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Research paper

Gumbel copula-based reliability assessment to describe the dependence of the multicomponent stress-strength model for Pareto distribution

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Abstract:
The stress-strength model is a commonly utilized topic in reliability studies. In many reliability analyses involving stress-strength models, it is typically assumed that the stress and strength variables are unrelated. Nevertheless, this assumption is often impractical in real-world scenarios. This research assumes that the strength and stress variables follow the Pareto distribution, and a Gumbel copula is employed to represent their relationship. Additionally, the data is gathered through the Type-I progressively hybrid censoring scheme. The method of maximum likelihood estimation is used for point estimation, while asymptotic and Bootstrap percentile confidence intervals are employed for interval estimation of the unknown parameters and system reliability. Simulation is employed to assess the effectiveness of the suggested estimators. Subsequently, an actual dataset is examined to showcase the practicality of the stress-strength model. Simulation is employed to assess the effectiveness of the suggested estimators. Subsequently, a real dataset is examined to demonstrate the practicality of the stress-strength model.

Keywords: Bootstrap percentile confidence interval, Gumbel copula, Pareto distribution, Multicomponent dependent stress-strength model, Type-I progressively hybrid censoring scheme.
Classification: 62N05; 62N01.

1 Introduction

The stress-strength model is a fundamental concept utilized in reliability engineering and statistics to evaluate the likelihood of system or component failure. Within this framework, stress denotes the applied load or demand on a system, whereas strength indicates the system's ability to endure that stress. The core principle of the stress-strength model posits that failure transpires when the applied stress surpasses the system's strength. The model postulates that stress and strength are

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stochastic variables, and their association can be characterized through probability distributions.

Engineers assess system reliability by comparing the distribution of stress and strength variables. If the stress distribution surpasses the strength distribution, the probability of system failure increases. Conversely, if the strength distribution exceeds the stress distribution, the system is more likely to endure. Understanding the interplay between stress and strength enables engineers to enhance system reliability and safety through informed decisions regarding design, maintenance, and risk management.

Numerous researchers have addressed the issue of evaluating the reliability of the stress-strength model. Initially, Birnbaum [5] as the first to studied into this area. Subsequently, several authors have explored stress-strength models using various distributions. They conducted classical and Bayesian assessments of R and its implications for distributions like the Exponentiated Burr distribution [1], Exponential distribution [19], Generalized Pareto distribution [27], Lindley distribution [2] and three-parameter generalized Rayleigh [13] were performed. The systems discussed in the articles mentioned are primarily single component systems, however, in numerous instances, they consist of two or more components, known as multi-component systems. Examples of such systems include mice, keyboards, computer hardware, air motors, and others. Mansoor et al. [15] and Dey et al. [7] conducted studies on estimating reliability in multi-component stress-strength models using classical and Bayesian approaches for Weibull and Kumaraswamy distributions, respectively. Rao [22], [24], [23] explored the reliability of the multi component stress-strength model using the generalized exponential distribution, the Burr distribution, and the two-parameter Weibull exponential distribution. More recently, Day and Moala [6] as well as Kayal and team [11] investigated the reliability of the multi component stress-strength model under the bathtub failure rate function and the Chen distribution, respectively.

In many research studies, it is commonly assumed that stress and strength variables are independent, even though in reality they are often dependent. For instance, household income and expenses can be seen as the strength variable X and the stress variable Y, respectively. In this scenario, $R = P(Y < X)$ represents the financial capability of the household, while $R = P(X < Y)$ indicates the financial vulnerability of the household [8]. The correlation between a household's income and expenses is evident. Therefore, it is crucial to assess the validity of the stress-strength model when there is a connection between stress and strength variables. One approach in this area is to employ bivariate distributions to characterize the relationship between stress and strength. For instance, Nadarajah [18], Kzlaslan and Nadar [12] and Nadar and Kzlaslan [17] utilized the bivariate gamma distribution, bivariate Kumaraswamy distribution, and bivariate Marshall-Olkin Weibull distribution, respectively, to evaluate the reliability of models incorporating stress-strength dependencies.

Certainly, the bivariate distribution model requires that the marginal distributions. To address this constraint, the copula function is employed, serving as the correlation function between the cumulative and marginal distributions. Recent research has delved into copula functions, for example Navarro and Durante [20] and Liu et al. [14]. Since in reality, it is difficult and sometimes impossible to obtain complete information about the failure time of all test units, it is recommended to use censored data in this case. Bai et al. [4] scrutinized the dependent inference $R = P(Y < X)$ utilizing Gamble's copula in the context of type-II progressively hybrid censoring. Additionally, Asgharzadeh et al. [3] presented a study on the reliability estimation of the multicomponent stress-strength model under a hybrid censoring scheme.

Based on the preliminaries discussed, this paper aims to assess the reliability of the multicomponent stress-strength model in the context of type-I progressively hybrid censoring where the stress and strength variables are dependent. The model is examined using the copula function.

The structure of the remainder of this article is as follows: In Section 2, copula theory is given. Section 3 outlines the model and fundamental assumptions. Section 4 covers the estimation of unknown parameters and the evaluation of stress-strength model reliability using the maximum likelihood method, as well as asymptotic and bootstrap percentile confidence intervals. Section 5 presents the results of simulation studies to elucidate the theoretical concepts, followed by a real data analysis in Section 6. Lastly, Section 7 provides the findings of the investigation.

2 Copula theory

Copulas are valuable tools for modeling and assessing the correlation between multiple dependent variables. For a deeper understanding of the underlying theory, please consult the works of Nelson [21] and Wei and Zhang [26]. The Archimedean copula, a significant category of copulas, is widely recognized and utilized for its straightforward construction, diverse range, and numerous favorable characteristics. In the following, a definition of the Archimedean copula is provided. The Gumbel's copula is one of the types of Archimedean copulas. In this paper Gumbel's copula is used to describe the dependence between strength and stress variables.

