The first order nonlinear autoregressive model with Ornstein Uhlenbeck processes driven by white noise

Parisa Nabati[1]

[1] Faculty of Science, Urmia University of Technology, Urmia, Iran
p.nabati@uut.ac.ir

Abstract:
This paper presents a nonlinear autoregressive model by Ornstein Uhlenbeck processes innovation driven with white noise. The presented notations and preliminaries about these processes, have important applications in finance. The parameter estimation for these processes is constructed from the time-continuous likelihood function that leads to an explicit maximum likelihood estimator. A semiparametric method is proposed to estimate the nonlinear autoregressive function using the conditional least square method for parametric estimation, and a nonparametric kernel approach by using the nonparametric factor that is derived by a local L2-fitting criterion for the regression adjustment estimation. Then the Monte Carlo numerical simulation studies are carried out to show the efficiency and accuracy of the present work. The mean square error (MSE) is a measure of the average squared deviation of the estimated function values from the actual ones. The values of MSE indicate that the innovation in noise structure is performed well in comparison with the existing noise in the nonlinear autoregressive models.

Keywords: Autoregressive model; Conditional nonlinear least squares method; Ornstein-Uhlenbeck processes; Semiparametric estimation.
MSC2010 Classifications: 60H10, 62G08, 62M10.

[1] Corresponding author
Received: 2020-06-25 Approved: 2020-09-25

1 Introduction

Stochastic differential equations (SDEs) are widely used in financial economics to analyze the time dependency of high-frequency data such as the ones measured in minutes or seconds. The most important type of SDEs is the Ornstein Uhlenbeck (OU) processes that have important applications in various fields. In mathematical finance, they are well known as the main building block of the Barndroff-Nielsen and Shephard stochastic volatility model [1]. The OU process is a diffusion process that was introduced as a model of the velocity of a particle undergoing Brownian motion. In recent years, the OU process has appeared in finance as a model of the volatility of the underlying asset price process. Estimation of the parameters of OU processes has been considered by many researchers [2, 10]. The maximum likelihood estimator is an efficient method. In many practical cases where a diffusion process has been observed at discrete time points, an explicit likelihood function is rarely available. Forecasting and capturing the trend of time series have been the object of numerous researches over the past decades [9]. The most widely used model for this case of models is the autoregressive (AR) model. Semiparametric time series models with the structure of nonlinear functions are proposed in [3, 8]. The common way of analyzing an AR model is based on the normality assumption of innovations that provides a Gaussian process for observations, whereas, in many empirical situations, the data violate this assumption, and therefore the traditional normal based theory leads to poor forecasts [6]. Recently many researchers have focused on developed methods in which the normality assumption does not hold. These methods allow replacing the normal innovation by a nonnormal innovation to build and analyze an AR model. Zhuoxi et al. considered a semiparametric method for estimating the nonlinear autoregressive model with independent errors [11]. Farnoosh and Mortazavi extend the Zhuoxi model to a nonlinear autoregressive model with dependent errors and investigate the asymptotic behaviors of the semiparametric estimators [4]. Hajrajabi and Fallah assume the skew-normal innovations instead of normal and estimate the parameters using the maximum likelihood approach and Expectation-Maximization type optimization [5]. Following these works, we sustain the semiparametric estimation for the nonlinear autoregressive model. The difference is that we allow the innovation to be a sequence of (i.i.d) random variables that are achieved from an OU process driven by the white noise. Also, the closed form for the likelihood function of the OU process likelihood function is obtained. The rest of this paper is organized as follows. In section 2 the preliminaries and notations about the OU processes are investigated. The maximum likelihood estimator for parameter estimation is presented in this section. The proposed semiparametric nonlinear autoregressive model with the OU process is discussed in section 3. Some numerical simulations are carried out in section 4. Finally, section 5 covers some concluding remarks.

