\kappa(T-t))}{\kappa}] + \frac{\sigma_r}{\sigma_c}(\frac{\gamma}{\kappa})^2[t - 2\frac{exp(\kappa(T-t))}{\kappa} + \frac{exp(2\kappa(T-t))}{2\kappa}])\sigma_c)\sigma_r]$$

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