Consider random variables $X_1, X_2, \ldots, X_p$ with continuous distribution functions $F_1, F_2, \ldots, F_p$, and their corresponding survival functions $S_1 = 1 - F_1, S_2 = 1-F_2, \ldots, S_p = 1-F_p$. Consequently, the well-known Scalar theorem [25] is defined as follows:

Theorem 2.1. *Let H be the distribution function with marginal functions $F_1, F_2, \ldots, F_p$. In such situations, there exists a p-dimensional copula function C such that for all $\mathbf{X} \in \overline{\mathbf{R}}^{\mathbf{P}}$ holds:*

$$H(x_1, x_2, \ldots, x_p) = C(F_1(x_1), F_2(x_2), \ldots, F_p(x_p)). \tag{1}$$

If all the functions $F_1, F_2, \ldots, F_p$ are continuous, the copula function C is unique. Otherwise, C is uniquely determined on the $RanF_1 \times RanF_2 \times \cdots \times RanF_p$. In addition, if C is a p-dimensional copula function and $F_1, F_2, \ldots, F_p$ are distribution functions, then the function H defined in equation (1) has marginal functions $F_1, F_2, \ldots, F_p$. If $F_1^{(-1)}, F_2^{(-1)}, \ldots, F_p^{(-1)}$ are quasi-inverse functions $F_1^{(-1)}, F_2^{(-1)}, \ldots, F_p^{(-1)}$, then for every $\upsilon \in \mathbf{I^p}$, we have:

$$C(\upsilon_1, \upsilon_2, \ldots, \upsilon_p) = H\big(F_1^{(-1)}(\upsilon_1), F_2^{(-1)}(\upsilon_2), \ldots, F_p^{(-1)}(\upsilon_p)\big) \tag{2}$$

Archimedes copulas have been particularly noted for their favorable characteristics among all copulas. The three primary types of Archimedean copulas include Clayton, Frank, and Gumbel. Here, to maintain generality, we present the two-dimensional Archimedean copula utilized in this paper. The Archimedean copula family is formed by its generator $\varphi_\theta(.)$, which is associated with the dependence parameter θ; i.e.

$$C_\theta(\upsilon_1, \upsilon_2) = \varphi_\theta\Big(\varphi_\theta^{-1}(\upsilon_1), \varphi_\theta^{-1}(\upsilon_2)\Big), (\upsilon_1, \upsilon_2) \in \mathbf{I^2}. \tag{3}$$

where $\varphi_\theta(.)$ is a continuously decreasing convex function. There is a one-to-one relationship between the dependence parameter and Kendall's tau. Kendall's tau provides a more intuitive measure of correlation compared to the dependence parameter. Kendall's tau is denoted as $\tau_{kendall}$ and $\tau_{kendall} \in [-1, 1]$. When $\tau_{kendall} = 0$ for random variables X_1 and X_2, it indicates that X_1 and X_2 are independent.

If $\tau_{kendall}$ equals 1 or -1, it signifies a complete positive or negative correlation between X_1 and X_2. The closer Kendall's absolute value is to 1, the greater the strength of the relationship.

To assess the level of dependence in a more intuitive manner, the dependence parameter θ is typically transformed into Kendall's tau. This paper employs the Gumbel copula to characterize the relationship between two random variables using the generator $\varphi_\theta(t) = (-\log t)^\theta$, as outlined below:

$$C_\theta(\upsilon, \nu) = \exp\Big\{ - [(-\log \upsilon)^\theta + (-\log \nu)^\theta]^{\frac{1}{\theta}} \Big\}, \quad \theta \geq 1, \tag{4}$$

And the relationship between $\tau_{kendall}$ and Gamble copula dependence parameter is $\tau_{kendall} = 1 - \theta^{-1}$.

3 Model assumptions

In a system comprising l components connected in series, the strength of the j th component is denoted by X_j, for $j = 1, 2, \ldots, l$; where X_j are independent random variables with a cumulative distribution function $F_{X_j}(x_j)$. All component strengths are exposed to an external stress Y, characterized by a cumulative distribution function $F_Y(y)$ with dependencies on each X_i. It is known that in any series system, the system's lifetime equals the minimum lifetime of its components, expressed as $X_{(1)} = \min_{1\leq j\leq l}(X_j)$. The cumulative distribution function and probability density function of $X_{(1)}$ are given by:

$$F_{X_{(1)}} = 1 - \prod_{j=1}^{l} \left[1 - F_{X_j(x)}\right], \tag{5}$$

$$f_{X_{(1)}} = \sum_{j=1}^{l} h_{X_j}(x) \prod_{j=1}^{l} \left[1 - F_{X_j(x)}\right], \tag{6}$$

And the function $h_{X_j}(x)$ represents the hazard rate of the variable X_j, defined as $h_{X_j}(x) = \frac{f_{X_j}(x)}{1-F_{X_j}(x)}$.