2 Preliminaries and notations

A stochastic process $\xi = \{\xi(t)\}$ is an OU process if it satisfies the linear stochastic differential equation

$$\begin{cases} d\xi(t) = -\lambda\xi(t)dt + \sigma dw(t) \\ \xi(0) = \xi_0 \end{cases} \tag{1}$$

where λ, σ are positive parameters and ξ_0 is a random variable is independent of a standard Brownian motion $w = \{w(t)\}$. The equation(1) has unique strong solution,

$$\xi(t) = \exp(-\lambda t)\Big(\xi_0 + \sigma \int_0^t \exp(\lambda s)dw(s)\Big) \tag{2}$$

The mean, variance and covariance of $\xi(t)$ with some algebraic manipulation can be obtained,

$$\begin{aligned} &E(\xi_t) = exp(-\lambda t)E(\xi_0) \\ &var(\xi_t) = \tfrac{\sigma^2}{2\lambda} + (var(\xi_0) - \tfrac{\sigma^2}{2\lambda})exp(-2\lambda t) \\ &cov(\xi(s), \xi(t+s)) = \Big(var(\xi_0) + \tfrac{\sigma^2}{2\lambda}(exp(2\lambda s) - 1)\Big)exp(-\lambda(2s+t)) \end{aligned} \tag{3}$$

respectively. If $\xi_0 \sim N(0, \frac{\sigma^2}{2\lambda})$ becomes a stationary Gaussian process with covariance function $c(t) = \frac{\sigma^2}{2\lambda}exp(-\lambda t)$. We modify Eq(1) to

$$\begin{cases} d\xi(t) = -\lambda\xi(t)dt + dZ(\lambda t) \\ \xi(0) = \xi_0 \end{cases} \tag{4}$$

where $\xi_0 \sim N(0, \frac{\sigma^2}{2})$ and $Z = Z(t)$ is an independent Gaussian process with $Z(t) \sim N(0, t\sigma^2)$. Now we want to study the maximum likelihood estimation of the vector parameter $\Theta = (\sigma^2, \lambda)$ in Eq(4). For this purpose, let $\xi_0, \xi_1, ..., \xi_n$ be the observed values of $\xi(t_0), \xi(t_1), ..., \xi(t_n)$ of the model(4). The likelihood function using the Markov and Gaussian properties is explicitly given by

$$\begin{aligned} L(\theta) &= f_{\xi(t_0)}(\xi_0) \prod_{k=1}^{n} f_{\xi(t_k)|\xi(t_{k-1})}(\xi_k) \\ &= \frac{1}{\sigma\sqrt{\pi}} \exp(\frac{-\xi_0^2}{\sigma^2}) \exp\Big(- \frac{(\xi_k - \exp(-\lambda\Delta_k)\xi_{k-1})^2}{\sigma^2\sqrt{\pi(1-\exp(-2\lambda\Delta_k)}}\Big) \end{aligned} \tag{5}$$

The log-likelihood function is given by:

$$\begin{aligned} K(\theta) &= -\frac{(n+1)}{2}\log(\pi\sigma^2) - \frac{\xi_0^2}{\sigma^2} - \frac{1}{2}\sum_{k=1}^{n}\log(1-\exp(-2\lambda\Delta_k)) \\ &- \sum_{k=1}^{n} \frac{(\xi_k - \exp(-\lambda\Delta_k)\xi_{k-1})^2}{\sigma^2(1-\exp(-2\lambda\Delta_k))} \end{aligned} \tag{6}$$

By maximizing this function, maximum likelihood estimator can be achieved [10].

3 The proposed Semiparametric model

Consider the nonlinear autoregressive model,

$$y_t = f(y_{t-1}) + \xi_t, \quad t = 1, ..., n \tag{7}$$

where $f(.)$ is an unknown autoregressive function and ξ_t is an OU process that satisfies in Eq(4). Note that $y(t)$ and $\xi(t)$ are independent for any t. We used a semiparametric method for estimation of the unknown autoregressive function $f(.)$ based on the work of Zhouxi et al. [11] and Farnoosh and mortazavi [4]. Suppose that $f(.)$ in Eq(9) has a parametric framework, namely parametric model as,

$$f(x) \in \{h(x, \beta), \beta \in \mathcal{B}\} \tag{8}$$

where $h(x, \beta)$ is a known function of x and β such that $\beta \in \mathcal{R}^m$ is a parametric space. In the model (9) the parameter vector β should be well estimated using the conditional nonlinear least square errors method as follows:

$$\hat{\beta} = argmin_{\beta\in\mathcal{B}} \sum_{t=1}^{n} \big(y_t - E(y_t|y_{t-1})\big)^2 \tag{9}$$

It is clear that

$$\begin{aligned} E(y_t|y_{t-1}) &= f(y_{t-1}) + E(\xi_t|y_{t-1}) \\ &= f(y_{t-1}) + e^{(-\lambda t)}E(\xi_0) \\ &= h(y_{t-1}, \beta) \end{aligned}$$

