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Research paper

A generation theorem for the perturbation of exponentially equicontinuous C_0-semigroups on locally convex spaces

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Abstract:
In this paper, we study the well-posedness of the evolution equation of the form $u'(t) = Au(t) + Cu(t)$, $t \geq 0$ where A is the infinitesimal generator of an exponentially equicontinuous C_0-semigroup and C is a (possibly unbounded) linear operator in a sequentially complete locally convex Hausdorff space X. In particular, we demonstrate that if A generates an exponentially equicontinuous C_0-semigroup $(T_A(t))_{t\geq 0}$ satisfying $p(T_A(t)x) \leq e^{\omega t}q(x)$ and C is a linear operator on X such that $D(A) \subset D(C)$ and $\{K^{-1}(\mu - \omega)^n(CR(\mu, A))^n; \mu > \omega, n \in \mathbb{N}\}$ is equicontinuous, then the above-mentioned evolution equation is well-posed, that is, $A + C$ generates an exponentially equicontinuous C_0-semigroup $(T_{A+C}(t))_{t\geq 0}$ satisfying $p(T_{A+C}(t)x) \leq e^{(\omega+K)t}q(x)$.

Keywords: C_0-semigroups, continuous linear operators, locally convex spaces
Classification: 47D03.

1 Introduction

In [12], Yosida initiated the study of equicontinuous C_0-semigroups on locally convex spaces. Recently, Singbal-Vedak [11] studied some properties of C_0-semigroups on a locally convex space. However, she proved a perturbation theorem of a C_0-semigroup on a sequentially complete locally convex Hausdorff space which generalized a result due to Phillips [10]. For more details, see [11]. On the other hand, Dembart [3] proved several results on the perturbation of a locally equicontinuous C_0-semigroup on a locally convex space.

Semigroups of continuous linear operators in locally convex spaces were studied by several researchers [5–7, 11]. As an application of C_0-semigroups of continuous linear operators on locally convex spaces is the abstract Cauchy problem for differential equations on a locally convex space X given by

$$ACP(A;x) \begin{cases} \frac{du(t)}{dt} = Au(t), \quad t \in \mathbb{R}_+, \\ u(0) = x, \end{cases}$$

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where $A : D(A) \subset X \to X$ is a densely defined closed linear operator on a sequentially complete locally convex Hausdorff space X and $x \in X$.

The aim of this paper is to prove some results on the perturbation of C_0-semigroups of continuous linear operators in sequentially complete locally convex Hausdorff spaces.

The perturbation of strongly continuous semigroups in a Banach space is one of central topics in the semigroup theory, e.g., [4, 9, 11].

Throughout this paper, X is a sequentially complete locally convex Hausdorff space over the field of complex numbers $\mathbb{C}$ under the family of seminorms Γ_X and $\mathcal{L}(X)$ denotes the collection of continuous linear operators on X. For $A \in \mathcal{L}(X)$, the domain of A is denoted by $D(A)$.

In this paper, we study the well-posedness of the evolution equation of the form

$$u'(t) = Au(t) + Cu(t), \quad t \geq 0, \tag{1}$$

where A is the infinitesimal generator of an exponentially equicontinuous C_0-semigroup and C is a (possibly unbounded) linear operator in a sequentially complete locally convex Hausdorff space X. Specifically, we will find conditions on A and C such that $A + C$ generates a C_0-semigroup of linear operators in X.

2 Preliminaries

Definition 2.1 ([8]). Let X be a vector space over $\mathbb{K}(= \mathbb{R}$ or $\mathbb{C})$. A function $p : X \to \mathbb{R}_+$ is called seminorm if

(i) For any $x \in X$ and for all $\lambda \in \mathbb{K}$, $p(\lambda x) = |\lambda| p(x)$,

(ii) For all $x, y \in X$, $p(x + y) \leq p(x) + p(y)$.

Example 2.2 ([8]). Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then we have

(i) If A is a linear map of a vector space X over $\mathbb{K}$ into a seminormed space (Y, p), then $p \circ A$ is a seminorm on X. In particular, if f is a linear functional on X, then $x \mapsto |f(x)|$ is a seminorm on X.

(ii) If X is a linear space of integrable $\mathbb{K}$-valued functions on some set T, then $p(x) = |\int_T x|$ is a seminorm on X.

(iii) If X is a vector space of $\mathbb{K}$-valued functions on a set T, then for any $t_0 \in T$, the map $x \mapsto |x(t_0)|$ is a seminorm on X.