If the stress Y surpasses the minimum strength $X_{(1)}$, the system will fail; otherwise, it will continue to operate correctly. Therefore, the variable δ is defined as follows:

$$\delta = \begin{cases} 1 & X_{(1)} \leq Y, \\ 0 & X_{(1)} > Y. \end{cases}$$

This paper assumes that the random variable X_j representing strength, where $j = 1, 2, \ldots, l$, follows a Pareto distribution with scale parameter α_j . Similarly, the random variable Y representing stress also follows a Pareto distribution with scale parameter α_0. The Pareto distribution is a probability distribution used to model phenomena where a small number of observations have significantly higher values than the majority of the data. It is commonly applied in various fields, including economics, finance, insurance, and reliability engineering. Here are some key applications of the Pareto distribution: income Distribution, failure rates or lifetimes of components or systems, risk Management, quality Control. The corresponding cumulative distribution functions for these variables are as follows:

$$\begin{aligned} F_{X_j}(x_j) &= 1 - x_j^{-\alpha_j}, \qquad x_j > 0,\ \ \alpha_j > 0, \\ F_Y(y) &= 1 - y^{-\alpha_0}, \qquad y > 0,\ \ \alpha_0 > 0, \end{aligned}$$

By utilizing equations (5) and (6), we can derive cumulative functions and determine the probability of system lifetime, which corresponds to the minimum strength of the system components.

$$F_{X_{(1)}}(x; \alpha_1, \ldots, \alpha_l) = 1 - x^{-\sum_{j=1}^{l} \alpha_j}, \qquad x > 0, \alpha_j > 0,$$

and

$$f_{X_{(1)}}(x; \alpha_1, \ldots, \alpha_l) = \sum_{j=1}^{l} \alpha_j x^{-1-\sum_{j=1}^{l} \alpha_j}, \qquad x > 0, \alpha_j > 0.$$

In this paper, we also consider the assumption that the relationship between stress Y and the minimum strength $(X_{(1)})$ is represented by a copula function. As per the scalar theorem 2.1, the two-dimensional joint distribution can be illustrated by the two-dimensional copula along with two marginal functions.

By utilizing equations (4) and (5), the survival function of the system's lifetime based on the Gumbel copula can be described as follows:

$$S(t) = e^{-\left[\left(\sum_{j=1}^{l} \alpha_j \log t\right)^{\theta} + \left(\alpha_0 \log t\right)^{\theta}\right]^{\frac{1}{\theta}}} \tag{7}$$

where θ is the dependence parameter. In the following, we consider that $\alpha = \sum_{j=1}^{l} \alpha_j$.

This study assumes that the strength elements operate autonomously, meaning the failure of the ith component does not impact the failure of the jth component for $i \neq j$. Given the assumptions made, the reliability of this system can be determined as follows:

$$\begin{aligned} R = P(Y < X_{(1)}) &= \int_0^{\infty} \int_0^{x_{(1)}} h(x_{(1)}, y) dy dx_{(1)}, \\ &= \int_0^{\infty} \int_0^{x_{(1)}} c(f(x_{(1)}, g(y)) f(x_{(1)}) g(y) dy dx_{(1)}, \end{aligned}$$

So

$$R = \int_0^{\infty} \frac{\partial C(\upsilon, \nu)}{\partial \nu} \mid_{\upsilon = f(x_{(1)}) \nu = g(x_{(1)})} g(x_{(1)}) dx_{(1)}. \tag{8}$$

In the equation mentioned, h represents the joint probability function, and C is the two-dimensional copula distribution function.

In order to reduce both time and cost, the data is gathered using a type-I progressively hybrid censoring method, as detailed further. Assuming there are n identical systems undergoing a lifetime test with an increasing censoring scheme $(R_1, R_2, \ldots, R_m)$ where $1 \leq m \leq n$. The experiment concludes at time τ_0, which is predetermined along with the values $(R_1, R_2, \ldots, R_m)$. Upon the first observation at time t_1, R_1 systems are randomly eliminated from the initial $n-1$ systems. Subsequently, at time t_2 with the second observation, R_2 systems are randomly removed from the remaining $n-2-R_1$ surviving systems. The censoring process persists in a similar manner until its completion. If the time of the mth observation, denoted as t_m, is earlier than the predetermined time τ_0, all surviving systems at time t_m ($R_m^* = n - m - \sum_{i=1}^{m-1} R_i$) are eliminated, and the experiment concludes at that time. If t_m happens after τ_0 and only r observations take place before tau_0 $(0 \leq r \leq m)$, then at τ_0, all the remaining survival systems ($R_r^* = n - r - \sum_{i=1}^{r} R_i$) are eliminated from the test, and the test ends at τ_0. These scenarios are illustrated below:

Case I: If $t_m \leq \tau_0$, then $t_1 < t_2 < \cdots < t_m < \tau_0$,

Case II: If $t_m > \tau_0$, then $t_1 < t_2 < \cdots < t_r < \tau_0 < t_{r+1} < t_m$.

Therefore, the observed data is obtained as $(t_1, \delta_1)(t_2, \delta_2)...(t_{m^*}, \delta_{m^*})$. In Case I: $t_{m^*} = t_m$ and $m^* = m$; and in Case II: $t_{m^*} = \tau_0$ and $m^* = r$.

4 Estimation of system reliability

4.1 Maximum likelihood estimation

In order to assess the reliability of the system outlined in (8), it is crucial to determine the unidentified parameters of the model. Initially, we estimate the dependency parameter. Consider $X_1, X_2, \ldots, X_m$ as a two-variable random vector, for $i = 1, 2, \ldots, m$, where $X_i = (x_{i1}, x_{i2})$. In this scenario, Kendall's tau is computed as follows:

$$\tau_{kendall} = \frac{4}{m(m-1)} \sum_{i \neq j} I_{\{X_{i1} \leq X_{j1}\}} I_{\{X_{i2} \leq X_{j2}\}} - 1. \tag{9}$$

As per the section provided, Kendall's tau in the Gamble copula is expressed as $\tau_{kendall} = 1 - \theta^{-1}$. With a direct relationship between $\tau_{kendall}$ and θ, the estimator of $\tau_{kendall}$ can be derived using equation (9), leading to the calculation of the estimate for θ as outlined below:

$$\hat{\theta} = \frac{1}{1 - \hat{\tau}_{kendall}}.$$

Next, we employ the maximum likelihood approach to estimate the parameters that are unknown. Let's assume $n_f = \sum_{i=1}^{m^*} \delta_i$.