Since $\xi_0 \sim N(0, \frac{\sigma^2}{2})$ then $E(\xi_0) = 0$. Zhuoxi obtain the strong consistency of $\hat{\beta}_n$ under a variety of conditions [11]. Using the similar idea as in Hjort and Jones we will adjust the initial approximation of $f(.)$ by the semiparametric form $h(x, \hat{\beta})\Gamma(x)$, where $\Gamma(x)$ is the adjustment factor [7]. The remaining problem is to determine $\Gamma(x)$. The local L2-fitting criterion is defined as

$$\mathcal{Q}_n(x, \Gamma) = \sum_{t=1}^{n} K(\frac{y_{t-1} - x}{b_n})\{f(y_{t-1}) - h(y_{t-1}, \hat{\beta}).\Gamma(x)\}^2, \qquad (10)$$

where $K(.)$ is a kernel and b_n is the band width depending on n. This kernel estimator is a special case of the local polynomial estimator proposed by Zhuoxi et al. So the estimator $\hat{\Gamma}(x)$ of $\Gamma(x)$ is obtained by minimizing the criterion in Eq(12) with respect to $\Gamma(x)$. Therefore a nonparametric estimator is calculated with smooth kernel method of $\Gamma(x)$ as

$$\hat{\Gamma}(x) = \frac{\sum_{t=1}^{n} f(y_{t-1})K(\frac{y_{t-1}-x}{b_n})h(y_{t-1}, \hat{\beta})}{\sum_{t=1}^{n} K(\frac{y_{t-1}-x}{b_n})h^2(y_{t-1}, \hat{\beta})} \qquad (11)$$

The above equation contains the unknown function $f(x)$, therefore by using $\xi_t = y_t - f(y_{t-1})$ since the error of the method are small values with expectation zero, then $y_t \sim f(y_{t-1})$ and we have,

$$\tilde{\Gamma}(x) = \frac{\sum_{t=1}^{y} {}_tK(\frac{y_{t-1}-x}{b_n})h(y_{t-1}, \hat{\beta})}{\sum_{t=1}^{n} K(\frac{y_{t-1}-x}{b_n})h^2(y_{t-1}, \hat{\beta})} \qquad (12)$$

Finally the autoregressive estimator is obtained by

$$\tilde{f}(x) = h(x, \hat{\beta}).\tilde{\Gamma}(x) \qquad (13)$$

4 Numerical Simulation

In this section a Monte Carlo simulation study is designed to show the efficiency and accuracy of present work. We consider a nonlinear AR(1) model of the form $y_t = f(y_{t-1}) + \xi_t$ where $d\xi_t = -0.1\xi(t)dt + 0.2dw(t)$. Let us consider the partition $0 = t_0 < t_1 < \cdots < t_n = T$ for interval $[0, T]$ and $\Delta t = \frac{T}{n}$, then the Euler Maruyama approximation for OU process is as follows:

$$\xi(t_{i+1}) = \xi(t_i) - \lambda\xi(t_i)\Delta t + \sigma\Delta w$$

where $\Delta w \sim N(0, \Delta t)$. At first the noise of system is simulated using the Monte Carlo simulation approach and then the autoregressive model is simulated with the nonlinear part in three case studies.

Case 1: $f_{11}(x) = 5\sin(x)$, $f_{12}(x) = 5\sin(x) + 0.1x$ by assuming $h_1(x, \beta) = \beta_1 \sin(x)$

Case 2: $f_{21}(x) = 3\exp(-x)$, $f_{22}(x) = 3\exp(-x) + 0.1x$ by assuming $h_2(x, \beta) = \beta_2 \exp(-x)$

Case 3: $f_{31}(x) = 2\log(x)$, $f_{32}(x) = 2\log(x) + 0.1x$ by assuming $h_3(x, \beta) = \beta_3 \log(x)$

Using the kernel function $K(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$ and $b_n = 0.3$. The mean values of the estimators of the parameters for n=100 observation in 1000 samples of the proposed algorithm are presented in table (1). The mean square error (MSE) is defined as

$$MSE = \frac{1}{n} \sum_{i=1}^{n} (f(x_i) - h(x_i, \hat{\beta}))^2$$

and earned for these case studies.