(iv) Let $C(X, \mathbb{K})$ be the linear space of all continuous maps of the topological space X into $\mathbb{K}$ and let K be a compact subset of X. Then $p_K(x) = \sup_{t \in K} |x(t)|$ $(x \in C(X, \mathbb{K}))$ is a seminorm on $C(X, \mathbb{K})$.

Definition 2.3 ([8]). A complex linear topological vector space X is called a locally convex space, if any if its open sets contains a convex, balanced and absorbing open set.

Definition 2.4 ([8]). A locally convex space X is Hausdorff if and only if for all non-zero $x \in X$, there is a continuous seminorm p on X such that $p(x) \neq 0$.

Definition 2.5 ([5]). Let X be a sequentially complete locally convex space and let $A \in \mathcal{L}(X)$. The resolvent set $\rho(A)$ of A is defined by

$$\rho(A) = \{\lambda \in \mathbb{C} : (\lambda - A)^{-1} \in \mathcal{L}(X)\}. \tag{2}$$

If $\lambda \in \rho(A)$, then the application $R(\lambda, A) = (\lambda - A)^{-1}$ is called the resolvent of A. The spectrum $\sigma(A)$ of A is defined by $\mathbb{C}\backslash\rho(A)$.

Definition 2.6 ([2]). Let X be a sequentially complete locally convex space. A one-parameter family $(J(t))_{t\in\mathbb{R}_+}$ of continuous linear operators on X is a C_0-semigroup if

(i) $J(0) = I$,

(ii) For any $t, s \in \mathbb{R}_+, J(t+s) = J(t)J(s)$,

(iii) For all $x \in X, \lim_{h\to 0^+} J(h)x = x$.

The linear operator A defined by

$$D(A) = \left\{ x \in X : \lim_{t\to 0^+} \frac{J(t)x - x}{t} \text{ exists} \right\} \tag{3}$$

and

$$Ax = \lim_{t\to 0^+} \tfrac{J(t)x-x}{t}, \text{ for any } x \in D(A),$$

is called the infinitesimal generator of the semigroup $(J(t))_{t\in\mathbb{R}_+}$.

Definition 2.7 ([2]). Let X be a sequentially complete locally convex Hausdorff space. A one-parameter C_0-semigroup $(J(t))_{t\in\mathbb{R}_+}$ on X is equicontinuous if for any continuous seminorm p on X, there exists a continuous seminorm q on X such that for all $x \in X$ and for each $t \in \mathbb{R}_+, p(J(t)x) \leq q(x)$.

Definition 2.8 ([2]). Let X be a sequentially complete locally convex Hausdorff space. A one-parameter C_0-semigroup $(J(t))_{t\in\mathbb{R}_+}$ on X is exponentially equicontinuous if there exists $\omega > 0$ such that $\{e^{-\omega t}J(t); t \in \mathbb{R}_+\}$ is equicontinuous.

Theorem 2.9 ([11]). *Let A be a closed linear densely defined operator on a sequentially complete locally convex Hausdorff space X. Then in order that A be the infinitesimal generator of an equicontinuous C_0-semigroup $(J(s))_{s\in\mathbb{R}_+}$ it is necessary and sufficient that the family*

$$\{\lambda^n R(\lambda, A)^n; \lambda > 0, n \in \mathbb{N}\} \tag{4}$$

is equicontinuous.

Theorem 2.10 ([11]). *Let A be a closed linear densely defined operator on a sequentially complete locally convex Hausdorff space X. Then in order that A be the infinitesimal generator of a continuous semigroup $(J(s))_{s\in\mathbb{R}_+}$ with $\{e^{-\omega s}J(s); s\in \mathbb{R}_+\}$ is equicontinuous for some $\omega > 0$ it is necessary and sufficient that the family*

$$\{(\lambda-\omega)^n R(\lambda, A)^n; \lambda > \omega, n \in \mathbb{N}\} \tag{5}$$

is equicontinuous.

Theorem 2.11 ([11]). *Let X be a sequentially complete locally convex Hausdorff space and A be a densely defined closed linear operator on X such that for some $\omega > 0$, the resolvent $R(\lambda, A)$ of A exists as a continuous linear operator on X for $\lambda > \omega$ and the family*

$$\{(\lambda-\omega)^n R(\lambda, A)^n; \lambda > \omega, n = 1, 2, \cdots\} \tag{6}$$

is equicontinuous. If B is a bounded operator on X, then there exists $\omega_1 > 0$ such that the resolvent $R(\lambda, A+B)$ of $A+B$ exists as a continuous linear operator X and the family

$$\{(\lambda-\omega_1)^n R(\lambda, A+B)^n; \lambda > \omega_1, n = 1, 2, \cdots\} \tag{7}$$

is equicontinuous.