The likelihood function for the stress-strength model given the observed data $(t_1, \delta_1), (t_2, \delta_2), \ldots, (t_{m^*}, \delta_{m^*})$ under type-I progressively hybrid censoring is as follows:

$$\begin{aligned} L = \prod_{i=1}^{m^*} & \Big[\frac{\partial C(\upsilon, \nu)}{\partial \upsilon} \,|_{\upsilon = S_{X_{(1)}(t_i)}, \nu = S_Y(t_i)} \; f_{X_{(1)}}(t_i) \Big]^{\delta_i} \times \\ & \Big[\frac{\partial C(\upsilon, \nu)}{\partial \nu} \,|_{\upsilon = S_{X_{(1)}}(t_i), \nu = S_Y(t_i)} \; f_Y(t_i) \Big]^{1-\delta_i} \Big[S(t_i) \Big]^{R_i} S(t_{m^*})^{n - m^* - \sum_{i=1}^{m^*} R_i}. \end{aligned} \tag{10}$$

The likelihood function is obtained by replacing equations (5) to (7) into equation (10). To simplify calculations, we utilize the logarithm of the likelihood function, which is expressed as follows:

$$\begin{aligned} \ell = \log L(\alpha_0, \alpha) \quad = \quad & (m^* - n_f) \log \alpha_0 + \alpha_0 (m^* - n_f) \sum_{i=1}^{m^*} \log t_i \\ & + \sum_{i=1}^{m^*} (1 - \delta_i) \log(\log t_i) + (1 - \theta) \sum_{i=1}^{m^*} \delta_i \log(\log t_i) \\ & + (\frac{1}{\theta} - 1) \sum_{i=1}^{m^*} \log \big[A(t_i) \big] + \sum_{i=1}^{m^*} \delta_i \log(t_i^{\alpha}) + (1 - \theta) \sum_{i=1}^{m^*} \delta_i \log \alpha \\ & - \sum_{i=1}^{m^*} (R_i + 1) \big[A(t_i) \big]^{\frac{1}{\theta}} - (n - m^* - \sum_{i=1}^{m^*} R_i) \big[A(t_{m^*}) \big]^{\frac{1}{\theta}}. \end{aligned}$$

where

$$A(x) = (\alpha \log x)^{\theta} + (\alpha_0 \log x)^{\theta}. \tag{11}$$

Now, by deriving the logarithm of the likelihood function and setting them equal to zero, we can calculate the estimate of the unknown parameters. We have:

$$\begin{aligned}
\frac{\partial \ell}{\partial \alpha_0} &= \frac{m^* - n_f}{\alpha_0} + (m^* - n_f)\sum_{i=1}^{m^*} \log t_i - \sum_{i=1}^{m^*}(R_i + 1)\big(A(t_i)\big)^{\frac{1}{\theta}-1}\alpha_0(\log t_i)^{\theta} \\
&\quad +(\frac{1}{\theta} - 1)\sum_{i=1}^{m^*}\frac{\theta\alpha_0^{\theta-1}(\log t_i)^{\theta}}{A(t_i)} - (n - m^* - \sum_{i=1}^{m^*} R_i)\big(A(t_{m^*})\big)^{\frac{1}{\theta}-1}\alpha_0(\log t_{m^*})^{\theta}, \\
\frac{\partial \ell}{\partial \alpha} &= \sum_{i=1}^{m^*}\delta_i \log t_i + \frac{1-\theta}{\alpha}\sum_{i=1}^{m^*}\delta_i - \sum_{i=1}^{m^*}(R_i + 1)\big(A(t_i)\big)^{\frac{1}{\theta}-1}\alpha(\log t_i)^{\theta} \\
&\quad +(\frac{1}{\theta} - 1)\sum_{i=1}^{m^*}\frac{\theta\alpha^{\theta-1}(\log t_i)^{\theta}}{A(t_i)} - (n - m^* - \sum_{i=1}^{m^*} R_i)\big(A(t_{m^*})\big)^{\frac{1}{\theta}-1}\alpha(\log t_{m^*})^{\theta}.
\end{aligned}$$

Clearly, solving the aforementioned equations analytically poses a challenge. Consequently, we have to use numerical methods to calculate $\hat{\alpha}_0$ and $\hat{\alpha}_1$; in this paper Newton-Raphson method is used.

4.2 Asymptotic confidence interval

Due to the lack of closed-form solutions for the maximum likelihood estimator (MLE) of the parameters, exact confidence intervals cannot be obtained. Hence, asymptotic confidence intervals are derived based on the asymptotically normal property of MLEs. As a result, the two-sided asymptotic confidence intervals for the unknown parameters α_0 and α at a confidence level of $100(1-\zeta)\%$ are as follows:

$$\alpha_0 : \hat{\alpha}_0 \pm z_{\zeta/2}\hat{\sigma}_{\alpha_0}, \qquad \alpha : \hat{\alpha} \pm z_{\zeta/2}\hat{\sigma}_{\alpha}$$

where $z_{\zeta/2}$ represents the $\zeta/2$th percentile point in the standard normal distribution. The estimation of the parameters σ_{α_0} and σ_{α}, denoted as $\hat{\sigma}_{\alpha_0}$ and $\hat{\sigma}_{\alpha}$ respectively. They can be easily computed using the inverse of the Fisher information matrix for the unknown parameters. Therefore, the next step is to obtain the Fisher information matrix.

