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Research paper

Stochastic portfolio optimization by diversity-weighted portfolio approach

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Abstract:
The portfolio optimization problem, including portfolio selection, typically aims to maximize return and minimize risk. In this paper, we discuss about increasing use of stochastic portfolios in investments and aim to create optimal portfolios. It follows the relative wealth process of these portfolios, outperforms the market portfolio over sufficiently long time-horizons. In this regard, initially, a model of the market is presented by the stochastic portfolio theory (SPT) and features like Growth rate, Excess growth rate are mentioned. Then, functionally-generated portfolios are defined by using diversity weighted portfolios with parameters $p \in (0, 1)$, $p < 0$ and combination of them. Finally, by obtaining the daily closing price of 10 stocks in Tehran Stock Exchange (TSE) ,the performance of diversity weighted portfolios is investigated.

Keywords: Diversity-weighted portfolios, Portfolio generating functions, Portfolio, Stochastic portfolio theory, Sharpe ratio.
Classification: Primary: 91G10; Secondary: 91B70, 91G80.

1 Introduction

Investing in financial markets is inherently risky and investors try to increase returns and reduce their risks. Stochastic Portfolio Theory (SPT), was introduced by Fernholz in 1999, which provided insights into the structure and behavior of stock portfolios and market arbitrage. In the early 1950s, Markowitz [11] developed a theoretical framework for the systematic combination of optimal asset portfolios, which became the basis for modern theories of portfolio management. Markowitz sought a portfolio that had less risk and produced greater returns. Subsequently, the stochastic portfolio theory began in 1995 with a linear version regarding stock market diversification, which was eventually published by Fernholz in 1999 in the Journal of Mathematical Economics [2]. This theory provides a flexible framework for analyzing the behavior of the portfolio and the structure of the financial market and is based on continuous-time market models. Also, the observable characteristics of the stock portfolio and real markets were examined by Fernholz and Karatzas

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in 2009 [5] and Vervuurt and Karatzas in 2015 [13]. The introduction of the model for the continuous process of asset prices by Karatzas in 1997 [7] and Karatzas and Shreve in 1998 [8] were presented. In 1998 Karatzas and Shreve showed that diversification is consistent with economic equilibrium [8]. Kardaras examined the behavior of semi-martingale market models in general in 2004 [9]. This finding is one of the most important topics in the stochastic portfolio theory , which Fernholz expanded upon with the concept of excess growth rate and the introduction of market diversity and entropy [3] , expanding on the work of Fernholz and Shay in 1982 [3]. To construct relative arbitrage, Fernholz introduced the theory of functionally generated portfolios in 1999 [2] and developed it in 2002 [1]. The most studied functionally generated portfolios are the diversified weighted portfolios with parameter p defined by Fernholz, Karatzas and Karadrad [6]. Kim in 2023 develops theory of functionally generated portfolios with market-to-book ratios of stocks [10]. In this paper, it focuses on creating portfolios using functionally generated methods, aiming for better long-term performance compared to market indices. The paper explores diversified portfolios weighted with positive and negative parameters, combining them to form a new portfolio theoretically superior based on specific market assumptions. The study includes practical applications, assessing portfolio performance using Sharpe Ratio and total growth rate, with examples from Tehran Stock Exchange.The rest of the paper is organized as follows. In section 2, we provide an overview of the preliminary concepts of stochastic portfolio theory. Furthermore, we give a brief introduction to the SPT models. Stochastic diversity weighted portfolios are proposed in section 3. Section 4 is devoted to the empirical example of Iran Stock Exchange market. Section 5 is concluded.