Table 1: The mean of estimator and MSE for the nonlinear autoregressive function

$f(x)$	$h(x, \beta)$	$\hat{\beta}$	MSE
$5\sin(x)$	$\beta\sin(x)$	5.000000	4.22×10^-6
$5\sin(x) + 0.1x$	$\beta\sin(x)$	4.999478	0.340420
$3\exp(-x)$	$\beta\exp(-x)$	3.001103	5.28×10^-6
$3\exp(-x) + 0.1x$	$\beta\exp(-x)$	3.001230	0.348497
$2\log(x)$	$\beta\log(x)$	2.000972	1.51×10^-10
$2\log(x) + 0.1x$	$\beta\log(x)$	1.999712	6.85×10^-6

The figures (1-7) shows the exact functions and simulated functions for three case studies. We can see that the proposed model performs well.

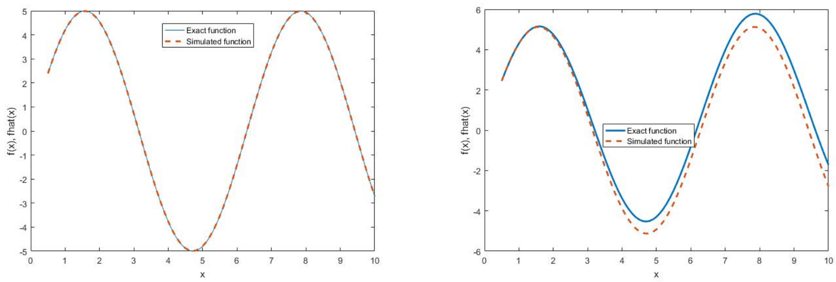

Figure 1: The exact function and simulated function for case 1, Right: for $f_{11}(x)$, Left: for $f_{12}(x)$.

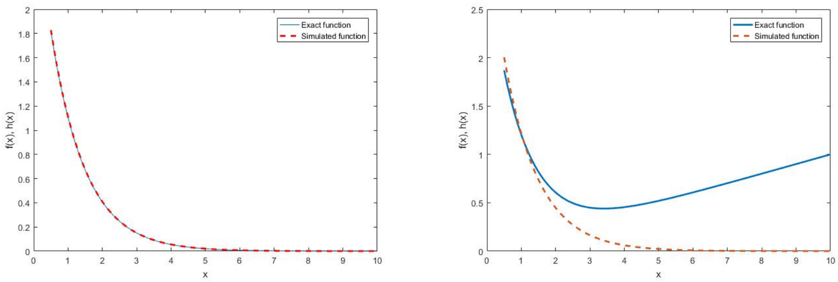

Figure 2: The exact function and simulated function for case 1, Right: for $f_{21}(x)$, Left: for $f_{22}(x)$.

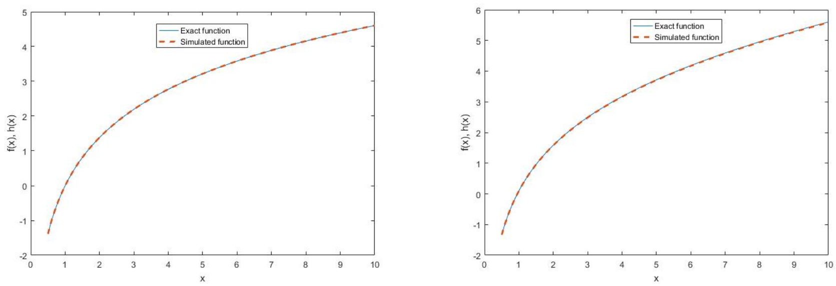

Figure 3: The exact function and simulated function for case 1, Right: for $f_{31}(x)$, Left: for $f_{32}(x)$.

5 Conclusion

In this paper a first order nonlinear autoregressive model with Ornstein Uhlenbeck processes innovation driven by the white noise is considered.

This assumption allows a flexible treatment of the observations.The simulated OU process by Matlab shows the trend of the noise toward the mean. The semiparametric method by using the conditional nonlinear least squares and the local L2-fitting criterion is used to estimate the nonlinear parts of the model. The numerical simulations show the accuracy of the present work. For the future works, researchers can use the flexible class of none Gaussian OU processes driven by levy noise with desirable properties for using in the time series models and semi-parametric methods in this paper.

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