Fisher information matrix is a matrix that contains key information about unknown parameters. It is determined as the negative of the second-order partial derivatives of the logarithm of the likelihood function with respect to the unknown parameters based on the collected data, as shown below.

$$
\begin{aligned}
I_{11} = &-\frac{\partial^2 \ell}{\partial \alpha_0^2} = -\frac{m^* - n_f}{\alpha_0^2} + (\frac{1}{\theta} - 1)\sum_{i=1}^{m^*} \frac{\theta(\theta-1)\alpha_0^{\theta-2} A(t_i)\big(\log t_i\big)^{\theta} - \theta^2 \alpha_0^{2\theta-2}\big(\log t_i\big)^{2\theta}}{A(t_i)^2} \\
&- \sum_{i=1}^{m^*}(R_i+1)\big(A(t_i)\big)^{\frac{1}{\theta}-1}\log t_i^{\theta} - \sum_{i=1}^{m^*}(R_i+1)(1-\theta)\big(A(t_i)\big)^{\frac{1}{\theta}-2}(\log t_i)^{2\theta}\alpha_0^{2\theta-1} \\
&- (n - m^* - \sum_{i=1}^{m^*} R_i)\Big\{(\theta-1)\alpha_0^{\theta-2}(\log t_{m^*})^{\theta}\big(A(t_{m^*})\big)^{\frac{1}{\theta}-1} + (1-\theta)\big(A(t_{m^*})\big)^{\frac{1}{\theta}-2} \\
&\alpha_0^{2\theta-2}\big(\log t_{m^*}\big)^{2\theta}\Big\} \\
I_{12} = I_{21} = &-\frac{\partial^2 \ell}{\partial \alpha \partial \alpha_0} = -(1-\theta)\alpha_0^{\theta-1}\alpha^{\theta-1}\sum_{i=1}^{m^*}(R_i+1)\big(A(t_i)\big)^{\frac{1}{\theta}-2}(\log t_i)^{2\theta+2} \\
&- (\frac{1}{\theta} - 1)\sum_{i=1}^{m^*}\frac{\theta^2\alpha_0^{\theta-1}\alpha^{\theta-1}\big(\log t_i\big)^{2\theta}}{\big(A(t_i)\big)^2} - (n - m^* - \sum_{i=1}^{m^*} R_i)(1-\theta)\big(\log t_{m^*}\big)^{2\theta}\alpha_0^{\theta-1}\alpha^{\theta-1}, \\
I_{22} = &-\frac{\partial^2 \ell}{\partial \alpha^2} = -(1-\theta)\sum_{i=1}^{m^*}\frac{\delta_i}{\alpha^2} + (\frac{1}{\theta} - 1)\sum_{i=1}^{m^*}\frac{\theta(\theta-1)\alpha^{\theta-2}(\log t_i)^{\theta}A(t_i) - \theta^2\alpha^{2\theta-2}(\log t_i)^{t\theta}}{A(t_i)^2} \\
&- \sum_{i=1}^{m^*}(R_i+1)\Big\{\big(A(t_i)\big)^{\frac{1}{\theta}-1}(\theta-1)\alpha^{\theta-2}\big(\log t_i\big)^{\theta} + (1-\theta)\big(A(t_i)\big)^{\frac{1}{\theta}-2}\alpha^{2\theta}\big(\log t_i\big)^{2\theta}\Big\} \\
&- (n - m^* - \sum_{i=1}^{m^*} R_i)\Big\{(\theta-1)\alpha^{\theta-2}(\log t_{m^*})^{\theta}\big(A(t_{m^*})\big)^{\frac{1}{\theta}-1} + (1-\theta)\big(A(t_{m^*})\big)^{\frac{1}{\theta}-2} \\
&\alpha^{2\theta-2}\big(\log t_{m^*}\big)^{2\theta}\Big\}.
\end{aligned}
$$

Substituting $\hat{\alpha}_0$ and $\hat{\alpha}$ for α_0 and α in I yields the observed Fisher information matrix. Inverting it provides the approximate asymptotic variance-covariance matrix for parameters α_0 and α. Following Bai et al. [4], the subsequent theorems are presented.

Theorem 4.1. *As $m \to \infty$ we have $\big(\sqrt{m}(\hat{\alpha}_0 - \hat{\alpha}_0), \sqrt{m}(\alpha_0 - \alpha)\big) \sim N_2(0, I^{-1}/m)$ where I^{-1} is the asymptotic variance-covariance matrix.*

Theorem 4.2. *As $m \to \infty$, we have $\sqrt{m}(\hat{R} - R) \sim N(0, B)$ where $B = b^T I^{-1} b$, $\xi = (\alpha_0, \alpha)^T$, $b = \left(\frac{\partial g}{\partial \alpha_0}, \frac{\partial g}{\partial \alpha}\right)^T$ and*

$$
R = g(\xi) = \int_0^{\infty}\int_0^{x_{(1)}} c\big(f(x_{(1)}), f(y)\big) f(x_{(1)}) f(y) dy dx_{(1)},
$$

Thus, the asymptotic confidence interval R is derived by replacing α_0 and α_0 whit $\hat{\alpha}_0$ and $\hat{\alpha}$ in B. Consequently, the asymptotic confidence interval at the $100(1-\zeta)\%$ confidence level for R is:

$$
R: \;\; \hat{R} \pm z_{\zeta/2}\sqrt{\hat{B}/m}.
$$

4.3 Bootstrap confidence interval

The requirement for utilizing an asymptotic confidence interval is a substantial sample size. Nevertheless, there are instances where the sample sizes may be insufficient. In such cases, the assumption of asymptotic normality of MLEs is not valid, making bootstrap confidence intervals more appropriate. The percentile bootstrap method, introduced by Efron [9], is derived as follows.