2 Preliminaries

SPT utilises the logarithmic representation for stocks and portfolios rather than the arithmetic representation used in classical mathematical finance. In the logarithmic representation, rate of return is replaced by the growth rate. The dynamics of the n stocks process (model M for a financial market) are given by

$$dX_i = X_i(t)\left(b_i(t)\,dt + \sum_{v=1}^{d} \sigma_{iv}(t)\,dw_v(t)\right), \quad t \geq 0\ , i = 1, \ldots, n.$$

The definitions and theorem for the logarithmic representation of assets and portfolios also methods for performance evaluation are stated below:

Definition 2.1 ([1]). Logarithmic representation of the stock price process is $d\log X_i(t) = \gamma_i(t)\,dt + \sum_{v=1}^{d} \sigma_{iv}(t)\,dw_v(t)$. where exess growth rate is $\gamma_i(t) := b_i(t) - \frac{1}{2}a_{ii}(t), \quad i = 1, \ldots, n.$

Theorem 2.2 ([1])**.** *Let π be a portfolio in M. Then the process V^{π} satisfies* $d\log V^{\pi}(t) = \gamma_{\pi}(t)\,dt + \sum_{v=1}^{d} \sigma_{\pi v}(t)\,dW_v(t)$ *. Where* $\gamma_{\pi}(t) := \sum_{i=1}^{n} \pi_i(t)\,\gamma_i(t) + \gamma_{\pi}^{*}(t)$ *and the growth rate of $\pi(\cdot)$ is*

$$\gamma_{\pi}^{*}(t) := \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\,a_{ii}(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t)\,a_{ij}(t)\,\pi_j(t)\right)$$

The stochastic portfolio optimization model has been introduced in terms of the portfolio growth rate by minimizing the variance of the portfolio [1]. This model is similar to the Markowitz model [11].

$$\begin{array}{ll} \min & a_{\pi\pi} = \sum_{i,j=1}^{n} \pi_i(t)\,a_{ij}(t)\,\pi_j(t) \\ \text{s.t.} & \gamma_{\pi}(t) = \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) \\ & \quad +\frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\,a_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\,a_{ij}(t)\,\pi_j(t)\right) \geq \gamma_0 \\ & \sum_{i=1}^{n} \pi_i(t) = 1 \\ & \pi_i(t) \geq 0; \quad i = 1, \ldots, n. \end{array}$$

Definition 2.3 ([1])**.** (Market portfolio): The value of the market portfolio represents the combined capitalization of all the stocks in the market and the weight of each stock is proportional to its total presence in the market. As a result, it is considered a suitable criterion for evaluating the performance of other portfolios produced by investors. The market portfolio weights are as follows:

$$\mu_i(t) := \frac{X_i(t)}{X(t)}, \quad t \in [0, \infty), \quad i = 1, \ldots, n\}]wherestockprice$$

X$_i(t)$ can be interpreted as the capitalization of the i^{th} company at time t and $X(t)$ is as total capitalization of the market.

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Definition 2.4 ([13])**.** (diversity wighted portfolio) diversity wighted portfolio with parameter $p \in \mathbb{R}$ is defined as follows:

$$\pi_i^{(p)}(t) := \frac{(\mu_i(t))^{(p)}}{\sum_{j=1}^{n} (\mu_j(t))^{(p)}}, \quad i = 1, \ldots, n. \tag{1}$$

Definition 2.5 ([12])**.** (Sharpe ratio): It is a measure that can be used to evaluate portfolio and asset performance and is defined as

$$SharpeRatio(\pi(\cdot)) = \frac{\overline{R^{\pi}} - \overline{RF}}{StdDev\,(R^{\pi})} \cdot \sqrt{\frac{T}{y}}$$

.

Definition 2.6 ([13]). (Total growth rate): It is another measure for performance evaluation.

$$\widetilde{\gamma}_{\pi}:=\frac{\gamma_{\pi}}{StdDev\left(R^{\pi}\right)}\cdot\frac{1}{y}$$

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3 stochastic diversity weighted portfolios

Stochastic Portfolio Theory offers a relevant novel approach to portfolio selection in stock markets, in order to construct portfolios which outperform an index, or benchmark portfolio, over a given time-horizon with probability one, whenever this might be possible. One such investment strategy is the so-called diversity-weighted portfolio. This re-calibrates the weights of the market portfolio, by raising them all to some given power p, and then re-normalizing. Diversity-weighted portfolios with parameters $p \in (0,1)$, $p<0$ and combination of them is produced and performance of these portfolios are assessed by two mentioned methods. In the following, one type of portfolio generating functions has been introduced [12]. Those portfolio generating functions that are smooth functions of the market weights can be used to create portfolios with returns that satisfy almost sure relationships relative to the market portfolio. Then, a particular class of portfolios, called functionally-generated portfolios, are mentioned. Consider function G that produces portfolio π with the following weights. So, π is functionally generated portfolio.