Theorem 2.12 ([11]). *Let X be a sequentially complete locally convex Hausdorff space and let A be the infinitesimal generator of a continuous semigroup $(T(s))_{s\in\mathbb{R}_+}$ such that for some $\omega > 0$ $(e^{-\omega s}T(s))_{s\in\mathbb{R}_+}$ is equicontinuous. If B is a bounded operator on X, then $A+B$ is the infinitesimal generator of a continuous semigroup $(S(s))_{s\in\mathbb{R}_+}$ such that for some $\omega_1 > 0$, $(e^{-\omega_1 s}S(s))_{s\in\mathbb{R}_+}$ is equicontinuous.*

3 Main results

In this section, we present our main results.

Theorem 3.1. *Let X be a sequentially complete locally convex Hausdorff space. Let A be the infinitesimal generator of an exponentially equicontinuous C_0-semigroup $(T_A(t))_{t\geq 0}$ on X such that for all $t \geq 0$ and all $x \in X$,*

$$p(T_A(t)x) \leq e^{\omega t} q(x).$$

Suppose further that C is a linear operator in X such that

(i) $D(A) \subset D(C)$;

(ii) There exists a constant $K > 0$ such that if $\mu > \omega$, then for all $p \in \Gamma_X$, there exists $q \in \Gamma_X$ such that for all $x \in X$ and all $n \in \mathbb{N}$,

$$p((CR(\mu, A))^n x) \leq \frac{K}{(\mu-\omega)^n} q(x). \tag{8}$$

Then, $A+C$ generates an exponentially equicontinuous C_0-semigroup, denoted by $(T_{A+C}(t))_{t\geq 0}$ that satisfies

$$p(T_{A+C}(t)x) \leq e^{(\omega+K)t}q(x), \quad t \geq 0 \text{ and } x \in X. \tag{9}$$

Proof. First, we will demonstrate that

$$(K+\omega, \infty) \subset \rho(A+C). \tag{10}$$

Indeed, for all $\mu > K+\omega$, as $K > 0$, we obtain $\mu \in \rho(A)$ and

$$(\mu - (A+C))R(\mu, A) = \mu R(\mu, A) - AR(\mu, A) - CR(\mu, A) = I - CR(\mu, A). \tag{11}$$

Then for all $\mu > K+\omega$, we have $\mu \in \rho(A)$ and

$$(\mu - (A+C)) = [I - CR(\mu, A)]\,(\mu - A). \tag{12}$$

From $\mu > K+\omega$, we obtain $K/(\mu-\omega) < 1$, then for all $\mu > K+\omega$ and for each $x \in X, p(CR(\mu, A)x) < q(x)$, hence for all $\mu > K+\omega, I - CR(\mu, A)$ is invertible. By (11), we obtain for all $\mu > K+\omega$, $(\mu - (A+C))$ is invertible, then (10) follows. In particular, this also yields that $A+C$ is a closed operator (as $\rho(A+C) \neq \emptyset$).

Now, for all $\mu > K+\omega$, one can see that

$$R(\mu, A+C) - R(\mu, A) = R(\mu, A+C)CR(\mu, A). \tag{13}$$

Then for all $\mu > K+\omega$,

$$R(\mu, A+C)\,(I - CR(\mu, A)) = R(\mu, A). \tag{14}$$

By assumption (8), for all $\mu > K+\omega$, we have for each $x \in X, p(CR(\mu, A)x) \leq K/(\mu-\omega)q(x) < q(x)$, then for all $\mu > K+\omega$, $I - CR(\mu, A)$ is invertible, hence for all $\mu > K+\omega, x \in X$ and for each $p \in \Gamma_X$, there exists $q \in \Gamma_X$ such that

$$p((I - CR(\mu, A))^{-1}\,x) \leq \frac{1}{1 - K/(\mu-\omega)}q(x). \tag{15}$$

Consequently, for all $\mu > K+\omega, x \in X$ and for each $p \in \Gamma_X$, there exists $q \in \Gamma_X$ such that

$$\begin{aligned} p(R(\mu, A+C)x) &= p(R(\mu, A)\,(I - CR(\mu, A))^{-1}\,x) \\ &\leq \frac{1}{\mu-\omega}q((I - CR(\mu, A))^{-1}\,x) \\ &\leq \frac{1}{\mu-\omega}\cdot\frac{1}{1 - K/(\mu-\omega)}q(x) \\ &\leq \frac{1}{\mu-(\omega+K)}q(x). \end{aligned} \tag{16}$$

By the Hille-Yosida's generation theorem, since $D(A) \subset D(A+C)$ is densely everywhere in X, (16) yields that the closed linear operator $A+C$ generate an exponentially equicontinuous C_0-semigroup, denoted by $(T_{A+C}(t))_{t\geq 0}$ that satisfies (9). This completes the proof. □

Similarly to the proof of Theorem 3.1, we deduce the following theorem.