(i) Calculate the $\hat{\alpha}_0$ and $\hat{\alpha}$ with regards to n, m, τ_0, $(R_1, R_2, \ldots, R_m)$ and based on the sample $(t_1, t_2, , t_{m^*})$ obtained under the type-I progressively hybrid censoring.

(ii) Generate the bootstrap sample $(t_1^*, t_2^*, \ldots, t_{m^*}^*)$ using n, m, τ_0, $(R_1, R_2, \ldots, R_m)$, $\hat{\alpha}_0$ and $\hat{\alpha}$. Then, based on the bootstrap sample $(t_1^*, t_2^*, \ldots, t_{m^*}^*)$, calculate the bootstrap estimates of α_0 and α denoted by $\hat{\alpha}_0^*$ and $\hat{\alpha}^*$.

(iii) Repeat step 2, N times and obtain N estimators of $\hat{\alpha}_0^{*(\nu)}$ and $\hat{\alpha}^{*(\nu)}$ for $\nu = 1, 2, \ldots, N$.

(iv) Substitute $\hat{\alpha}_0^*$ and $\hat{\alpha}^*$ for α_0 and α in equation (8) to derive bootstrap estimators R, denoted by $\hat{R}^{*(\nu)}$ for $\nu = 1, 2, \ldots, N$.

(v) Sort the bootstrap estimators $\{\hat{\alpha}_0^{*(\nu)}, \hat{\alpha}^{*(\nu)}, \hat{R}^{*(\nu)}\}$ in ascending order to obtain the following bootstrap sample

$$\{\hat{\alpha}_0^{*[1]}, \ldots, \hat{\alpha}_0^{*[N]}; \hat{\alpha}^{*[1]}, \ldots, \hat{\alpha}^{*[N]}; \hat{R}^{*[1]}, \ldots, \hat{R}^{*[N]}\}.$$

(vi) Obtain $100(1-\zeta)\%$ two-sided confidence intervals for the parameters α_0, α and R in the following manner:

$$\begin{aligned}
\alpha_0 &: \left(\hat{\alpha}_{0,L}^*, \hat{\alpha}_{0,U}^*\right) = \left(\hat{\alpha}_0^{*[N(\zeta/2)]}, \hat{\alpha}_0^{*[N(1-\zeta/2)]}\right), \\
\alpha &: \left(\hat{\alpha}_L^*, \hat{\alpha}_U^*\right) = \left(\hat{\alpha}^{*[N(\zeta/2)]}, \hat{\alpha}^{*[N(1-\zeta/2)]}\right), \\
R &: \left(\hat{R}_L^*, \hat{R}_U^*\right) = \left(\hat{R}^{*[N(\zeta/2)]}, \hat{R}^{*[N(1-\zeta/2)]}\right).
\end{aligned}$$

5 Simulation studies

In this section, numerical methods are employed to assess the performance of reliability estimation methods and unknown parameters of the stress-strength model. One aim of this analysis is to assess how R varies with adjustments in the initial values. Consider the n two-component series system, where $\alpha_0 = 0.9$, $\alpha_1 = 1.4$, and $\alpha_2 = 2$, and therefore resulting in $\alpha = 3.4$. In addition, assume $\theta = 3, 4, 5$ as the dependence parameter, which leading to $\tau_{Kendall} = \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$, respectively. Table 1 is utilized for the prefixed progressively sampling scheme. Henceforth, the symbol $i - j$ denotes the i-th scheme with j-progressive samples for $i, j = 1, 2, 3$. For example, $1 - 1$ represents scheme 1 with incremental censoring samples 1, i.e.

Table 1: Censoring schemes.

Scheme	n	m	τ_0	Progressively censoring scheme		
				1	2	3
1	50	20	0.3	$(1,1,\ldots,1,1)$	$(2,0,\ldots,2,0)$	$(3,0,0,\ldots,3,0,0)$
2	80	30	0.3	$(1,1,\ldots,1,1)$	$(2,0,\ldots,2,0)$	$(3,0,0,\ldots,3,0,0)$
3	100	45	0.3	$(1,1,\ldots,1,1)$	$(2,0,\ldots,2,0)$	$(3,0,0,\ldots,3,0,0)$

$(n, m, \tau_0) = (50, 20, 0.3)$ and $(R_1, R_2, \ldots, R_m) = (1, 1, \ldots, 1)$. By conducting a thousand simulations and generating data based on various censoring schemes outlined in Table 1, the MLE of the model's unknown parameters, the Mean Square Error (MSE), the length of the asymptotic confidence interval (LA) at 95%, and the length of the bootstrap percentile confidence interval (LB) at 95% are computed. Furthermore, the coverage percentage (CP) is determined for each confidence interval, and the outcomes for various censoring schemes are showcased in Tables 2 to 4.

The analysis of Tables 2 to 4 indicates that, as anticipated, the MLEs of the unknown parameters approach their true values as the sample size increases, resulting in smaller MSEs. A comparison of the approximate and bootstrap confidence intervals reveals that the length of the asymptotic confidence interval for the effective sample size (m) is greater than the length of the bootstrap percentile confidence interval. As the effective sample size increases, the lengths of both confidence intervals become closer. Moreover, with small sample sizes, the coverage percentage of the bootstrap confidence interval slightly exceeds that of the approximate confidence interval. Naturally, as the sample size grows, all coverage percentages approach their nominal values, i.e. 95%. On the other hand, according to the criteria calculated, there are no significant differences between the three different censorship plans that have been examined in the paper. Furthermore, altering the dependence parameter does not significantly impact the estimation of unknown parameters for identical sample sizes and censoring designs. Also, the reliability estimator declines as the dependence parameter value rises under the same sample size and censoring schemes.