$$\pi_i\left(t\right):=\left(D_i\log G\left(\mu\left(t\right)\right)+1-\sum_{j=1}^{n}\mu_j\left(t\right)D_j\log G\left(\mu\left(t\right)\right)\right)\cdot\mu_i\left(t\right). \tag{2}$$

The portfolio weights of (2) depend only on the market weights $\mu_1(t)$, ů ů ů , $\mu_n(t)$, not on the covariance structure of the market. Thus the portfolio of (2) can be implemented, and its associated wealth process$V^{\pi}(.)$ observed through time, only in terms of the evolution of these market weights over [0, T]. Theorem 3.1 of Fernholz (1999) [2]asserts that the performance of the wealth process corresponding to π, when measured relative to the market, satisfies the decomposition. It can be shown that the relative wealth process of this portfolio, with respect to the market, is given by the following formula

$$\log\left(\frac{V^{\pi}(t)}{V^{\mu}(t)}\right)=\log G\left(\mu\left(t\right)\right)+\mathfrak{g}\left(t\right). \tag{3}$$

where the so-called drift process $\mathfrak{g}$ is given by

$$\mathfrak{g}\left(t\right):=-\frac{-1}{2G\left(\mu\left(t\right)\right)}\sum_{i,j=1}^{n}D_{ij}G\left(\mu\left(t\right)\right)\mu_i\left(t\right)\mu_j\left(t\right)\mathcal{T}_{ij}{}^{\mu}\left(t\right)dt, \tag{4}$$

We are also looking for a portfolio whose value is higher than the market portfolio. As a result, the strategy sells the companys stock as its value increases, and buys it as its value decreases.
For example, $G_p(x) := (\sum_{i=1}^{n} x_i^p)^{\frac{1}{p}}, \quad x \in \Delta_+^n$, generates the diversity-weighted portfolio (2) (that can be) $\pi_i^{(p)}(t) := \frac{(\mu_i(t))^{(p)}}{\sum_{j=1}^{n}(\mu_j(t))^{(p)}}, \quad i = 1, \ldots, n$ with drift process

$$\mathfrak{g}_p(\cdot) = (1-p)\,\gamma^*_{\pi^{(p)}}(\cdot)\,. \tag{5}$$

In this paper, an attempt is made to increase the diversity of the portfolio as much as possible. Following this, the condition of non-failure is stated:

Definition 3.1 (Non-failure condition)**.**

$$\exists \varphi \epsilon \left(0, \frac{1}{n}\right) \text{ s.t : } \mathbb{P}\left(\mu_{(n)}(t) > \varphi, \forall t \in [0,T]\right) = 1. \tag{NF}$$

According to the definition, equation (5) , and Fernholzs master equation (3), it follows that relative wealth process of this portfolio, $\pi^{(p)}(\cdot)$ in a period $[0,T]$, outperforms with respect to the market portfolio over sufficiently long time-horizons.

$$\begin{aligned}\log\left(\frac{V^{\pi^{(p)}}(T)}{V^{\mu}(T)}\right) &= \log\left(\frac{G_p(\mu(T))}{G_p(\mu(0))}\right) + (1-p)\int_0^T \gamma^*_{\pi^{(p)}}(t)\,dt \\ &> \log(n\varphi) + (1-p)\frac{\varepsilon}{2}T\left(1 - \frac{(n\varphi)^p}{n}\right) > 0. \end{aligned} \tag{6}$$

Parametr P can accept $p \in (0,1)$, $p < 0$ and combination of them. It is proved that in the market model M, The diversity weigted portfolios $\pi^{(p)}$, with negative parameter

$$p^- \in \left(\frac{\log n}{\log(n\varphi)}, 0\right),$$

have a better performance than the market portfolio [13].

4 Numerical example

We consider data set of Iran Stock Exchange as a case study. All of the price on the stock market are publicly available from the official website of Tehran Stock Exchange Market (TSE). Moreover, trading days are recorded according to the market calendar, with all weekends and holidays removed from dataset. There is a ten-group of companies (industries) which they are randomly chosen from 27/01/1392 to 18/12/1400 from TSE. Figure 1 and Figure 2 demonstrates the stock price behavior and return volatility in the market respectively . With the help of the return of these 10 stocks, a favorite portfolio $\pi(\cdot)$ and two efficient portfolios ($\rho(\cdot)$ and $\eta(\cdot)$)are considered that are created by GAMS software.