Theorem 3.2. *Let X be a sequentially complete locally convex Hausdorff space. Let A be the infinitesimal generator of an equicontinuous C_0-semigroup $(T_A(t))_{t\geq 0}$ on X such that for all $t \geq 0$ and all $x \in X$,*

$$p(T_A(t)x) \leq q(x).$$

Suppose that further that C is a linear operator in X such that

(i) $D(A) \subset D(C)$;

(ii) There exists a constant $K > 0$ such that if $\mu > 0$, then for all $p \in \Gamma_X$, there exists $q \in \Gamma_X$ such that for all $x \in X$ and all $n \in \mathbb{N}$,

$$p((CR(\mu, A))^n x) \leq \frac{K}{\mu^n} q(x). \tag{17}$$

Then, $A + C$ generates an equicontinuous C_0-semigroup, denoted by $(T_{A+C}(t))_{t\geq 0}$.

If C is a bounded linear operator on X, we conclude from Theorem 3.1 that.

Theorem 3.3. *Let X be a sequentially complete locally convex Hausdorff space. Let A be the infinitesimal generator of an exponentially equicontinuous C_0-semigroup $(T_A(t))_{t\geq 0}$ on X such that for all $t \geq 0$ and all $x \in X$,*

$$p(T_A(t)x) \leq e^{\omega t} q(x).$$

Suppose further that C is a bounded linear operator in X such that there exists a constant $K > 0$ and if $\mu > \omega$, then for all $p \in \Gamma_X$, there exists $q \in \Gamma_X$ such that for all $x \in X$ and all $n \in \mathbb{N}$,

$$p((CR(\mu, A))^n x) \leq \frac{K}{(\mu - \omega)^n} q(x). \tag{18}$$

Then, $A + C$ generates an exponentially equicontinuous C_0-semigroup, denoted by $(T_{A+C}(t))_{t\geq 0}$ that satisfies

$$p(T_{A+C}(t)x) \leq e^{(\omega+K)t} q(x), \quad t \geq 0 \text{ and } x \in X. \tag{19}$$

Similarly to the proof of Theorem 3.2, we obtain:

Theorem 3.4. *Let X be a sequentially complete locally convex Hausdorff space. Let A be the infinitesimal generator of an equicontinuous C_0-semigroup $(T_A(t))_{t\geq 0}$ on X such that for all $t \geq 0$ and all $x \in X$,*

$$p(T_A(t)x) \leq q(x).$$

Suppose further that C is a bounded linear operator in X such that there exists a constant $K > 0$ and if $\mu > 0$, then for all $p \in \Gamma_X$, there exists $q \in \Gamma_X$ such that for all $x \in X$ and all $n \in \mathbb{N}$,

$$p((CR(\mu, A))^n x) \leq \frac{K}{\mu^n} q(x). \tag{20}$$

Then, $A + C$ generates an equicontinuous C_0-semigroup, denoted by $(T_{A+C}(t))_{t\geq 0}$.

We finish with the following application.

Application 1. Let X be a sequentially complete locally convex Hausdorff space. Consider the Cauchy problem of the form

$$\begin{cases} u'(t) = Au(t) + Cu(t), & t \in \mathbb{R}_+, \\ u(0) = x, & x \in X \end{cases} \tag{21}$$

where A is the infinitesimal generator of an exponentially equicontinuous C_0-semigroup and C is an unbounded linear operator on X. If A generates an exponentially equicontinuous C_0-semigroup $(T_A(t))_{t\geq 0}$ satisfying $p(T_A(t)x) \leq e^{\omega t} q(x)$ and C is an unbounded linear operator on X such that $D(A) \subset D(C)$ and $p((CR(\mu, A))^n x) \leq K/(\mu - \omega)^n q(x)$ for any $\mu > \omega$, then by Theorem 3.1, the Cauchy problem (21) is well-posed.

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