6 Real data

In this part, we examine an real dataset. The dataset utilized here was initially documented by McGilchrist and Aisbett [16]. It presents the timing of infection recurrence in kidney patients using portable dialysis equipment. Data from 30 patients were gathered, where X and Y denoting the timing of the first and second recurrences, respectively. The data is displayed in Table 5. Data transformation is

Table 2: MLEs, MSEs, LAs, LBs and CPs for unknown parameters and R when $\theta = 3$

Scheme	Parameter	MLE	MSE	LA	CP-A	LB	CP-B
1-1	α_0	0.6839	0.1530	0.8040	0.9277	0.7656	0.9256
	α	2.8725	0.1332	0.9222	0.9357	0.9591	0.9295
	R	0.4921	0.0018	0.1915	0.9557	0.1853	0.9505
1-2	α_0	0.7051	0.1492	0.7716	0.9247	0.7419	0.9365
	α	2.9141	0.1356	0.8667	0.9377	0.8725	0.9315
	R	0.4835	0.0015	0.1929	0.9587	0.2036	0.9535
1-3	α_0	0.6955	0.1598	0.7463	0.9367	0.7337	0.9452
	α	2.9735	0.1384	0.8802	0.9487	0.9041	0.9345
	R	0.4812	0.0013	0.1905	0.9547	0.2058	0.9485
2-1	α_0	0.6892	0.1295	0.3018	0.9437	0.2291	0.9335
	α	2.9801	0.1245	0.4743	0.9497	0.4329	0.9365
	R	0.4831	0.0017	0.1539	0.9627	0.1670	0.9565
2-2	α_0	0.7367	0.1357	0.2910	0.9427	0.2424	0.9425
	α	3.0085	0.1225	0.4801	0.9467	0.4461	0.9385
	R	0.4527	0.0013	0.1505	0.9547	0.1669	0.9565
2-3	α_0	0.6938	0.1203	0.3164	0.9557	0.2737	0.9465
	α	3.0461	0.1193	0.4525	0.9537	0.4468	0.9355
	R	0.4302	0.0012	0.1461	0.9577	0.1625	0.9465
3-1	α_0	0.7318	0.1191	0.1439	0.9637	0.1306	0.9535
	α	3.1651	0.1005	0.2515	0.9647	0.2151	0.9425
	R	0.4641	0.0014	0.1252	0.9777	0.1395	0.9595
3-2	α_0	0.7439	0.1128	0.1251	0.9647	0.1027	0.9575
	α	3.2285	0.0923	0.2667	0.9677	0.2356	0.9615
	R	0.4293	0.0011	0.1464	0.9787	0.1636	0.9635
3-3	α_0	0.7561	0.1095	0.1234	0.9634	0.1045	0.9625
	α	3.3019	0.0856	0.2535	0.9587	0.2456	0.9575
	R	0.4351	0.0010	0.1472	0.9747	0.1706	0.9695

Table 3: MLEs, MSEs, LAs, LBs and CPs for unknown parameters and R when $\theta = 4$

Scheme	Parameter	MLE	MSE	LA	CP-A	LB	CP-B
1-1	α_0	0.7018	0.1245	0.7731	0.9407	0.7089	0.9255
	α	3.0115	0.1185	0.9535	0.9377	0.8629	0.9325
	R	0.4521	0.0013	0.2089	0.9537	0.2085	0.9455
1-2	α_0	0.7231	0.1233	0.7915	0.9337	0.8062	0.9335
	α	3.0417	0.1152	0.8555	0.9447	1.8037	0.9375
	R	0.4533	0.0012	0.2010	0.9617	0.2169	0.9505
1-3	α_0	0.7345	0.1187	0.7225	0.9447	0.7358	0.9405
	α	3.0552	0.1120	1.8764	0.9376	0.7968	0.9355
	R	0.4566	0.0013	0.1949	0.9507	0.2089	0.9455
2-1	α_0	0.7518	0.1134	0.2973	0.9417	0.2549	0.9425
	α	3.1451	0.1102	0.4822	0.9557	0.4460	0.9405
	R	0.4251	0.0010	0.1611	0.9527	0.1713	0.9445
2-2	α_0	0.7631	0.1104	0.3046	0.9497	0.2584	0.9475
	α	3.1838	0.1082	0.5011	0.9427	0.4651	0.9385
	R	0.4271	0.0011	0.1572	0.9527	0.1606	0.9435
2-3	α_0	0.7574	0.1122	0.3036	0.9517	0.2802	0.9435
	α	3.2141	0.1054	0.4625	0.9427	0.4352	0.9315
	R	0.4218	0.0009	0.1361	0.9497	0.1611	0.9375
3-1	α_0	0.7711	0.1072	0.1373	0.9557	0.1046	0.9525
	α	3.0741	0.1100	0.2266	0.9627	0.2082	0.9505
	R	0.4189	0.0008	0.1205	0.9757	0.1308	0.9595
3-2	α_0	0.7784	0.1054	0.1137	0.9607	0.0985	0.9565
	α	3.1425	0.1080	0.2302	0.9627	0.2182	0.9555
	R	0.4112	0.0007	0.1112	0.9827	0.1297	0.9685
3-3	α_0	0.7591	0.1032	0.1081	0.9597	0.0937	0.9555
	α	3.1635	0.1015	0.2222	0.9637	0.2069	0.9515
	R	0.4154	0.0008	0.1070	0.9837	0.1239	0.9725