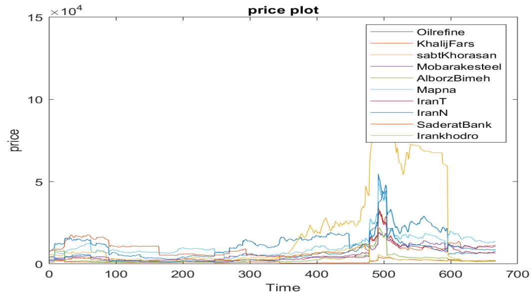

Figure 1: Daily prices of ten different companies of Iranian stock market

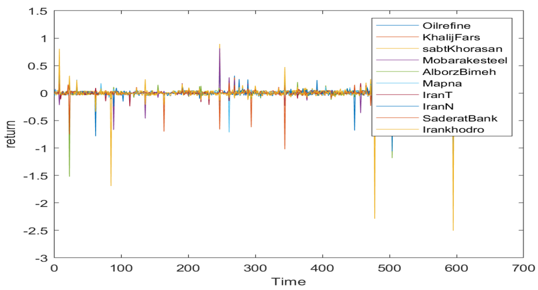

Figure 2: The return volatility of ten different companies of Iranian stock market

Now, with the assumed parameters and their combination according portfolio (2.4) and equation (2) with generating function $G_p(x) := \left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}}$, we have made diversity weighted portfolios according to Table 1.

Table 1: favorite portfolio and optimal portfolio

	portfolio	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$	$i=10$
1.	$\pi(\cdot)$	0.1317	0.0576	0.0905	0.1029	0.1070	0.0700	0.0823	0.1193	0.1399	0.0988
2.	$\pi^{(p)}(\cdot)$,p=0.4	0.1125	0.0808	0.0969	0.1019	0.1035	0.0874	0.0932	0.1082	0.1153	0.1003
3.	$\pi^{(p)}(\cdot)$,$p=-3$	0.0283	0.3384	0.0872	0.0594	0.0528	0.1890	0.1161	0.0381	0.0236	0.0672
4.	$\widehat{\pi}(\cdot)$,p^+=0.4,$p^-=-3$	0.1316	0.0577	0.0906	0.1029	0.1070	0.0700	0.0823	0.1193	0.1399	0.0988
5.	$\pi^{(p)}(\cdot)$,$p=-0.4$	0.0879	0.1223	0.1021	0.0970	0.0955	0.1132	0.1061	0.0914	0.0858	0.0986
6.	$\rho(\cdot)$	0.0000	0.0000	0.1391	0.1016	0.0260	0.0131	0.2448	0.0399	0.1898	0.2458
7.	$\rho^{(p)}(\cdot)$,p=0.4	0.0000	0.0000	0.1423	0.1254	0.0727	0.0553	0.1783	0.0863	0.1611	0.1786
8.	$\rho^{(p)}(\cdot)$,$p=-3$	0.0000	0.0000	0.0007	0.0018	0.1091	0.8576	0.0001	0.0302	0.0003	0.0001
9.	$\widehat{\rho}(\cdot)$,p^+=0.4,$p^-=-3$	0.0000	0.0000	0.1392	0.1016	0.0260	0.0131	0.2447	0.0399	0.1897	0.2457
10.	$\rho^{(p)}(\cdot)$,$p=-0.4$	0.0000	0.0000	0.0927	0.1051	0.1813	0.2386	0.0739	0.1527	0.0819	0.0738
11.	$\eta(\cdot)$	0.0000	0.0000	0.1369	0.1270	0.0000	0.0000	0.2035	0.1171	0.1841	0.2314
12.	$\eta^{(p)}(\cdot)$,p=0.4	0.0000	0.0000	0.1553	0.1507	0.0000	0.0000	0.1819	0.1459	0.1748	0.1915
13.	$\eta^{(p)}(\cdot)$,$p=-3$	0.0000	0.0000	0.2095	0.2624	0.0000	0.0000	0.0638	0.3346	0.0862	0.0434
14.	$\widehat{\eta}(\cdot)$,p^+=0.4,$p^-=-3$	0.0000	0.0000	0.1371	0.1272	0.0000	0.0000	0.2033	0.1174	0.1840	0.2310
15.	$\eta^{(p)}(\cdot)$,$p=-0.4$	0.0000	0.0000	0.1771	0.1825	0.0000	0.0000	0.1511	0.1885	0.1573	0.1436