Table 4: MLEs, MSEs, LAs, LBs and CPs for unknown parameters and R when $\theta = 4$

Scheme	Parameter	MLE	MSE	LA	CP-A	LB	CP-B
1-1	α_0	0.7612	0.1020	0.6930	0.9357	0.6381	0.9355
	α	4.1130	0.1008	0.9542	0.9470	0.8848	0.9385
	R	0.3978	0.0007	0.1951	0.9577	0.1976	0.9465
1-2	α_0	0.7893	0.1008	0.6869	0.9387	0.6287	0.9305
	α	3.1780	0.0985	0.8935	0.9467	0.7661	0.9395
	R	0.3912	0.0007	0.1906	0.9627	0.1989	0.9515
1-3	α_0	0.8012	0.0961	0.7221	0.9397	0.6476	0.9365
	α	3.1990	0.0820	0.9123	0.9427	0.8590	0.9375
	R	0.3925	0.0006	0.1990	0.9587	0.2058	0.9555
2-1	α_0	0.8218	0.0842	0.2802	0.9397	0.2428	0.9375
	α	3.3015	0.0750	0.4645	0.9497	0.4367	0.9425
	R	0.3921	0.0004	0.1566	0.9498	0.1682	0.9475
2-2	α_0	0.8312	0.0804	0.2621	0.9467	0.2389	0.9355
	α	3.3315	0.0710	0.4919	0.9457	0.4861	0.9415
	R	0.3905	0.0005	0.1463	0.9637	0.1682	0.9505
2-3	α_0	0.8418	0.0725	0.2522	0.9537	0.2400	0.9487
	α	3.3654	0.0640	0.4501	0.9527	0.4281	0.9405
	R	0.3910	0.0005	0.1403	0.9667	0.1511	0.9625
3-1	α_0	0.8591	0.0420	0.0510	0.9397	0.0378	0.9505
	α	3.4391	0.0512	0.1879	0.9497	0.1681	0.9535
	R	0.3871	0.0003	0.1095	0.9398	0.1314	0.9695
3-2	α_0	0.8851	0.0311	0.0653	0.9467	0.0456	0.9585
	α	4.4251	0.0508	0.2001	0.9457	0.1821	0.9515
	R	0.3821	0.0003	0.1043	0.9637	0.1246	0.9695
3-3	α_0	0.8920	0.0285	0.0643	0.9537	0.0477	0.9595
	α	3.3918	0.0402	0.1888	0.9527	0.1706	0.9645
	R	0.3790	0.0002	0.1034	0.9667	0.1207	0.9755

applied in this model by taking the square root of the data and dividing all values by 10. This adjustment aligns the data well with the proposed model. The dependence parameter is estimated using Kendall's tau value as $\hat{\theta} = 1.13$. As a result, the data for X and Y can be deemed dependent. By employing numerical techniques, the model's unknown parameters were estimated, and the system reliability was calculated, with the outcomes displayed in Table 6. Furthermore, Table 7 presents the results of the Kolmogorov-Smirnov test assessing the data's goodness of fit. Based on the information in Table 7, it is observed that the Pareto distribution is appropriate for both sets of data. By estimating the parameters $\hat{\alpha}_0$ $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\hat{\theta}$ and applying the invariant property of the MLEs, $\hat{R} = 0.5247$ for the dependent condition and $\hat{R} = 0.5773$ for the independent condition, representing a 10.02% increase in the dependent mood.

Table 5: Real dataset from McGilchrist and Aisbett [16].

Variable	Data														
X	8	23	22	447	30	24	7	511	53	15	7	141	96	149	536
	17	185	292	22	15	152	402	13	39	12	113	132	34	2	130
Y	16	13	28	318	12	245	9	30	196	154	333	8	38	70	25
	4	117	114	159	108	362	24	66	46	40	201	156	30	25	26

Table 6: MLEs, asymptotic and bootstrap confidence intervals of unknown parameters and reliability for real data.

Parameter	α_0	α	R
MLE	0.5107	0.7048	0.5247
ACI	(0.5567 0.4819)	(0.7476 0.6935)	(0.5562 0.4986)
BCI	(0.5413 0.4900)	(0.7220 0.6901)	(0.5436 0.5021)

Table 7: Goodness of fit test results.

Parameter	K-S	P-value
X	01425	0.8754
Y	0.1281	0.9381

7 Conclusion

In this research, we examined the reliability of the multicomponent stress-strength model. While many studies typically treat stress and strength variables as independent, this paper takes into account real-world scenarios where there is a presumed relationship between stress and strength variables. The Gamble copula function was employed to assess this dependency. Furthermore, it was hypothesized that the stress and strength variables adhere to the Pareto distribution with distinct parameters, and the data was collected using the type-I progressively hybrid censoring. We calculated ML estimators, along with asymptotic confidence intervals and bootstrap confidence intervals for the unknown parameters and the reliability of the stress-strength model. Nine distinct datasets were generated from the analyzed model by varying censoring schemes and adjusting the dependence parameter. The simulation findings indicated that as the sample size grows, the MSE of estimating unknown parameters decreases, along with an improvement in the model's reliability and an increase in the coverage percentage of the calculated confidence intervals. When dealing with a small effective sample size, the bootstrap confidence interval outperforms the asymptotic confidence interval. As the effective sample size increases, both confidence intervals approach each other. Nevertheless, the significance of the dependence parameter should not be overlooked as it can lead to a significant deviation in the model's reliability estimate. The greater the dependence parameter, the weaker the model's reliability. Then, examination of the real dataset demonstrated that the estimation method employed is practical for assessing the reliability of the multicomponent stress-strength model using the copula function.

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