Table 2: performance evaluation of generatad portfolio

	portfolio	standard diviation	expected return	sharpe ratio	total growth rate
1.	$\pi(\cdot)$	0.0505	0.0020	0.3391	0.495239
2.	$\pi^{(p)}(\cdot), p = 0.4$	0.0495	0.0020	0.3428	0.527249
3.	$\pi^{(p)}(\cdot), p = -3$	0.0502	0.0021	0.3608	0.605654
4.	$\widehat{\pi}(\cdot), p^{+} = 0.4, p^{-} = -3$	0.0505	0.0020	0.3391	0.495308
5.	$\pi^{(p)}(\cdot), p = -0.4$	0.048668	0.0020	0.348236	0.566517
6.	$\rho(\cdot)$	0.0434	0.0019	0.3754	0.605654
7.	$\rho^{(p)}(\cdot), p = 0.4$	0.0449	0.0020	0.3799	0.668824
8.	$\rho^{(p)}(\cdot), p = -3$	0.0780	0.0047	0.5176	0.685288
9.	$\widehat{\rho}(\cdot), p^{+} = 0.4, p^{-} = -3$	0.0434	0.0019	0.3754	0.605654
10.	$\rho^{(p)}(\cdot), p = -0.4$	0.053263	0.0025	0.411952	0.66885
11.	$\eta(\cdot)$	0.0438	0.0021	0.4034	0.683057
12.	$\eta^{(p)}(\cdot), p = 0.4$	0.0453	0.0021	0.4037	0.739208
13.	$\eta^{(p)}(\cdot), p = -3$	0.0536	0.0026	0.4197	0.772284
14.	$\widehat{\eta}(\cdot), p^{+} = 0.4, p^{-} = -3$	0.0438	0.0021	0.4034	0.704723
15.	$\eta^{(p)}(\cdot), p = -0.4$	0.047587	0.0022	0.405564	0.771956

In this section, we consider portfolios $\pi(\cdot)$, $\rho(\cdot)$ and $\eta(\cdot)$ as the market portfolio and we analyze performance of positive and negative parameter of the diversity-weighted portfolio as $p = -0.4$ $p = 0.4$ $p = -3$ and combination of them by sharp ratio and total growth rate.
In table 2, performance evaluation of generated portfolio with negative, positive and their combination parametrs are shown.
From Table 2 we can deduce that performance of diversity weighted portfolios $\pi^p(\cdot)$, $\rho^p(\cdot)$, $\eta^p(\cdot)$ with positive and negative parameters are better than $\pi(\cdot)$, $\rho(\cdot)$, $\eta(\cdot)$. Also, performance of diversity weighted portfolios $\pi^p(\cdot)$, $\rho^p(\cdot)$, $\eta^p(\cdot)$ with negative parameters $p = -0.4$, $p = -3$ are the best. For data exploration, draw plots, constructing of diversity weighed portfolios and calculating mean return, standard deviation, sharp ratio and total growth rate, the Matlab and Excel software are employed.

5 conclusion

In this section, we describe the dynamics of asset prices in a stochastic process using the growth rate process instead of the rate of return. Since the growth rate process examines the behavior of assets over the long term, it allows investors to predict the behavior of assets and consequently forecast the behavior of asset portfolios in the future. We also introduce another approach called the portfolio generating functions , which helps us obtain optimal weighted portfolios. By extending this theory, we generate diversified weighted portfolios with positive and negative parameters , and a combination of these two parameters. Using the Sharpe ratio and the total growth rate as evaluation criteria, we assess the performance of these weighted portfolios. As observed, negatively parameterized weighted portfolios outperform the market index or the market portfolio over the long term.

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